15-750 Graduate Algorithms

Homework # 7 **due: May 1, 2001**

Groundrules: (as usual)

Exercises:

1. Consider a random walk on the d-dimensional cube $\{1, \ldots, n\}^d$. Let's say we start at the point $(1, 1, \ldots, 1)$. At each time step we pick one of the d coordinate directions at random, and then flip a coin to decide whether to move up or down in that direction. As in the line example from class, if our coin says to move down from 1 or move up from n, then we just stay put. Notice that this space has n^d points. Your job will be to show this walk mixes in time $O(n^2 d \log(d/\epsilon))$. So, when d is large (e.g., $d \geq n$) the mixing time is polynomial in the log of the number points — we get "rapid mixing". Notice also that the mixing rate is much smaller than the cover time.

   (a) Prove that the uniform distribution (equal probability on all points) is the stationary distribution.

   (b) Give a coupling argument to prove that this walk achieves variation distance ϵ in time $O(n^2 d \log(d/\epsilon))$.

   Hints: You will want to think about what is going on one dimension at a time. Can you describe the walk in any given dimension as a walk on a simple graph with self-loops?

2. A common approach for attacking discrete optimization problems is randomized hill-climbing. Specifically, think of a state space with $N = 2^n$ states (e.g., all possible assignments for n variables), where each state has a value (e.g., the number of clauses satisfied in a given CNF formula). What we want to do is find the state of highest value. The way that hill-climbing works is we define some natural neighborhood structure so that each state has a small (polynomial in n) number of neighbors (e.g., we might allow flipping a single bit in any given assignment). Then we start at some random point and walk according to the following rules:

   - If any neighbor has a higher value than our current position, go to the one of highest value.
   - Otherwise, if there are neighbors with the same value as our current position, go to one of them at random.
   - Otherwise stop.

   Let's assume that all states have the same number of neighbors, and that values are integers in a small (polynomial in n) range (this is true for MAX-SAT, for instance).

   Argue that this procedure will find the optimal solution in polynomial expected time if the following conditions are true:

   (1) For any value v, the number of nodes with value larger than v (if there are any at all) is at least $1/poly(n)$ times the number of nodes of value equal to v.

   (2) For any v, the graph restricted to nodes of value at least v has conductance at least $1/poly(n)$.

Problems:

3. In the coupon collector's problem, there are n different coupons. At each step, the coupon collector gets another coupon chosen at random (with equal probability) from all n possible coupons. As you know, the expected time for the collector to collect all n coupons is nH_n. Let $f(t)$ be the probability that the coupon collector fails to collect all n coupons in t steps.

   (a) Consider the reverse of the shuffling algorithm on the previous assignment. That is, in each step pick a card at random from the deck, and move it to the top of the deck. Use a coupling argument to show that the variation distance of this Markov chain after t shuffling steps is at most $f(t)$.

   (b) Here's a nice way to compute an upper bound on $f(t)$. Let $I(t)$ be a random variable which is the set of coupons that the collector has collected after t steps. Let $\overline{I}(t)$ be the ones that she has not collected. Using this notation, $f(t) = \Pr[|\overline{I}(t)| \neq 0]$. Now, because $|\overline{I}(t)|$ is a non-negative integer random variable we know that:

   $$\Pr[|\overline{I}(t)| \neq 0] \leq \mathrm{E}[|\overline{I}(t)|]$$

   Prove that this expectation is at most $ne^{-t/n}$.

4. In an n-node binary tree, there are $n-1$ different possible rotations. (To see this, note that you can pick any node that is not the root, and rotate it up above its parent. Any possible rotation can be specified in this fashion.)

   Consider the markov chain that consists of starting from any n-node tree (with $n \geq 3$) and doing random rotations.

   (a) Prove that this markov chain is irreducible. (Hint: define some "canonical" tree and show you can get from any tree to that one).

   (b) Prove that this markov chain is aperiodic.

   (c) Prove that stationary distribution of this markov chain is uniform.