15-750 Graduate algorithms
Additional notes for 03/20/01

Randomized routing

Given an undirected graph and a set of pairs $\{(s_i, t_i)\}$ we want to route these pairs to minimize the maximum congestion. E.g., think of routing paths in a VLSI chip. This problem is NP-hard. Can we find an approximate solution?

Idea: (Raghavan & Thompson)

1. Solve fractionally. Think of as multi-commodity flow (e.g., allow s_i to route to t_i by sending 1/2 down one path, 1/4 down another path, and 1/4 down another). Can solve with linear programming: for each (directed) edge e, and each commodity i, have variable X_{ei}. Constraints for inflow = outflow. Constraints $\forall e, \sum_i X_{ie} \leq C$, and minimize C.

2. For each pair (s_i, t_i) we have a flow. Now what we do is view these fractional values as probabilities and select a path such that the probability we pick edge e is equal to the flow of this commodity on e. How can we do this algorithmically? (Give proof that greedy approach works.)

Analysis: fix some edge. Let f_i be the flow of commodity i on this edge. This also means that f_i is the probability that we picked this edge for routing (s_i, t_i). So, for a given edge, can think of $\{0, 1\}$ random variables X_i corresponding to event that we picked this edge for commodity i, where $\mathbf{Pr}[X_i = 1] = f_i$. For a given edge, these X_i are all INDEPENDENT. (Not independent for the same i across different edges, but that's OK). Expected value of sum is at most C. Now apply Chernoff bounds (see below).

$$\mathbf{Pr}[total > (1 + \epsilon)C] < e^{-\epsilon^2 C/3}$$

The point now is if this is small enough (e.g., $o(1/n^2)$) then the probability that *there exists* an edge whose congestion exceeds this bound is also small ($o(1)$).
So, if $C \gg \log(n)$, then w.h.p., maximum is only $1 + \epsilon$ times larger than the expectation, and we have a $(1 + \epsilon)$ approximation. For small C we can apply the bound:

$$Pr[total > kC] < (e^{k-1}/k^k)^C$$

This is small enough for $k = O(\log(n)/\log\log(n))$. Open problem: can you get a constant-factor approximation?

Chernoff bounds

Let $X_1, \ldots, X_n$ be a sequence of n independent $\{0,1\}$ random variables with $\mathbf{Pr}[X_i = 1] = p_i$ not necessarily the same. Let S be the sum of the r.v., and $\mu = \mathbf{E}[S]$. Then, for $0 \leq \delta \leq 1$, the following inequalities hold:

- $\mathbf{Pr}[S > (1+\delta)\mu] \leq e^{-\delta^2 \mu/3}$,
- $\mathbf{Pr}[S < (1-\delta)\mu] \leq e^{-\delta^2 \mu/2}$.

Additive bounds:

- $\mathbf{Pr}[S - \mu > \delta n] \leq e^{-2n\delta^2}$.
- $\mathbf{Pr}[S - \mu < -\delta n] \leq e^{-2n\delta^2}$.

Also, for any $k > 1$, we get:

- $\mathbf{Pr}[S > k\mu] < \left(\frac{e^{k-1}}{k^k}\right)^{\mu}$.