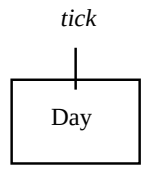
Mod7 Counter Satisfies Days

Garlan & Wing Handout 10 9 October 1995

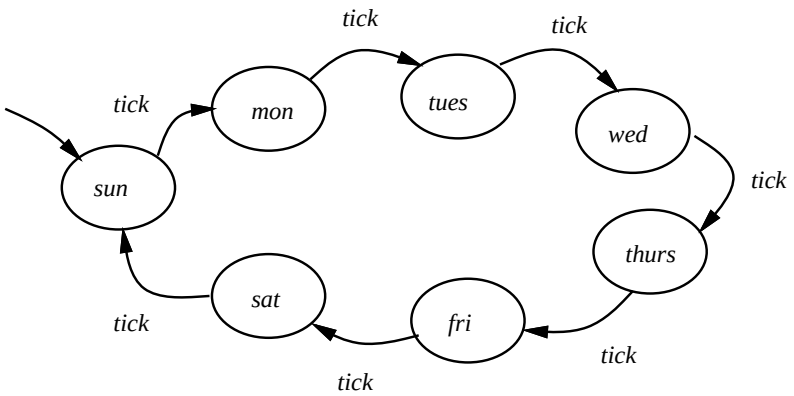
Here's a simple example to show how an "integer mod-7" counter state machine satisfies a "days of the week" machine. Both the abstract and concrete machines have a finite number of states, a finite number of transitions, and infinite traces. My proof of correctness uses an abstraction function that is one-to-one.

1 The Abstract Machine

The Day abstract machine, has one *tick* action.



Day's Interface



Day's State Transition Diagram

Day is like an enumerated type in Pascal or Ada; its set of states is just the days of the week:

$$
\begin{array}{l}
\text{Day} = (\\
\{sun, mon, tues, wed, thurs, fri, sat\}, \\
\{sun\}, \\
\{tick\}, \\
\delta_A = \{(sun, tick, mon), (mon, tick, tues), (tues, tick, wed), (wed, tick, thurs), \\
\quad (thurs, tick, fri), (fri, tick, sat), (sat, tick, sun)\} \\
).
\end{array}
$$

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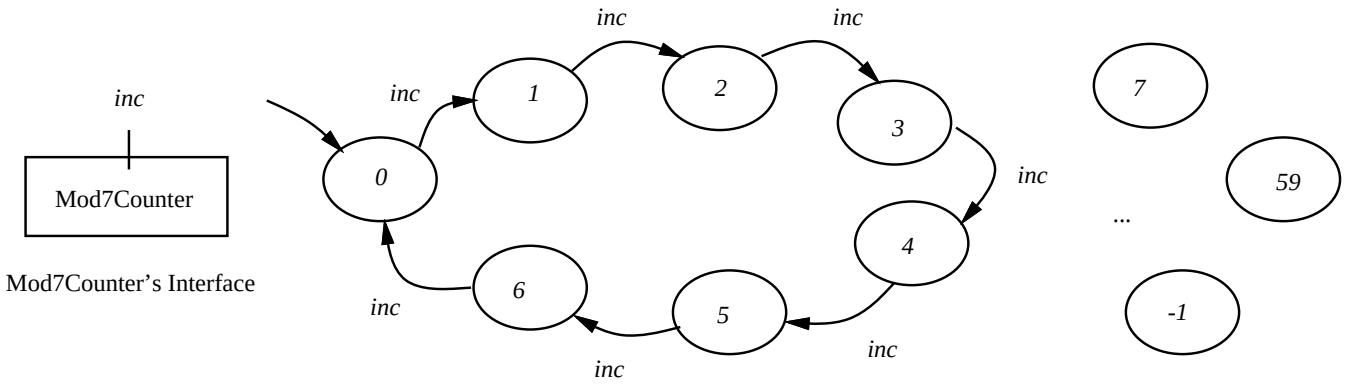
2 The Concrete Machine

Most programming languages do not support something like Day as a built-in type. But fortunately, they usually support integers. So I'm going to represent the Day machine in terms of integers modulo-7.



Mod7Counter's State Transition Diagram

$$
\begin{array}{l}
\text{Mod7Counter} = (\\
\{x : int\}, \\
\{0\}, \\
\{inc\}, \\
\delta_C = \{(0, inc, 1), (1, inc, 2), (2, inc, 3), (3, inc, 4), (4, inc, 5), \\
\quad (5, inc, 6), (6, inc, 0)\} \\
).
\end{array}
$$

My Mod7Counter concrete machine has a single action, *inc*, which is defined in the obvious way. I've intentionally defined Mod7Counter's set of states to be the set of integers rather than just the integers between 0 and 6, inclusive. (What are the reachable states of Mod7Counter?)

3 Proof of Correctness

It should be pretty obvious that Mod7Counter machine satisfies the Day machine. But, let's go through the steps.

1. Step 1: Define an abstraction function and representation invariant.

   - First, I define an abstraction function.

     $$
     \begin{array}{l}
     AF : int \nrightarrow \{sun, mon, tues, wed, thurs, fri, sat\} \\
     AF(0) = sun \\
     AF(1) = mon \\
     \ldots \\
     AF(6) = sat
     \end{array}
     $$

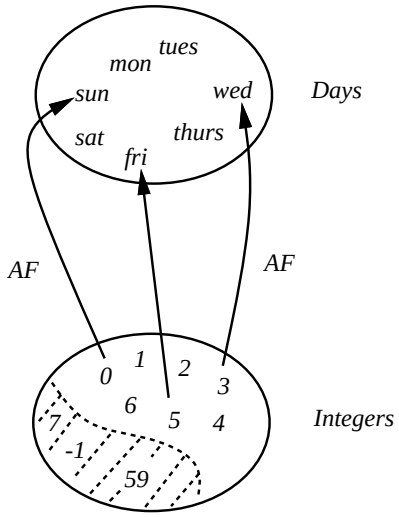
     Notice that, as promised, it's one-to-one.

   - Next, I define the representation invariant. I note that I don't need all integers to represent the days of the week. I only need seven. Moreover, the transition function, δ_C is defined for only those seven.

     $$
     \begin{array}{l}
     RI : int \rightarrow bool \\
     RI(i) = 0 \leq i \leq 6
     \end{array}
     $$

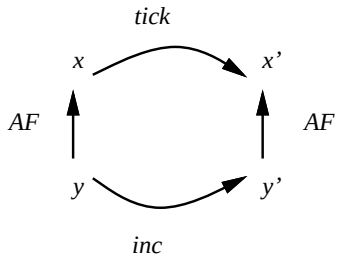
The representation invariant just says that the only integer values I have to worry about are between 0 and 6 inclusive. It characterizes the domain of AF.

The picture relating concrete to abstract states looks like:



2. Step 2: Initial conditions and commuting diagram for each action.

   - Initial conditions: I need to show that each initial state of Mod7Counter is an initial state of Day (under the abstraction function).
     - 0 is the initial state for the Mod7Counter. Thus, $AF(0)$ had better be some initial state of Day. Indeed, $AF(0) = sun$, the initial state of Day.
   - Commuting diagram: I need to show it holds for each action of Mod7Counter.



$$\delta_A(AF(y), tick) = AF(\delta_C(y, inc))$$

In other words, I need to show[1] that $\delta_A(AF(y), tick) = AF(\delta_C(y, inc))$ for all y that satisfy the representation invariant, i.e., for all $y \in \{i : int \mid 0 \leq i \leq 6\}$. I need to show the commuting diagram for only those states that satisfy RI. The simplest proof is to do an exhaustive case analysis. y can take on only seven values so there are seven cases. I'll do the most "interesting" case ($y = 6$).

- Case: $y = 6$

  $\delta_A(AF(6), tick)$
  $= \delta_A(sat, tick)$ def'n of AF

[1] Note that I use functional notation for the state transition relations δ_A and δ_C; more importantly, because they are both functions, it's sound to use equational reasoning in the proof.

$= sun$ def'n of δ_A
$= AF(0)$ def'n of AF
$= AF(\delta_C(6, inc))$ def'n of δ_C

When I did the last part of the proof above, I didn't really do it as shown, but rather I reduced both sides of the equation at the same time yielding $sun = sun$. I can do that because equality is bi-directional:

$\delta_A(AF(6), tick) = AF(\delta_C(6, inc))$
$\delta_A(sat, tick) = AF(0)$ def'ns of AF and δ_C
$sun = sun$ def'ns of δ_A and AF

I think this proof is more readable and it's perfectly acceptable. Just remember to give your justification next to each proof step if it's not obvious or clear from context.