Continuous Functionals of Dependent Types and Equilogical Spaces

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Abstract. We show that dependent sums and dependent products of continuous parametrizations on domains with dense, codense, and natural totalities agree with dependent sums and dependent products in equilogical spaces, and thus also in the realizability topos $\mathsf{RT}(\mathcal{P}\omega)$.

Keywords: continuous functionals, dependent type theory, domain theory, equilogical spaces.

1 Introduction

Recently there has been a lot of interest in understanding notions of totality for domains [3, 23, 4, 18, 21]. There are several reasons for this. Totality is the semantic analogue of termination, and one is naturally interested in understanding not only termination properties of programs but also how notions of program equivalence depend on assumptions regarding termination [21]. Another reason for studying totality on domains is to obtain generalizations of the finite-type hierarchy of total continuous functionals by Kleene and Kreisel [11], see [8] and [19] for good accounts of this subject. Ershov [7] showed how the Kleene-Kreisel functionals arise in a domain-theoretic setting as the *total* elements of domains of *partial* continuous functionals. This work has been pursued further by Normann, Berger and others, who have studied both inductive types and dependent types with universe operators [3, 23, 4, 18, 12, 26]. The aims of their work include both finding models of Martin-Löf type theory [16, 26] and also extending the *density theorems* to transfinite hierarchies. The density theorems are used in the study of higher-type recursion theory and in order-theoretic characterizations of extensionality for total objects [4, 17]

It is important to understand how different models of computation relate. Indeed, a number of results demonstrate that the Kleene-Kreisel functionals arise in various computational models [7, 10, 15, 3, 13], which is good evidence that this class of functionals is an important and robust model of higher-type computation. We proved one such result in [2], where we related domains with totality to *equilogical spaces*, introduced by Dana Scott [2]: the so-called dense and codense totalities on domains [3] embed fully and faithfully into the category of equilogical spaces and the embedding preserves the cartesian-closed structure implicit in the totalities for products and function spaces. From this it follows easily that the Kleene-Kreisel functionals of finite type can be constructed in the category Equ of equilogical spaces by repeated exponentiation, starting from the natural numbers object. In this paper we extend these results to dependent types.

We build on Berger's Habilitationsschrift [4], in which Berger generalized density and codensity on domains from simple types to dependent types with universe operators and proved the corresponding Density Theorems. We show that, in a precise sense, the dependent types of dense, codense, and natural totalities on consistent parametrizations coincide with the dependent types of equilogical spaces. It follows that the dependent type hierarchies over the natural numbers and the booleans coincide in four settings: equilogical spaces, domains with totality, limit spaces [20], and filter spaces [22, 9]. We also note recent work by Menni and Simpson [14], which relate locally cartesian closed subcategories of equilogical spaces, sequential spaces, and limit spaces. All these results taken together provide a satisfactory "goodness of fit" picture, at the level of *dependent* type structures.

More precisely, domains here are algebraic, countably based, consistently-complete dcpos. Since the domains are countably based, we only need to consider countably based equilogical spaces, which form a full locally cartesian closed subcategory of the category of all equilogical spaces. The category of countably based equilogical spaces is equivalent to the category of modest sets $\mathsf{Mod}(\mathcal{P}\omega)$ over the graph model $\mathcal{P}\omega$ of the untyped λ-calculus, and since the modest sets form a full locally cartesian closed subcategory of the realizability topos $\mathsf{RT}(\mathcal{P}\omega)$ over the graph model, it follows that the domain-theoretic total continuous functionals of dependent types are the same as the ones in the realizability topos $\mathsf{RT}(\mathcal{P}\omega)$.

The plan of the paper is as follows. In the following section we present an overview of the technical work, and explain the main idea of the proof of our main theorem, Theorem 1. In Sect. 3 we recall the definition of the category of equilogical spaces and the construction of dependent sums and products of equilogical spaces. In Sect. 4 we briefly review domains with totality, and refer you to [4] for more details. Sect. 5 contains the Main Theorem and its proof, which relates dependent types in Equ to dependent types in domains with totality. As an example of how the Main Theorem can be used, we translate Berger's Continuous Choice Principle for dependent totalities [4] into a choice principle expressed in the internal logic of Equ. Finally, Sect. 7 contains some concluding remarks and suggestions for future work.

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2 Overview of Technical Work

In this section we give a brief overview of the rather technical theorems and proofs from Sect. 5. We do not provide any proofs or references for the claims made in this overview, because they are repeated in more detail in the rest of the paper. Please consult Sects. 3 and 4 for basic definitions and explanation of the notation. Berger [4, 5] contains material on totalities for parametrizations on domains, and [2] can serve as a reference on equilogical spaces.

The category of countably based equilogical spaces, as defined originally by Dana Scott, is equivalent to $\mathsf{PER}(\omega\mathsf{ALat})$, the category of partial equivalence relations on countably based algebraic lattices. We work exclusively with $\mathsf{PER}(\omega\mathsf{ALat})$ and so we set $\mathsf{Equ} = \mathsf{PER}(\omega\mathsf{ALat})$.

If $M \subseteq D$ is a codense subset of a domain D, then the consistency relation $\uparrow$ (which relates two elements when they are bounded) restricted to M is a partial equivalence relation on D. Thus, a codense subset of a domain D can be viewed as a partial equivalence relation, induced by the consistency relation on M, on the algebraic lattice $D^\top$, the domain D with a compact top element $\top$ added to it.

Let $F = (|F|, ||F||)$ be a dense, codense and consistent totality on $D = (|D|, ||D||)$, i.e., $(|F|, |D|)$ is a consistent parametrization on the domain $|D|$, $||D|| \subseteq |D|$ is a dense and codense totality on $|D|$, and $(||D||, ||F||)$ is a dense and codense dependent totality for $|F|$. We can explain the main point of the proof that the dependent types in domains with totality agree with dependent types in equilogical spaces by looking at how the dependent products are constructed in both setting. In the domain-theoretic setting a total element of the dependent product $P = \Pi(D, F)$ is a continuous map $f = \langle f_1, f_2 \rangle \colon |D| \to |\Sigma(D, F)|$ that maps total elements to total elements and satisfies, for all $x \in ||D||$, $f_1 x = x$. In $\mathsf{PER}(\omega\mathsf{ALat})$ a total element of the dependent product $Q = \prod_D F$ is a continuous map $g = \langle g_1, g_2 \rangle \colon |D|^\top \to |\Sigma(D, F)|^\top$ that preserves the partial equivalence relations and satisfies, for all $x \in ||D||$, $g_1 x \uparrow_D x$. Here $\uparrow_D$ is the consistency relation on domain $|D|$, restricted to the totality $||D||$. In order to prove that P and Q are isomorphic we need to be able to translate an element $f \in ||P||$ to one in $||Q||$, and vice versa. It is easy enough to translate $f \in ||P||$ since we can just use f itself again. This is so because $f_1 x = x$ implies $f_1 x \uparrow_D x$. However, given a $g \in ||Q||$, it is not obvious how to get a corresponding function in $||P||$. We need a way of continuously *transporting* 'level' $||F(g_1 x)||$ to 'level' $||Fx||$. In other words, we need a continuous map t such that whenever $x, y \in ||D||$, $x \uparrow y$, and $u \in ||Fy||$ then $t(y, x)u \in ||Fx||$ and $\langle x, t(y, x)u \rangle \uparrow \langle y, u \rangle$ in $|\Sigma(D, F)|$. Given such a map t, the element of $||P||$ corresponding to $g \in ||Q||$ is the map

$$x \mapsto \langle x, t(g_1 x, x)(g_2 x) \rangle .$$

The theory of totality for parametrizations on domains provides exactly what we need. Every consistent parametrization F has a *transporter* t, which has the desired properties. In addition, we must also require that the parametrization F be *natural*, which guarantees that $t(y, x)$ maps $||Fy||$ to $||Fx||$ whenever x and y

are total and consistent. Berger [4] used the naturality conditions for dependent totalities to show that the consistency relation coincides with extensional equality. As equality of functions in equilogical spaces is defined extensionally, it is not surprising that naturality is needed in order to show the correspondence between the equilogical and domain-theoretic settings.

Finally, let us comment on the significance of the density and codensity theorems [4] for the results presented in this paper. We define a translation from dependent totalities to equilogical spaces, and show that it preserves dependent sums and products. The density theorems for dependent totalities ensure that the translation is well defined in the first place. Thus, density plays a fundamental role, which is further supported by the observation that the category of equilogical spaces is equivalent to the category of *dense* partial equivalence relations on Scott domains, see [2]. The effect of codensity is that the translation of domain-theoretic totalities into equilogical spaces gives a rather special kind of *totally disconnected equilogical spaces*, which we comment on further in Sect. 7.

3 Equilogical Spaces

In this paper, we take an *equilogical space* $A = (|A|, \approx_A)$ to be a partial equivalence relation $\approx_A$ on an algebraic lattice $|A|$. The category $\mathsf{PER}(\omega\mathsf{ALat})$ of such objects and equivalence classes of equivalence preserving continuous maps between them is equivalent to the original definition of equilogical spaces [2].

The *support* of an equilogical space A is the set

$$\|A\| = \left\{x \in |A| \;\middle|\; x \approx_A x\right\} .$$

We explicitly describe the locally cartesian closed structure of $\mathsf{PER}(\omega\mathsf{ALat})$.

Let $r \colon J \to I$ be a morphism in $\mathsf{PER}(\omega\mathsf{ALat})$. The *pullback along* r^* is the functor

$$r^* \colon \mathsf{PER}(\omega\mathsf{ALat})/I \to \mathsf{PER}(\omega\mathsf{ALat})/J$$

that maps an object $a \colon A \to I$ over I to an object $r^*a \colon r^*A \to J$ over J, as in the pullback diagram

$$\begin{tikzcd} r^*A \arrow[r] \arrow[d, "r^*a"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "a"] \\ J \arrow[r, "r"'] & I \end{tikzcd}$$

The pullback functor r^* has left and right adjoints. The left adjoint is the *dependent sum along* r

$$\textstyle\sum_r \colon \mathsf{PER}(\omega\mathsf{ALat})/J \to \mathsf{PER}(\omega\mathsf{ALat})/I$$

that maps an object $b\colon B \to J$ over J to the the object $\sum_r b = r \circ b\colon B \to I$ over I. The right adjoint to the pullback functor r^* is the *dependent product along r*

$$\prod_r : \mathsf{PER}(\omega\mathsf{ALat})/J \to \mathsf{PER}(\omega\mathsf{ALat})/I,$$

defined as follows. Let $b\colon B \to J$ be an object in the slice over J. Let $\sim$ be a partial equivalence relation on the algebraic lattice $|I| \times (|J| \to |B|)$ defined by

$$\langle i, f\rangle \sim \langle i', f'\rangle$$
$$\text{if and only if}$$
$$i \approx_I i' \wedge \forall\, j, j' \in |J| \,.\, (j \approx_J j' \wedge r(j) \approx_I i \Longrightarrow f(j) \approx_B f'(j') \wedge b(f(j)) \approx_J j)$$

The dependent product $\prod_r b$ is the object $(|\prod_r b|, \sim)$, where

$$|\textstyle\prod_r b| = |I| \times (|J| \to |B|) \quad . \tag{1}$$

The map $\prod_r b\colon \prod_r b \to I$ is the obvious projection $\langle i, f\rangle \mapsto i$. See [2] for more details about the locally cartesian closed structure of $\mathsf{PER}(\omega\mathsf{ALat})$.

For background material on domain theory we suggest [24] or [1]. A *Scott domain* is a countably based, algebraic, consistently-complete dcpo. Let $\omega\mathsf{Dom}$ be the category of Scott domains and continuous maps between them. This category is cartesian closed and contains the category $\omega\mathsf{ALat}$ as a full cartesian closed subcategory. We define the 'top' functor $\Box^\top : \omega\mathsf{Dom} \to \omega\mathsf{ALat}$ by setting $D^\top$ to be the domain D with a new compact top element added to it. Given a map $f\colon D \to E$, let $f^\top : D^\top \to E^\top$ be defined by

$$f^\top x = \begin{cases} fx & \text{if } x \neq \top_D \\ \top_E & \text{if } x = \top_D \end{cases} .$$

It is is easily checked that $f^\top$ is a continuous map. We are going to use the following two lemmas and corollary later on. The easy proofs are omitted.

Lemma 1. *Let C, D, and E be Scott domains and $f : C \to (D \to E^\top)$ a continuous map. Then the map $f' : C \to (D^\top \to E)$ defined by*

$$f'\,xy = \begin{cases} fxy & \textit{if } y \neq \top_D \\ \top_E & \textit{if } y = \top_D \end{cases}$$

is also continuous.

Corollary 1. *Let D, and E be Scott domains and $f\colon D \to E^\top$ a continuous map. Then the map $f'\colon D^\top \to E$ defined by*

$$f'\,y = \begin{cases} fy & \textit{if } y \neq \top_D \\ \top_E & \textit{if } y = \top_D \end{cases}$$

is also continuous.

Lemma 2. *Suppose D and E are Scott domains, $S \subseteq D$ is an open subset, and $f\colon D \setminus S \to E^{\top}$ is a continuous map from the Scott domain $D \setminus S$ to the algebraic lattice $E^{\top}$. Then the map $f' : D \to E^{\top}$ defined by*

$$f'x = \begin{cases} fx & \text{if } x \notin S \\ \top_E & \text{if } x \in S \end{cases}$$

is also continuous.

4 Domains and Totality

We review some basic definitions about domains with totality from Berger [3, 4]. Let $\mathbb{B}_\perp$ be the flat domain on the Booleans $\mathbb{B} = \{\mathsf{false}, \mathsf{true}\}$. Given a domain D and a subset $M \subseteq D$, let $\mathcal{E}_D(M)$ be the family

$$\mathcal{E}_D(M) = \left\{p\colon D \to \mathbb{B}_\perp \;\middle|\; \forall\, x \in M \,.\, px \neq \perp\right\} .$$

In words, $\mathcal{E}_D(M)$ is the set of those continuous predicates on D which only take on values true and false on elements of M. The family $\mathcal{E}_D(M)$ is *separating* when for every unbounded finite set $\{x_0, \dots, x_n\} \subseteq D$, there exist $p_0, \dots, p_n \in \mathcal{E}_D(M)$ such that $p_i x_i = \mathsf{true}$ for $i = 0, \dots, n$ and $p_0^*(\mathsf{true}) \cap \dots \cap p_n^*(\mathsf{true}) = \emptyset$.

A *totality* on a domain is a pair $D = (|D|, \|D\|)$ where $|D|$ is a domain and $\|D\|$ is a subset of $|D|$. Often the set $\|D\|$ itself is called a totality as well. A totality is *dense* when $\|D\|$ is a topologically dense subset of $|D|$. A totality is *codense* when the family $\mathcal{E}_{|D|}(\|D\|)$ is separating. The consistency relation $\uparrow$ restricted to a codense totality $\|D\|$ is symmetric and transitive.

To each dense and codense totality D we assign an equilogical space

$$\mathrm{Q}D = (|D|^{\top}, \uparrow_D) \tag{2}$$

where $\uparrow_D$ is the consistency relation restricted to the totality $\|D\|$, i.e., $x \uparrow_D y$ if, and only if, $x, y \in \|D\| \wedge x \uparrow y$. We consider only dense and codense totalities from now on.

A *parametrization* on a domain $|D|$ is a co-continuous functor $F\colon |D| \to \omega\mathsf{Dom}^{\mathrm{ep}}$ from $|D|$, viewed as a category, to the category $\omega\mathsf{Dom}^{\mathrm{ep}}$ of Scott domains and *good* embeddings. Recall from [4] that an embedding-projection pair is good when the projection preserves arbitrary suprema. Whenever $x, y \in |D|$, $x \leq y$, there is an embedding $F(x \leq y)^+ : Fx \to Fy$ and a projection $F(x \leq y)^- : Fy \to Fx$. We abbreviate these as follows, for $u \in Fx$ and $v \in Fy$:

$$u^{[y]} = F(x \leq y)^+(u) \;\; ,$$
$$v_{[x]} = F(x \leq y)^-(v) \;\; .$$

A parametrization F on $|D|$ is *consistent* when it has a *transporter*. A transporter is a continuous map t such that for every $x, y \in |D|$, $t(x, y)$ is a map from Fx to Fy, satisfying:

(1) if $x \leq y$ then $F(x \leq y)^+ \leq t(x,y)$ and $F(x \leq y)^- \leq t(y,x)$,
(2) $t(x,y)$ is strict,
(3) $t(y,z) \circ t(x,y) \leq t(x,z)$.

Let D be a totality. A *dependent totality* on D is a pair $F = (|F|, \|F\|)$ where $|F|\colon |D| \to \omega\mathsf{Dom}^{\mathrm{ep}}$ is a parametrization and $(\|D\|, \|F\|)$ is a totality for the parametrization $(|D|, |F|)$. Just like for totalities on domains, there are notions of *dense* and *codense* dependent totalities. See Berger [4] for definitions of these and also for definitions of *dependent sum* $\Sigma(D,F)$ and *dependent product* $\Pi(D,F)$. From now on we only consider dense and codense dependent totalities on consistent parametrizations.

A dependent totality F on D is *natural* if $\|D\|$ is upward closed in $|D|$, $\|Fx\|$ is upward closed in $|Fx|$ for all $x \in \|D\|$, and whenever $x \leq y \in \|D\|$ then

$$\forall\, v \in |Fy|\,.\,\big(v \in \|Fy\| \iff v_{[x]} \in \|Fx\|\big) \ .$$

Note that the above condition implies

$$\forall\, u \in |Fx|\,.\,\Big(u \in \|Fx\| \iff u^{[y]} \in \|Fy\|\Big) \ .$$

Lemma 3. *Let F be a natural dependent totality on D. Since F is consistent, it has a transporter t. Let $x, y \in \|D\|$, $x \uparrow y$, and $u \in \|Fy\|$. Then $t(y,x)u \in \|Fx\|$ and $\langle y, u\rangle \uparrow \langle x, t(y,x)u\rangle$ in $|\Sigma(D,F)|$.*

Proof. By naturality of F we have $(u^{[x\vee y]})_{[x]} \in \|Fx\|$, and since

$$(u^{[x\vee y]})_{[x]} \leq t(x \vee y, x)(t(y, x \vee y)u) \leq t(y,x)u$$

also $t(y,x)u \in \|Fx\|$. Furthermore, $\langle y, u\rangle \uparrow \langle x, t(y,x)u\rangle$ in $|\Sigma(D,F)|$ because $x \uparrow y$ and $u^{[x\vee y]} \uparrow (t(y,x)u)^{[x\vee y]}$, which follows from the common upper bound

$$\begin{aligned} u^{[x\vee y]} &\leq t(y, x\vee y)u, \\ (t(y,x)u)^{[x\vee y]} &\leq (t(x, x\vee y) \circ t(y,x))u \leq t(y, x\vee y)u \ . \end{aligned}$$

This completes the proof.

Let F be a dependent totality on D and let G be a dependent totality on $\Sigma(D,F)$. Define a *parametrized* dependent totality $\widetilde{G}$, i.e., a co-continuous functor from D to the category of parametrizations [4], by

$$\widetilde{G}x = \lambda u \in Fx\,.\,G(x,u) \ .$$

More precisely, for each $x \in D$, $\widetilde{G}x$ is a dependent totality on Fx, defined by the curried form of G as above. In [4], which provides more details, $\widetilde{G}$ is called the *large currying* of G. Given such a $\widetilde{G}$, there are parametrized versions of dependent sum $\Sigma(F,G)$ and dependent product $\Pi(F,G)$, which are dependent totalities on D, defined for $x \in D$ by

$$\begin{aligned} \Pi(F,G)x &= \Pi(Fx, \widetilde{G}x) \ , \\ \Sigma(F,G)x &= \Sigma(Fx, \widetilde{G}x) \ . \end{aligned}$$

To each natural dependent totality F on D we assign an equilogical space

$$\mathrm{q}(D,F)\colon \mathrm{Q}(D,F) \to \mathrm{Q}D$$

in the slice over $\mathrm{Q}D$ by defining

$$\mathrm{Q}(D,F) = \mathrm{Q}(\Sigma(D,F)) \tag{3}$$

$$\mathrm{q}(D,F) = \pi_1^{\top} \ , \tag{4}$$

where π_1 is the first projection $\pi_1\colon |\Sigma(D,F)| \to |D|$, $\pi_1\colon \langle x, u\rangle \mapsto x$.

5 Comparison of Dependent Types

We show that dependent sums and products on totalities coincide with those on equilogical spaces.

Theorem 1 (Main Theorem). *Let F be a dependent totality on D, and let G be a dependent totality on $\Sigma(D,F)$. The construction of dependent sum $\Sigma(F,G)$ and dependent product $\Pi(F,G)$ agrees with the construction of dependent sum and dependent product in* $\mathsf{PER}(\omega\mathsf{ALat})$*, i.e.,*

$$\mathrm{Q}(D,\Sigma(F,G)) \cong \textstyle\sum_{\mathrm{q}(D,F)} \mathrm{q}(\Sigma(D,F),G) \ ,$$
$$\mathrm{Q}(D,\Pi(F,G)) \cong \textstyle\prod_{\mathrm{q}(D,F)} \mathrm{q}(\Sigma(D,F),G)$$

in the slice over $\mathrm{Q}D$*.*

The rest of this section constitutes a proof of the Main Theorem, but before we embark on it, let us explain its significance. We have defined a translation Q from domain-theoretic dependent totalities to equilogical spaces. The Main Theorem says that this translation commutes with the construction of dependent sums and products. Thus, Q preserves the implicit local cartesian closed structure of totalities $\Sigma(F,G)$ and $\Pi(F,G)$. It may seem odd that we did not define a *functor* Q that would embed the dependent totalities into $\mathsf{PER}(\omega\mathsf{ALat})$ and preserve the locally cartesian closed structure. This can be done easily enough, by defining the morphisms $(D,F) \to (E,G)$ to be (equivalence classes of) equivalence-preserving continuous maps $\mathrm{Q}(D,F) \to \mathrm{Q}(E,G)$, i.e., essentially as the morphisms in $\mathsf{PER}(\omega\mathsf{ALat})$. Note that this is different from the definition of morphisms between parametrizations, as defined in Berger [4], where the motivation was to build the hierarchies in the first place, rather than to study an interpretation of dependent type theory. Thus, a notion of morphism suitable for the interpretation of dependent type theory was never explicitly given, although it is fairly obvious what it should be. In this manner we trivially obtain a full and faithful functor Q. The crux of the matter is that with such a choice of morphisms, the domain-theoretic constructions $\Sigma(F,G)$ and $\Pi(F,G)$ indeed yield the category-theoretic dependent sums and products. This is the main purpose of our work—to show that the domain theoretic constructions of dependent functionals, which has at times been judged arcane and ad hoc, is essentially the same as the dependent functionals arising in the realizability topos $\mathsf{RT}(\mathcal{P}\omega)$, which is much smoother and better understood from the category-theoretic point of view. The benefits of this correspondence go both ways. On the one hand, the domain-theoretic construction, which was conceived through a sharp conceptual analysis of the underlying domain-theoretic notions, is more easily understood and accepted by a category theorist. On the other hand, we can transfer the domain-theoretic results about the dependent functionals to Equ and $\mathsf{RT}(\mathcal{P}\omega)$, e.g., the Continuous Choice Principle from Sect 6. It is not clear how to obtain the Continuous Choice Principle directly in the realizability setting.

Lastly, we note that the Main Theorem is formulated for dependent sums and products with *parameters*, i.e., for parametrizations of parametrizations on domains; a parameter-free formulation states only that $Q(\Pi(D,F)) \cong \prod q(D,F)$. We need the theorem with parameters in order to establish the full correspondence between the lccc structures. We now proceed with the proof of the Main Theorem.

Dependent Sums. Dependent sums are easily dealt with because all we have to do is unravel all the definitions. For this purpose, let $X = \mathrm{Q}(D, \Sigma(F,G))$ and $Y = \sum_{\mathrm{q}(D,F)} \mathrm{q}(\Sigma(D,F),G)$. In order to simplify the presentation we assume that ordered pairs and tuples satisfy the identities $\langle x,y,z\rangle = \langle\langle x,y\rangle,z\rangle = \langle x,\langle y,z\rangle\rangle$. This does affect the correctness of the proof, since it just amounts to leaving out the appropriate canonical isomorphisms. In particular, this assumption implies the equality $|\Sigma(\Sigma(D,F),G)| = |\Sigma(D,\Sigma(F,G))|$. From this it follows that the underlying lattices $|X|$ and $|Y|$ agree because

$$|Y| = |\Sigma(\Sigma(D,F),G)|^{\top} = |\Sigma(D,\Sigma(F,G))|^{\top} = |X| \ .$$

It remains to show that the partial equivalence relations on X and Y agree as well. We omit the straightforward verification of this fact.

Dependent Products. Dependent products are more complicated. There seems to be no way around it, since we are dealing with rather heavy domain-theoretic machinery. Let

$$\begin{aligned} U &= \mathrm{Q}(D, \Pi(F,G)) \ , \\ V &= \textstyle\prod_{\mathrm{q}(D,F)} \mathrm{q}(\Sigma(D,F),G) \ . \end{aligned}$$

Let us explicitly describe U and V. The underlying lattice of U is

$$|U| = |\Sigma(D,\Pi(F,G))|^{\top} \ . \tag{5}$$

The partial equivalence relation on U relates $\langle x,f\rangle \in |U|$ and $\langle y,g\rangle \in |U|$ if, and only if,

$$\begin{aligned} & x \uparrow_D y \ \wedge \\ & (\forall\, u \in \|Fx\| \,.\, fu \in \|G(x,u)\|) \wedge (\forall\, v \in \|Fy\| \,.\, gv \in \|G(y,v)\|) \ \wedge \\ & \forall\, w \in |F(x \vee y)| \,.\, \Big((f(w_{[x]}))^{[\langle x\vee y, w\rangle]} \uparrow (g(w_{[y]}))^{[\langle x\vee y, w\rangle]} \Big) \ . \end{aligned}$$

By (1), the underlying lattice of V is

$$|V| = |D|^{\top} \times \left(|\Sigma(D,F)|^{\top} \to |\Sigma(\Sigma(D,F),G)|^{\top}\right) \ . \tag{6}$$

Elements $\langle x, y\rangle \in |V|$ and $\langle y, g\rangle \in |V|$ are related if, and only if, the following holds: $x \uparrow_D y$, and for all $z, z' \in |D|$ such that $z \uparrow_D x$ and $z' \uparrow_D x$, and for all $w \in |Fz|$, $w' \in |Fz'|$ such that $w^{[z\vee z']} \uparrow_{F(z\vee z')} w'^{[z\vee z']}$,

$$\begin{aligned}
&f\langle z, w\rangle \uparrow_{\Sigma(\Sigma(D,F),G)} g(z', w') \ \wedge \\
&\pi_1(f\langle z, w\rangle) \uparrow_{\Sigma(D,F)} \langle z, w\rangle \ \wedge \ \pi_1(g\langle z', w'\rangle) \uparrow_{\Sigma(D,F)} \langle z', w'\rangle \ .
\end{aligned}$$

We define maps $\phi\colon |U| \to |V|$ and $\theta\colon |V| \to |U|$, and verify that they represent isomorphisms between U and V. Let t be a transporter for the parametrization F. Define the map $\phi\colon |U| \to |V|$ by

$$\phi\top = \top \ , \qquad \phi(x, f) = \langle x, \phi_2(x, f)\rangle \ ,$$

where $\phi_2(x, f)\colon |\Sigma(D,F)|^{\top} \to |\Sigma(\Sigma(D,F),G)|^{\top}$ is

$$\phi_2(x, f)\top = \top \ , \qquad \phi_2(x, f)(y, u) = \langle x, t(y, x)u, f(t(y, x)u)\rangle \ .$$

Let s be a transporter for the parametrization G on $\Sigma(D,F)$. Define the map $\theta\colon |V| \to |U|$ by

$$\begin{aligned}
\theta(\top, g) &= \top \\
\theta(x, g) &= \texttt{if}\ \ \exists\, u \in |Fx| \,.\, g(x, u) = \top \\
&\ \ \texttt{then}\ \ \top \\
&\ \ \texttt{else}\ \ \langle x, \lambda u \in |Fx| \,.\, s(g_1(x, u), \langle x, u\rangle)(g_2(x, u))\rangle
\end{aligned}$$

where $g = \langle g_1, g_2\rangle\colon |\Sigma(D,F)| \to |\Sigma(\Sigma(D,F),G)|$.

It is easy and tedious to verify that ϕ and θ have the intended types. Continuity of ϕ follows directly from Corollary 1 and Lemma 1. Continuity of θ follows from Lemmas 1 and 2. We can apply Lemma 2 because the set

$$\left\{\langle x, g\rangle \mid \exists\, u \in |Fx| \,.\, g(x, u) = \top\right\} \subseteq |D| \times \left(|\Sigma(D,F)|^{\top} \to |\Sigma(\Sigma(D,F),G)|^{\top}\right)$$

is open, as it is a projection of the open set

$$\left\{\langle x, u, g\rangle \mid g(x, u) = \top\right\} \subseteq |\Sigma(D,F)| \times \left(|\Sigma(D,F)|^{\top} \to |\Sigma(\Sigma(D,F),G)|^{\top}\right) \ .$$

Next we verify that ϕ and θ represent morphisms and that they are inverses of each other. Since we only work with total elements from now on, we do not have to worry about the cases when $\top$ appears as an argument or a result of an application.

(1) ϕ represents a morphism $U \to V$ in the slice over QD. Let $\langle x, f\rangle, \langle x', f'\rangle \in \|U\|$ and suppose $\langle x, f\rangle \uparrow \langle x', f'\rangle$. This means that $x \uparrow x'$ and $f^{[x\vee x']} \uparrow f'^{[x\vee x']}$, i.e., for every $w \in |F(x \vee x')|$

$$(f(w_{[x]}))^{[\langle x\vee x', w\rangle]} \uparrow (f'(w_{[x']}))^{[\langle x\vee x', w\rangle]} \ .$$

We prove that $\phi(x, f) \approx_V \phi(x', f')$. Clearly, $x \uparrow_D x'$ since $x \uparrow x'$ and $x, x' \in \|D\|$. Let

$$\begin{aligned} g &= \pi_2(\phi(x, f)) = \lambda\langle y, u\rangle \in |\Sigma(D, F)| \,.\, \langle x, t(y, x)u, f(t(y, x)u)\rangle \\ g' &= \pi_2(\phi(x', f')) = \lambda\langle y, u\rangle \in |\Sigma(D, F)| \,.\, \langle x', t(y, x')u, f'(t(y, x')u)\rangle \ . \end{aligned}$$

Let $y, y' \in \|D\|$ such that $y \uparrow y'$ and $y \uparrow x$. Let $u \in \|Fy\|$ and $u' \in \|Fy'\|$ such that $u^{[y\vee y']} \uparrow u'^{[y\vee y']}$. We need to show the following:

(a) $\langle y, u\rangle \uparrow \langle x, t(y, x)u\rangle$
(b) $g(y, u) \in \|\Sigma(\Sigma(D, F), G)\|$
(c) $g'(y', u') \in \|\Sigma(\Sigma(D, F), G)\|$
(d) $(g(y, u))^{[\langle y,u\rangle\vee\langle y',u'\rangle]} \uparrow (g'(y', u'))^{[\langle y,u\rangle\vee\langle y',u'\rangle]}$.

Proof of (a): by assumption $y \uparrow x$, and $u^{[x\vee y]} \uparrow t(y, x)(u)^{[x\vee y]}$ holds because of the common upper bound:

$$\begin{aligned} u^{[x\vee y]} &\leq t(y, x \vee y)u \\ (t(y, x)u)^{[x\vee y]} &\leq (t(x, x \vee y) \circ t(y, x))u \leq t(y, x \vee y)u \ . \end{aligned}$$

Proof of (b): by assumption $x \in \|D\|$, and also $t(y, x)u \in \|Fx\|$ because $x, y \in \|D\|$, $x \uparrow y$ and $u \in \|Fy\|$. Finally, $f(t(y, x)u) \in \|G(x, t(y, x)u)\|$ because $f \in \|\Pi(Fx, \widetilde{G}x)\|$. The proof of (c) is analogous to the proof (b).
Proof of (d): by assumption $x \uparrow x'$, and $(t(y, x)u)^{[x\vee x']} \uparrow (t(y', x')u')^{[x\vee x']}$ holds because

$$\begin{aligned} (t(y, x)u)^{[x\vee x']} &\leq t(y, x \vee x')u \leq t(y \vee y', x \vee x')(u^{[y\vee y']}) \\ (t(y', x')u')^{[x\vee x']} &\leq t(y', x \vee x')u' \leq t(y \vee y', x \vee x')(u'^{[y\vee y']}) \end{aligned}$$

and $u^{[y\vee y']} \uparrow u'^{[y\vee y']}$. Let $z = t(y, x)u$ and $z' = t(y', x')u'$, and let $w = z^{[x\vee x']} \vee z'^{[x\vee x']}$. We claim that

$$(fz)^{[\langle x\vee x', w\rangle]} = (fz)^{[\langle x,z\rangle\vee\langle x',z'\rangle]} \uparrow (f'z')^{[\langle x,z\rangle\vee\langle x',z'\rangle]} = (f'z')^{[\langle x\vee x', w\rangle]} \ .$$

From $z \leq w_{[x]}$ it follows that $fz \leq f(w_{[x]})$, hence

$$(fz)^{[\langle x\vee x', w\rangle]} \leq (f(w_{[x]}))^{[\langle x\vee x', w\rangle]} \ ,$$

and similarly,

$$(f'z')^{[\langle x\vee x', w\rangle]} \leq (f'(w_{[x']}))^{[\langle x\vee x', w\rangle]} \ .$$

The claim holds because $f(w_{[x]})^{[\langle x\vee x', w\rangle]} \uparrow f'(w_{[x']})^{[\langle x\vee x', w\rangle]}$.

(2) θ represents a morphism $V \to U$ in the slice over QD*.* The proof goes along the same lines as the proof of (1) and is omitted.

(3) $\theta \circ \phi \approx_{U \to U} 1_U$. Let $\langle x, f\rangle \in \|U\|$. We need to show that $\theta(\phi(x, f)) \uparrow \langle x, f\rangle$. The first component is obvious since $\pi_1(\theta(\phi(x, f))) = x$. As for the second component, for any $v \in \|Fx\|$,

$$
\begin{aligned}
(\pi_2(\theta(\phi(x, f))))v &= s(\langle x, t(x, x)v\rangle, \langle x, v\rangle)(f(t(x, x)v)) \\
&\geq s(\langle x, v\rangle, \langle x, v\rangle)(fv) \\
&\geq fv \ ,
\end{aligned}
$$

hence $\pi_2(\theta(\phi(x, f))) \uparrow f$.

(4) $\phi \circ \theta \approx_{V \to V} 1_V$. Let $\langle x, g\rangle \in \|V\|$. We need to show that $\phi(\theta(x, g)) \approx_V \langle x, g\rangle$. Again, the first component is obvious since $\pi_1(\phi(\theta(x, g))) = x$. For the second component, given any $\langle y, u\rangle \in \|\Sigma(D, F, \|)$ such that $x \uparrow y$, what has to be shown is

$$\langle x, t(y, x)u, s(g_1(x, t(y, x)u), \langle x, t(y, x)u\rangle)(g_2(x, t(y, x)u))\rangle \uparrow g(y, u) \ .$$

First, we have

$$\langle x, t(y, x)u\rangle \uparrow \langle y, u\rangle \text{ and } \langle y, u\rangle \uparrow g_1(y, u),$$

and since these are elements of a codense totality, we may conclude by transitivity that $\langle x, t(y, x)u\rangle \uparrow g_1(y, u)$. Let $z = g_1(y, u)$ and $w = \langle x, t(y, x)u\rangle$. The relation

$$(g_2(y, u))^{[z \vee w]} \uparrow (s(g_1 w, w)(g_2 w))^{[z \vee w]}$$

holds because

$$
\begin{aligned}
(g_2(y, u))^{[z \vee w]} &\leq s(z, z \vee w)(g_2(y, u)) \\
s(g_1 w, w)(g_2 w)^{[z \vee w]} &\leq s(g_1 w, z \vee w)(g_2 w) \ ,
\end{aligned}
$$

and $(y, u) \uparrow w$ together with monotonicity of the function $s(g_1\square, z \vee w)(g_2\square)$ imply that

$$s(z, z \vee w)(g_2(y, u)) \uparrow s(g_1 w, z \vee w)(g_2 w) \ .$$

This concludes the proof of the Main Theorem.

Let $\mathcal{B}$ be the full subcategory of Equ on objects QD where D is a natural totality, i.e., $\|D\|$ is a dense, codense, and upward closed subset of $|D|$. It is the case that $\mathcal{B}$ is a cartesian closed subcategory of Equ, see [2]. However, note that the Main Theorem does not imply that $\mathcal{B}$ is a locally cartesian closed subcategory of Equ. We only showed that $\mathcal{B}$ is closed under those dependent sums and products that correspond to parametrizations on domains. In order to resolve the question whether $\mathcal{B}$ is locally cartesian closed it would be useful to have a good characterization of $\mathcal{B}$ in terms of the categorical structure of Equ.

6 Continuous Choice Principle

As an application of the Main Theorem, we translate Berger's Continuous Choice Principle for dependent totalities [4] into a Choice Principle expressed in the *internal* logic of Equ. The internal logic of Equ is a predicative version of intuitionistic first-order logic with dependent types, subset types, and regular quotient types. It is the logic that Equ inherits as a subcategory of the realizability topos $\mathsf{RT}(\mathcal{P}\omega)$, see [6] for details. In this section we use obvious and customary notational simplifications for dependent products and sums.

Let (D, F) be a dependent totality. By [4, Proposition 3.5.2] there is a continuous functional

$$\mathsf{choose} \in |\Pi(x : D, (Fx \to \mathbb{B}_\bot) \to Fx)|$$

such that for all $x \in \|D\|$ and $p \in \|Fx \to \mathbb{B}\|$, if $p^*(\mathsf{true}) \neq \emptyset$, then $(\mathsf{choose}\, x)p \in p^*(\mathsf{true}) \cap \|Fx\|$. Let $X = \mathrm{Q}D$, $Y = \mathrm{Q}(D, F)$ and $2 = \mathrm{Q}(\mathbb{B}_\bot)$. By looking at the proof of [4, Proposition 3.5.2], we see that choose is *not* a *total* functional of type $\|\Pi(x : D, (Fx \to \mathbb{B}_\bot) \to Fx)\|$ because choose applied to the constant function $\lambda x.\,\mathsf{false}$ yields $\bot$, which is not total. This means that choose does not represent a morphism in Equ. Nevertheless we can use it to construct a realizer for the following Choice Principle, stated in the internal logic of Equ:

$$\begin{aligned} \forall\, p \in (\textstyle\sum_{x\,:\,X} Yx) \to 2\,.\, \Big(&(\forall\, x \in X\,.\, \neg\neg\exists\, y \in Yx\,.\,(p(x, y) = \mathsf{true})\,) \implies \\ &(\exists\, h \in \textstyle\prod_{x\,:\,X} Yx\,.\,\forall\, x \in X\,.\, p(x, hx) = \mathsf{true})\Big) \end{aligned} \tag{7}$$

We omit the proof. Suffice it to say that (7) is realized using choose in much the same way as in the proof of [4, Corollary 3.5.3].

If we specialize (7) by setting $X = 1$ and $Y = \mathbb{N}$, we obtain

$$\forall\, p \in \mathbb{N} \to 2\,.\,\big((\neg\neg\exists\, y \in \mathbb{N}\,.\, py = \mathsf{true}) \implies \exists\, z \in \mathbb{N}\,.\, pz = \mathsf{true}\big)$$

This is a form of *Markov's Principle*, see for example [25, Vol. 1, Chap. 4, Sect. 5]. Thus, (7) is a generalization of Markov's Principle. This view is in accordance with the construction of the choose functional in [4], which works by *searching* for a witness.

7 Concluding Remarks

We have shown that dependent sums and dependent products of continuous parametrizations on domains with dense, codense, and natural totalities agree with dependent sums and dependent products in Equ. This subsumes our result from [2] and gives further support to Dana Scott's remark that Equ is a theory of *total* functions. Our result can be combined with the result by Normann and Waggbø, who related dependent types in domains with totality and dependent types in limit spaces [20], and with the results by Rosolini, who related dependent types in Equ to dependent types in various categories of filter spaces [22]. The conclusion is that the dependent-type hierarchies over the natural numbers agree in four settings: domains with totality, equilogical spaces, and thus also in the realizability topos $\mathsf{RT}(\mathcal{P}\omega)$, limit spaces, and filter spaces.

Once the Main Theorem was established, we could use the Continuous Choice Principle of Berger from the setting of domains with totality to show the validity of a Choice Principle in Equ. The Choice Principle in Equ is most concisely stated in the internal logic of Equ, and it would be interesting to prove it directly in Equ. It is likely that such a proof requires better understanding of what codensity corresponds to in Equ. It is not clear how to express codensity in terms of the categorical or the internal logical structure of Equ. We remark that every dense and codense totality D translates into a totally disconnected equilogical space $\mathrm{Q}D$. An equilogical space X is totally disconnected when the curried form of the evaluation map $X \to 2^{2^X}$ is monic, or equivalently, when the topological quotient $\|X\|/{\approx_X}$ is a totally disconnected space. There are totally disconnected equilogical spaces that do not arise as dense and codense totalities. The subcategory of totally disconnected equilogical spaces is a locally cartesian closed subcategory of Equ. Perhaps the notion of total disconnectedness, or some refinement of it, can be useful for this purpose.

The Main Theorem can be used to infer another consequence about equilogical spaces. Berger [4, 5] showed that extensional equality on the dependent-type hierarchy over the natural numbers coincides with the partial equivalence relation induced by the consistency relation on the underlying domains. This is important because the logical complexity of extensional equality is as complicated as the type at which it is defined, whereas consistency can be expressed as a Π^0_1 statement and has bounded logical complexity. The Main Theorem implies an analogous result for equality in Equ.

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