15-750 Graduate Algorithms

Homework # 6 **due: April 19, 2001**

Groundrules:

- Homeworks will generally consist of easier *exercises* and harder *problems*. You should do the exercises by yourself, but you may work with a friend or two on the problems if you want — just write down who you worked with. One exception: no fair working with someone who has already figured out (or already knows) the answer. Of course, your writeup should be in your own words.
- If you've seen a problem before (sometimes we will give problems that are "famous"), then say that in your solution (it won't affect your score, we just want to know). Also, if you use any sources other than the book or those on the web page, write that down too. It's fine to look up a complicated sum or inequality, but don't look up an entire solution.

Exercises:

1. Prove that any exactly-k-CNF formula that has less than 2^k clauses must be satisfiable. (We mentioned this in class, but we didn't give a formal proof.)

2. The *Lovasz Local Lemma* considers the following situation. Suppose we have a collection of "bad" events $B_1, B_2, \ldots, B_n$ that might happen. Let G be the event that none of the bad events occur. Our goal is to say that $\mathbf{Pr}(G) > 0$. If the bad events were all independent, then $\mathbf{Pr}(G) = \prod_i (1 - \mathbf{Pr}(B_i))$, which is greater than 0 so long as each B_i has probability less than 1. But what if the events are dependent? In the worst-case (the B_i are actually *disjoint*), it is possible that $\mathbf{Pr}(G) = 0$ even though all B_i have probability just $1/n$. Intuitively, however, if each of the B_i depends only on a small number d of the other B_j's, then it should be sufficient for all B_i to have probability less than $1/d$ or so to ensure that $\mathbf{Pr}(G) > 0$.

   The Lovasz Local Lemma makes this precise. Consider a graph (called the *dependency graph*) in which the events B_i are vertices, and we draw edges such that each event B_i is independent of all its non-neighbors. That is, we can condition on anything at all about the non-neighbors of B_i, and this still doesn't change $\mathbf{Pr}(B_i)$. The Lovasz Local Lemma (simplified form) says that if this graph has maximum degree d, and if all B_i satisfy $\mathbf{Pr}(B_i) < 1/(e(d+1))$, then $\mathbf{Pr}(G) > 0$. (Here e is 2.718...)

   Use the Lovasz Local Lemma to prove the following: for $k \geq 10$, any exactly-k-CNF formula, no matter how many clauses it has, is satisfiable so long as each variable appears in at most $2^{k/2}$ clauses. Be clear about what the bad events are, etc.

3. Consider a casino with a single fair game: each time you play, with probability $1/2$ you win a dollar, and with probability $1/2$ you lose a dollar. Suppose you enter this casino with exactly one dollar and play until you are broke or you have n dollars, whichever comes first. What is the probability you leave with n dollars? Explain.

Problems:

4. In this problem we will consider the graph consisting of a clique on $n/2$ nodes attatched to a line of $n/2$ nodes (a "ball-and-chain" or "lollipop" graph), and prove that the cover time is $\Omega(n^3)$.

   (a) Let node 0 be the node of the clique that is connected to the line, and node 1 be the neighbor of node 0 on the line. (So, node 1 has degree 2, node 0 has degree $n/2$, and all other nodes of the clique have degree $n/2-1$.) Prove that the hitting time H_{01} is $\Omega(n^2)$.

   (b) Now solve the whole problem, showing the cover time is $\Omega(n^3)$. Hint: think about the exercise with the casino.

5. Consider the following shuffling step for a deck of n cards:

   > Take the top card, and put it randomly into the remaining deck. (There are n ways to do this considering the top and bottom positions.)

   Prove that the rate of convergence of this markov chain to uniformity is:

   $$\tau(\epsilon) \quad \leq \quad (n \ln n)^2 (\log 1/\epsilon)$$

   (Rate of convergence is defined on page 3 of the Jerrum paper.)