A Relationship between Equilogical Spaces and Type Two Effectivity

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Abstract

In this paper I compare two well studied approaches to topological semantics—the domain-theoretic approach, exemplified by the category of countably based equilogical spaces, Equ, and Type Two Effectivity, exemplified by the category of Baire space representations, $\mathsf{Rep}(\mathbb{B})$. These two categories are both locally cartesian closed extensions of countably based T_0-spaces. A natural question to ask is how they are related.

First, we show that $\mathsf{Rep}(\mathbb{B})$ is equivalent to a full coreflective subcategory of Equ, consisting of the so-called 0-equilogical spaces. This establishes a pair of adjoint functors between $\mathsf{Rep}(\mathbb{B})$ and Equ. The inclusion $\mathsf{Rep}(\mathbb{B}) \to \mathsf{Equ}$ and its coreflection have many desirable properties, but they do not preserve exponentials in general. This means that the cartesian closed structures of $\mathsf{Rep}(\mathbb{B})$ and Equ are essentially different. However, in a second comparison we show that $\mathsf{Rep}(\mathbb{B})$ and Equ do share a common cartesian closed subcategory that contains all countably based T_0-spaces. Therefore, the domain-theoretic approach and TTE yield equivalent topological semantics of computation for all higher-order types over countably based T_0-spaces. We consider several examples involving the natural numbers and the real numbers to demonstrate how these comparisons make it possible to transfer results from one setting to another.

1 Introduction

In this paper I compare two approaches to topological semantics—the domain-theoretic approach, exemplified by the category of countably based *equilogical spaces* [6,23], Equ, and *Type Two Effectivity* (TTE) [27,26,25,14], exemplified by the category of *Baire space representations*, $\mathsf{Rep}(\mathbb{B})$. These frameworks have been extensively studied, albeit by two somewhat separate research communities. The present paper relates the two approaches and helps transfer results between them.

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Domain-theoretic models of computation arise from the idea that the result of a (possibly infinite) computation is *approximated* by the *finite* stages of the computation. As the computation progresses, the finite stages approximate the final result ever so better. This leads to a formulation of partially ordered spaces, called *domains*, in which every element is the supremum of the distinguished "finite" elements that are below it. We recommend [1] and [24] for an introduction to domain theory.

The TTE framework arises from the study of (possibly infinite) computations performed by Turing machines that read infinite input tapes and write results on infinite output tapes. If we view input and output tapes as a sequences of natural numbers, then Turing machines correspond to computable partial operators on the Baire space $\mathbb{B} = \mathbb{N}^{\mathbb{N}}$. We obtain a purely topological model of computation by considering all *continuous* partial operators on $\mathbb{B}$, not just the computable ones. We recommend [27] for an introduction to TTE.

The use of equilogical spaces as an exemplification of the domain-theoretic approach to topological semantics needs an explanation. Already in the original manuscript [23] Scott showed that equilogical spaces are equivalent to partial equivalence relations (PERs) on algebraic lattices. He also proved that the category of algebraic domains is a cartesian closed subcategory of equilogical spaces, and it is not hard to see that the same holds for continuous lattices. In [6,5] we showed that equilogical spaces are a generalization of domain theory with totality [9,8,7,20,21]. The crucial observation needed for those results is that equilogical spaces are equivalent to the category of *dense* PERs on algebraic domains (a PER on a domain is said to be dense if its extension is a dense subset of the domain). The equivalence remains if we take dense PERs on continuous domains instead. In this sense, it is fair to say that equilogical spaces generalize several domain-theoretic frameworks and contain a number of important categories of domains that have been studied, but of course not all of them. In this paper we focus solely on the countably based equilogical spaces, and call them simply "equilogical spaces".

As the ambient category of TTE we take the category of Baire space representations, $\mathsf{Rep}(\mathbb{B})$, which is defined in Section 3. Contemporary formulations of TTE often use the Cantor space in place of the Baire space, but since we are not concerned with computational complexity here, it does not matter which one we use because they yield in equivalent categories. We call Baire space representations just "representations".

Equilogical spaces and representations both form locally cartesian closed extensions of the category of countably based T_0-spaces, $\omega\mathsf{Top}_0$. Thus they are both appealing models of computation on topological spaces. This is why it is important from the programming semantics point of view to understand precisely how they are related.

The general framework within which we carry out the comparison is realizability theory, since Equ and $\mathsf{PER}(\mathbb{B})$ are just realizability models; the former is equivalent to the PER model on the Scott-Plotkin graph model $\mathcal{P}\mathbb{N}$, whereas the latter is equivalent to the PER model on the Second Kleene Algebra $\mathbb{B}$. We can then use Longley's theory of applicative morphisms between partial combinatory algebras (PCAs) to compare the two PER models [17]. While this may be the most general and elegant technique that could be used to compare other semantic frameworks as well, it has a distinctly anti-topological flavor. But we can translate all the results from realizability back into the language of topology, which is precisely what we do. This immediately gives us the first result: a simple topological description of $\mathsf{Rep}(\mathbb{B})$, without any mention of the partial combinatory structure of the Second Kleene Algebra.

From the topological description of $\mathsf{Rep}(\mathbb{B})$ so obtained, it is apparent that $\mathsf{Rep}(\mathbb{B})$ is equivalent to a full subcategory of Equ. This subcategory is denoted by $\mathsf{0Equ}$ and consists of all the 0-*equilogical spaces*, which are those equilogical spaces whose underlying topological spaces are 0-dimensional. The inclusion $I \colon \mathsf{0Equ} \to \mathsf{Equ}$ has a coreflection $D \colon \mathsf{Equ} \to \mathsf{0Equ}$. These two functors have many desirable properties, but they do *not* preserve the function spaces in general.

We compare Equ and $\mathsf{Rep}(\mathbb{B})$ in another way, by demonstrating that they share a common cartesian closed subcategory that contains all countably based T_0-spaces. This subcategory was discovered by Menni and Simpson [19,18] as the category of *ω-projecting T_0-quotients*, and by Schröder [22] as the category of *sequential T_0-spaces with admissible representations*. We prove that these two categories coincide. Therefore, the domain-theoretic approach and TTE yield equivalent topological semantics of computation for all higher-order types over countably based T_0-spaces.

Finally, we discuss various consequences and the potential for transfer of results between the two settings, in particular with respect to the natural numbers, the real numbers, and their higher-order function spaces.

The paper is organized as follows. In Section 1 we review the basic definitions and facts about equilogical spaces and ω-projecting quotients. In Section 3 we review Baire space representations and admissible representations. Sections 4 and 5 contain the two comparisons of Equ and $\mathsf{Rep}(\mathbb{B})$. In Section 6 we obtain various transfer results between the two settings.

The material presented here is part of my Ph.D. dissertation [4], written under the supervision of Dana Scott. The omitted proofs can be found in the dissertation.

I gratefully acknowledge helpful discussions about this topic with Steven Awodey, Lars Birkedal, Peter Lietz, Alex Simpson, Matthias Schröder, and Dana Scott. Peter and I found the equivalence of 0-equilogical spaces and Baire space representations together. I could have never proved the coincidence of ω-projecting quotients and admissible representations without talking to Matthias and Alex. I also thank the knowledgeable anonymous referee for helpful suggestions on how to better present the material.

2 Equilogical Spaces and ω-projecting Quotients

An *equilogical space* was defined by Scott [23,6] to be a T_0-space with an equivalence relation. Here we are only interested in *countably based equilogical spaces*, which are countably based T_0-spaces with equivalence relations. We denote the category of countably based T_0-spaces and continuous maps by $\omega\mathsf{Top}_0$. We omit the qualifier "countably based" from now on, unless we are explicitly dealing with spaces that are not countably based.

More precisely, an equilogical space is a pair $X = (|X|, \equiv_X)$ where $|X| \in \omega\mathsf{Top}_0$ and $\equiv_X$ is an equivalence relation on the underlying set of $|X|$. The *associated quotient* of an equilogical space X is the topological quotient $\|X\| = |X|/{\equiv_X}$. The canonical quotient map $|X| \to \|X\|$ is denoted by q_X. Note that $\|X\|$ need not be T_0 or countably based. A morphism $f\colon X \to Y$ between equilogical spaces X and Y is a continuous map $f\colon \|X\| \to \|Y\|$ that is *tracked* by some (not necessarily unique) continuous map $g\colon |X| \to |Y|$, which means that the following diagram commutes:

$$\begin{tikzcd} |X| \arrow[r, "g"] \arrow[d, "q_X"'] & |Y| \arrow[d, "q_Y"] \\ \|X\| \arrow[r, "f"'] & \|Y\| \end{tikzcd}$$

Any map g that appears in the top row of such a diagram is *equivariant*, or *extensional*, meaning that, for all $x, y \in |X|$, $x \equiv_X y$ implies $gx \equiv_Y gy$.[2] The category of equilogical spaces and morphisms between them is denoted by Equ.

An *exponential* of X and Y is an object $E = Y^X$ with a morphism $e\colon E \times X \to Y$, called the *evaluation map*, such that, for all Z and $f\colon Z \times X \to Y$, there exists a unique map $\widetilde{f}\colon Z \to E$, called the *transpose* of f, such that the following diagram commutes:

$$\begin{tikzcd} E \times X \arrow[dr, "e"] & \\ Z \times X \arrow[u, "\widetilde{f} \times 1_X"] \arrow[r, "f"'] & Y \end{tikzcd}$$

A *weak exponential* is defined in the same way but without the uniqueness requirement for $\widetilde{f}$. A category is said to be *cartesian closed* when it has the terminal object, finite products, and all exponentials. It is *locally cartesian closed* when every slice is cartesian closed.

[2] We could define morphisms between equilogical spaces to be equivalence classes of equivariant maps, which is the original definition from [23].

The category Equ is equivalent to the PER model $\mathsf{PER}(\mathcal{P}\mathbb{N})$ [4, Theorem 4.1.3], which is a regular locally cartesian closed category. This equivalence gives us a description of exponentials in Equ, though a very impractical one. A somewhat better description can be obtained as follows. Suppose X and Y are equilogical spaces, and (W, e) is a weak exponential of $|X|$ and $|Y|$ in $\omega\mathsf{Top}_0$. Define a relation $\equiv_E$ on W by

$$f \equiv_E g \iff \forall\, x, y \in |X| \,.\, (x \equiv_X y \Longrightarrow e(f, x) \equiv_Y e(g, y)) \ .$$

Let $E = (|E|, \equiv_E)$ be the equilogical space whose underlying space is

$$|E| = \left\{ f \in W \;\middle|\; f \equiv_E f \right\} \subseteq W \ .$$

It is easy to check that E with the morphism induced by the evaluation map $e \colon |E| \times |X| \to |Y|$ is the exponential of X and Y [4, Proposition 4.1.7]. The category $\omega\mathsf{Top}_0$ has weak exponentials, thus the following construction shows that Equ has exponentials. It would be desirable to have a good theory of weak exponentials of topological spaces, as that would give us better descriptions of exponentials in Equ. In certain cases (weak) exponentials have good descriptions. For example, if $|X|$ is locally compact and Hausdorff, then the space of continuous maps $W = \mathcal{C}(|X|, |Y|)$ with the compact-open topology together with the usual evaluation map is an exponential of $|X|$ and $|Y|$ in $\omega\mathsf{Top}_0$.

Every countably based T_0-space X can be viewed as an equilogical space $(X, =_X)$ where $=_X$ is equality on X. This defines a full and faithful inclusion functor $I \colon \omega\mathsf{Top}_0 \to \mathsf{Equ}$. The inclusion preserves finite limits, coproducts, and all exponentials that already exist in $\omega\mathsf{Top}_0$. Preservation of exponentials follows directly from the above description of exponentials in Equ.

There is the *associated quotient* functor $Q \colon \mathsf{Equ} \to \mathsf{Top}$ that maps an equilogical space X to the associated quotient $QX = \|X\|$ and a morphism $f \colon X \to Y$ to the continuous map $Qf = f \colon \|X\| \to \|Y\|$. Here Top is the category of *all* topological spaces and continuous maps, because the associated quotient need not be countably based or T_0. Clearly, Q is a faithful functor, and it is not hard too see that it is not full. Menni and Simpson [19,18] showed that there is a largest subcategory $\mathcal{C}$ of Equ such that Q restricted to $\mathcal{C}$ is full. They worked with equilogical spaces built from all countably based topological spaces, as opposed to just T_0-spaces, but their results hold when we restrict them to T_0-spaces. We are restricting to T_0-spaces because Schröder proved his results for T_0-spaces. Below we summarize the relevant findings from [19,18].

Definition 2.1 A subset $S \subseteq X$ of a topological space X is *sequentially open* when every sequence with limit in S is eventually in S. A topological space X is a *sequential space* when every sequentially open set $V \subseteq X$ is open in X. The category of sequential spaces and continuous maps between them is denoted by Seq.

Theorem 2.2 *Sequential spaces form a cartesian closed category that contains $\omega\mathsf{Top}_0$. The inclusion $\omega\mathsf{Top}_0 \to \mathsf{Seq}$ preserves finite limits and all exponentials that already exist in $\omega\mathsf{Top}_0$.*

Proof. This is well known and follows from the fact that Seq is a reflective subcategory of the cartesian-closed category Lim of *limit spaces* [15], and the reflection preserves products. □

Definition 2.3 Let $X \in \omega\mathsf{Top}_0$ and $q \colon X \to Y$ be a continuous map. Then q is said to be *ω-projecting* when for every $Z \in \omega\mathsf{Top}_0$ and every continuous map $f \colon Z \to Y$ there exists a lifting $g \colon Z \to X$ such that $f = q \circ g$.

An equilogical space X is *ω-projecting* when the canonical quotient map $q_X \colon |X| \to \|X\|$ is ω-projecting. The full subcategory of Equ on the ω-projecting equilogical spaces is denoted by EPQ_0. Let PQ_0 be the category of those T_0-spaces Y for which there exists an ω-projecting map $q \colon X \to Y$.

The name PQ_0 stands for "ω-projecting quotient", and EPQ_0 stands for "equilogical ω-projecting quotient".

Theorem 2.4 (Menni & Simpson [19]) *The category PQ_0 is a cartesian closed subcategory of Seq, EPQ_0 is a cartesian closed subcategory of Equ, and the categories PQ_0 and EPQ_0 are equivalent via the restriction of the associated quotient functor $Q \colon \mathsf{EPQ}_0 \to \mathsf{PQ}_0$.*

Proof. See [19]. In fact, Menni and Simpson prove that PQ_0 is the largest common subcategory $\mathcal{C}$ of Equ and Top such that Q restricted to $\mathcal{C}$ is full. □

3 Type Two Effectivity

In this section we review the basic setup of Type Two Effectivity. The Baire space $\mathbb{B} = \mathbb{N}^\mathbb{N}$ is the set of all infinite sequences of natural numbers, equipped with the product topology. Let $\mathbb{N}^*$ be the set of all finite sequences of natural numbers. The length of a finite sequence a is denoted by $|a|$. If $a, b \in \mathbb{N}^*$ we write $a \sqsubseteq b$ when a is a prefix of b. Similarly, we write $a \sqsubseteq \alpha$ when a is a prefix of an infinite sequence $\alpha \in \mathbb{B}$. A countable topological base for $\mathbb{B}$ consists of the basic open sets, for $a \in \mathbb{N}^*$,

$$a{::}\mathbb{B} = \left\{ a{::}\beta \;\middle|\; \beta \in \mathbb{B} \right\} = \left\{ \alpha \in \mathbb{B} \;\middle|\; a \sqsubseteq \alpha \right\} \;.$$

The expression $a{::}\beta$ denotes the concatenation of the finite sequence $a \in \mathbb{N}^*$ with the infinite sequence $\beta \in \mathbb{B}$. We write $n{::}\beta$ instead of $[n]{::}\beta$ for $n \in \mathbb{N}$ and $\beta \in \mathbb{B}$. The base $\left\{ a{::}\mathbb{B} \;\middle|\; a \in \mathbb{N}^* \right\}$ is a clopen countable base for the topology of $\mathbb{B}$, which means that $\mathbb{B}$ is a countably based 0-dimensional T_0-space. Recall that a space is 0-*dimensional* when its clopen subsets form a base for its topology. A 0-dimensional T_0-space is always Hausdorff.

In order to obtain a simple topological description of Baire space representations, we need to characterize subspaces of $\mathbb{B}$ and those partial continuous maps $\mathbb{B} \rightharpoonup \mathbb{B}$ that can be encoded as elements of $\mathbb{B}$. This is accomplished by the Embedding and Extension Theorems for $\mathbb{B}$, which we prove next.

Theorem 3.1 (Embedding Theorem for $\mathbb{B}$) *A topological space is a 0-dimensional countably based T_0-space if, and only if, it embeds into $\mathbb{B}$.*

Proof. Clearly, every subspace of $\mathbb{B}$ is a countably based 0-dimensional T_0-space. Suppose X is a countably based 0-dimensional T_0-space with a countable base $\{U_k \mid k \in \mathbb{N}\}$ of clopen sets. Define the map $e \colon X \to \mathbb{B}$ by

$$ex = \lambda n \in \mathbb{N}\,.\,(\texttt{if } x \in U_n \texttt{ then } 1 \texttt{ else } 0)\ .$$

It is easy to check that e is a topological embedding. $\square$

For topological spaces X and Y, a partial map $f \colon X \rightharpoonup Y$ is said to be *continuous* when the restriction to its domain $f \colon \mathsf{dom}(f) \to Y$ is a continuous (total) map, where $\mathsf{dom}(f)$ is equipped with the subspace topology inherited from X. There is no requirement that $\mathsf{dom}(f)$ be an open subset of X. We consider partial continuous maps $\mathbb{B} \rightharpoonup \mathbb{B}$ and characterize those that can be encoded as elements of $\mathbb{B}$.

Given a finite sequence of numbers $a = [a_0, \dots, a_{k-1}]$, let $\mathsf{seq}\, a$ be the encoding of a as a natural number, for example

$$\mathsf{seq}\,[a_0, \dots, a_{k-1}] = \prod_{i=0}^{k-1} {p_i}^{1+a_i}\ ,$$

where p_i is the i-th prime number. For $\alpha \in \mathbb{B}$ let $\overline{\alpha}n = \mathsf{seq}\,[\alpha 0, \dots, \alpha(n-1)]$. For $\alpha, \beta \in \mathbb{B}$, define $\alpha \star \beta$ by

$$\alpha \star \beta = n \iff \exists\, m \in \mathbb{N}\,.\,\big(\alpha(\overline{\beta}m) = n + 1 \wedge \forall\, k < m\,.\,\alpha(\overline{\beta}k) = 0\big)\ .$$

If there is no $m \in \mathbb{N}$ that satisfies the above condition, then $\alpha \star \beta$ is undefined. Thus, $\star$ is a partial operation $\mathbb{B} \times \mathbb{B} \rightharpoonup \mathbb{N}$. It is continuous because the value of $\alpha \star \beta$ depends only on finite prefixes of α and β. The *continuous function application* $\Box \mid \Box \colon \mathbb{B} \times \mathbb{B} \to \mathbb{N} \rightharpoonup \mathbb{N}$ is defined by

$$(\alpha \mid \beta)n = \alpha \star (n{::}\beta)\ .$$

The Baire space $\mathbb{B}$ together with $\mid$ is a partial combinatory algebra, where $\alpha \mid \beta$ is considered to be undefined when $\alpha \mid \beta$ is not a total function, see [13] for details. Every $\alpha \in \mathbb{B}$ represents a partial function $\boldsymbol{\eta}_\alpha \colon \mathbb{B} \rightharpoonup \mathbb{B}$ defined by

$$\boldsymbol{\eta}_\alpha \beta = \alpha \mid \beta\ .$$

We say that a partial map $f \colon \mathbb{B} \rightharpoonup \mathbb{B}$ is *realized* when there exists $\alpha \in \mathbb{B}$ such that $f = \boldsymbol{\eta}_\alpha$. Such an α is called a *realizer* for f. Because $\mid$ is a continuous operation, a realized map is always continuous, although not every partial continuous map is realized. Recall that a G_δ-set is a set that is equal to a countable intersection of open sets.

Proposition 3.2 *If $U \subseteq \mathbb{B}$ is a G_δ-set then the function $u\colon \mathbb{B} \rightharpoonup \mathbb{B}$ defined by*

$$u\alpha = \begin{cases} \lambda n : \mathbb{N} . 1 & \alpha \in U , \\ \textit{undefined} & \textit{otherwise} \end{cases}$$

is realized.

Proof. The set U is a countable intersection of countable unions of basic open sets, $U = \bigcap_{i \in \mathbb{N}} \bigcup_{j \in \mathbb{N}} a_{i,j}{::}\mathbb{B}$. Define a sequence $v \in \mathbb{B}$ for all $i, j \in \mathbb{N}$ by $v(\mathsf{seq}\,(i{::}a_{i,j})) = 2$, and set $vn = 0$ for all other arguments n. Clearly, if $\boldsymbol{\eta}_v\alpha$ is total then its value is $\lambda n.\, 1$, so we only need to verify that $\mathsf{dom}(\boldsymbol{\eta}_v) = U$. If $\alpha \in \mathsf{dom}(\boldsymbol{\eta}_v)$ then $v \star (i{::}\alpha)$ is defined for every $i \in \mathbb{N}$, therefore there exists $ci \in \mathbb{N}$ such that $v(\mathsf{seq}\,(i{::}[\alpha 0, \dots, \alpha(ci)])) = 2$, which implies that $\alpha \in a_{i,ci}$. Hence $\alpha \in \bigcap_{i \in \mathbb{N}} a_{i,ci}{::}\mathbb{B} \subseteq U$. Conversely, if $\alpha \in U$ then for every $i \in \mathbb{N}$ there exists some $ci \in \mathbb{N}$ such that $\alpha \in a_{i,ci}$. For every $i \in \mathbb{N}$, $v(\mathsf{seq}\,(i{::}[\alpha 0, \dots, \alpha(ci)])) = 2$, therefore $(\boldsymbol{\eta}_v\alpha)i = v \star (i{::}\alpha) = 1$. Hence $\alpha \in \mathsf{dom}(\boldsymbol{\eta}_v)$. $\square$

Corollary 3.3 *Suppose $\alpha \in \mathbb{B}$ and $U \subseteq \mathbb{B}$ is a G_δ-set. Then there exists $\beta \in \mathbb{B}$ such that $\boldsymbol{\eta}_\alpha\gamma = \boldsymbol{\eta}_\beta\gamma$ for all $\gamma \in \mathsf{dom}(\boldsymbol{\eta}_\alpha) \cap U$ and $\mathsf{dom}(\boldsymbol{\eta}_\beta) = U \cap \mathsf{dom}(\boldsymbol{\eta}_\alpha)$.*

Proof. By Proposition 3.2 there exists $v \in \mathbb{B}$ such that for all $\beta \in \mathbb{B}$

$$\boldsymbol{\eta}_v\beta = \begin{cases} \lambda n : \mathbb{N} . 1 & \beta \in U , \\ \text{undefined} & \text{otherwise} . \end{cases}$$

It suffices to show that the function $f : \mathbb{B} \rightharpoonup \mathbb{B}$ defined by

$$(f\beta)n = ((\eta_v\beta)n) \cdot ((\eta_\alpha\beta)n)$$

is realized. This is so because coordinate-wise multiplication of sequences is realized, and so are pairing and composition. $\square$

Theorem 3.4 (Extension Theorem for $\mathbb{B}$) *(a) Every partial continuous map $\mathbb{B} \rightharpoonup \mathbb{B}$ can be extended to a realized one. (b) The realized partial maps $\mathbb{B} \rightharpoonup \mathbb{B}$ are precisely those continuous partial maps whose domains are G_δ-sets.*

Proof. (a) Suppose $f : \mathbb{B} \rightharpoonup \mathbb{B}$ is a partial continuous map. Consider the set $A \subseteq \mathbb{N}^* \times \mathbb{N}^2$ defined by

$$\begin{aligned} A = \big\{ \langle a, i, j \rangle \in \mathbb{N}^* \times \mathbb{N}^2 \;\big|\; & \\ & a{::}\mathbb{B} \cap \mathsf{dom}(f) \neq \emptyset \text{ and } \forall\, \alpha \in (a{::}\mathbb{B} \cap \mathsf{dom}(f))\,.\, ((f\alpha)i = j) \big\} . \end{aligned}$$

If $\langle a, i, j\rangle \in A$, $\langle a', i, j'\rangle \in A$ and $a \sqsubseteq a'$ then $j = j'$ because there exists $\alpha \in a'{::}\mathbb{B} \cap \mathsf{dom}(f) \subseteq a{::}\mathbb{B} \cap \mathsf{dom}(f)$ such that $j = (f\alpha)i = j'$. We define a sequence $\phi \in \mathbb{B}$ as follows. For every $\langle a, i, j\rangle \in A$ let $\phi(\mathsf{seq}\,(i{::}a)) = j + 1$, and for all other arguments let $\phi n = 0$. Suppose that $\phi(\mathsf{seq}\,(i{::}a)) = j + 1$ for some $i, j \in \mathbb{N}$ and $a \in \mathbb{N}^*$. Then for every prefix $a' \sqsubseteq a$, $\phi(\mathsf{seq}\,(i{::}a')) = 0$ or $\phi(\mathsf{seq}\,(i{::}a')) = j + 1$. Thus, if $\langle a, i, j\rangle \in A$ and $a \sqsubseteq \alpha$ then $\phi \star (i{::}\alpha) = j$. We show that $(\boldsymbol{\eta}_\phi\alpha)i = (f\alpha)i$ for all $\alpha \in \mathsf{dom}(f)$ and all $i \in \mathbb{N}$. Because f is continuous, for all $\alpha \in \mathsf{dom}(f)$ and $i \in \mathbb{N}$ there exists $\langle a, i, j\rangle \in A$ such that $a \sqsubseteq \alpha$ and $(f\alpha)i = j$. Now we get $(\boldsymbol{\eta}_\phi\alpha)i = (\phi \mid \alpha)i = \phi \star (i{::}\alpha) = j = (f\alpha)i$.

(b) First we show that $\boldsymbol{\eta}_\alpha$ is a continuous map whose domain is a G_δ-set. It is continuous because the value of $(\boldsymbol{\eta}_\alpha\beta)n$ depends only on n and finite prefixes of α and β. The domain of $\boldsymbol{\eta}_\alpha$ is the G_δ-set

$$\begin{aligned}\mathsf{dom}(\boldsymbol{\eta}_\alpha) &= \left\{\beta \in \mathbb{B} \;\middle|\; \forall\, n \in \mathbb{N}.\,((\alpha \mid \beta)n \text{ defined})\right\} \\ &= \bigcap_{n\in\mathbb{N}} \left\{\beta \in \mathbb{B} \;\middle|\; (\alpha \mid \beta)n \text{ defined}\right\} = \bigcap_{n\in\mathbb{N}} \bigcup_{m\in\mathbb{N}} \left\{\beta \in \mathbb{B} \;\middle|\; \alpha \star (n{::}\beta) = m\right\} .\end{aligned}$$

Each of the sets $\left\{\beta \in \mathbb{B} \;\middle|\; \alpha \star (n{::}\beta) = m\right\}$ is open because $\star$ and $::$ are continuous operations. Now let $f \colon \mathbb{B} \rightharpoonup \mathbb{B}$ be a partial continuous function whose domain is a G_δ-set. By part (a) of this theorem there exists $\phi \in \mathbb{B}$ such that $f\alpha = \boldsymbol{\eta}_\phi\alpha$ for all $\alpha \in \mathsf{dom}(f)$. By Corollary 3.3 there exists $\psi \in \mathbb{B}$ such that $\mathsf{dom}(\boldsymbol{\eta}_\psi) = \mathsf{dom}(f)$ and $\boldsymbol{\eta}_\psi\alpha = \boldsymbol{\eta}_\phi\alpha$ for every $\alpha \in \mathsf{dom}(f)$. □

A *Baire space representation*, or simply a *representation*, is a partial surjection $\delta_S \colon \mathbb{B} \rightharpoonup S$, where S is a set. A representation $\delta_S \colon \mathbb{B} \rightharpoonup S$ of a set S induces a quotient topology on S, defined by

$$U \subseteq S \text{ open} \iff \delta_S^{-1}(U) \text{ open in } \mathsf{dom}(\delta_S) \;.$$

We denote by $\|S\|$ the topological space S with the quotient topology induced by δ_S. A *realized map* $f \colon (S, \delta_S) \to (T, \delta_T)$ is a function $f \colon S \to T$ such that there exists a partial continuous map $g \colon \mathbb{B} \rightharpoonup \mathbb{B}$ which tracks f, meaning that $\mathsf{dom}(f) \subseteq \mathsf{dom}(g)$ and that, for every $\alpha \in \mathsf{dom}(f)$, $f(\delta_S\alpha) = \delta_T(g\alpha)$. A realized map f is always continuous as map $f \colon \|S\| \to \|T\|$. The category of Baire space representations and realized maps is denoted by $\mathsf{Rep}(\mathbb{B})$.

The category $\mathsf{Rep}(\mathbb{B})$ is equivalent to the PER model $\mathsf{PER}(\mathbb{B})$ where $\mathbb{B}$ is equipped with the structure of the Second Kleene Algebra. The objects of $\mathsf{PER}(\mathbb{B})$ are partial equivalence relations on $\mathbb{B}$. If A is a PER on $\mathbb{B}$ we denote it by A when we think of it as an object and by $=_A$ when we think of it as a binary relation. For $A, B \in \mathsf{PER}(\mathbb{B})$, we say that $\alpha \in \mathbb{B}$ *realizes* a morphism $[\alpha] \colon A \to B$ when, for all $\beta, \gamma \in \mathbb{B}$, if $\beta =_A \gamma$, then $\alpha \mid \beta$ and $\alpha \mid \gamma$ are defined, and $\alpha \mid \beta =_B \alpha \mid \gamma$. Here α and α' realize the same morphism, $[\alpha] = [\alpha']$, when, for all $\beta, \gamma \in \mathbb{B}$, $\beta =_A \gamma$ implies $\alpha \mid \beta =_B \alpha' \mid \gamma$. The equivalence of $\mathsf{Rep}(\mathbb{B})$ and

$\mathsf{PER}(\mathbb{B})$ assigns to each representation $\delta_S \colon \mathbb{B} \rightharpoonup S$ the PER $=_S$ defined by

$$\alpha =_S \beta \iff \delta_S(\alpha) = \delta_S(\beta) \ .$$

If $f \colon (S, \delta_S) \to (T, \delta_T)$ is a realized map in $\mathsf{Rep}(\mathbb{B})$, tracked by $g \colon \mathbb{B} \rightharpoonup \mathbb{B}$, then by Extension Theorem 3.4 there exists $\alpha \in \mathbb{B}$ such that $\boldsymbol{\eta}_\alpha$ is a continuous extension of g. Under the equivalence $\mathsf{Rep}(\mathbb{B}) \simeq \mathsf{PER}(\mathbb{B})$, the morphism f corresponds to the morphism $[\boldsymbol{\eta}_\alpha]$. The most relevant consequence of this equivalence is that $\mathsf{Rep}(\mathbb{B})$ is a regular locally cartesian closed category, since every PER model on a PCA is such a category [4]. For example, the exponential B^A of PERs $A, B \in \mathsf{PER}(\mathbb{B})$ is defined by

$$\alpha =_{B^A} \alpha' \iff \forall\, \beta, \gamma \in \mathbb{B} \,.\, (\beta =_A \gamma \Longrightarrow (\alpha \mid \beta){\downarrow} =_B (\alpha' \mid \gamma){\downarrow}) \ .$$

Unfortunately, this description of exponentials in not very helpful in particular cases, and it completely obscures the topological properties of exponentials. In many important cases better descriptions are available, cf. Theorem 4.5.

In TTE we are typically interested in representations of topological spaces, rather than arbitrary sets. For this reason it is important to represent a topological space X with a representation (X, δ_X) which has a reasonable relation to the topology of X. An obvious requirement is that the original topology of X should coincide with the quotient topology of $\|X\|$. However, as is well known by the school of TTE, this requirement is too weak because it allows ill-behaved representations. A desirable condition on representations of topological spaces is that all continuous maps between them be realized. Thus, we are led to further restricting the allowable representations of topological spaces as follows.

Definition 3.5 An *admissible representation* of a topological space X is a partial continuous quotient map $\delta \colon \mathbb{B} \rightharpoonup X$ such that every partial continuous map $f \colon \mathbb{B} \rightharpoonup X$ can be factored through δ. This means that there exists $g \colon \mathbb{B} \rightharpoonup \mathbb{B}$ such that $f\alpha = \delta(g\alpha)$ for all $\alpha \in \mathsf{dom}(f)$.

The main effect of this definition is that if $\delta_X : \mathbb{B} \rightharpoonup X$ and $\delta_Y : \mathbb{B} \rightharpoonup Y$ are admissible representations, then every continuous map $f \colon X \to Y$ is realized, and conversely, every realizer that respects δ_X and δ_Y induces a continuous map $X \to Y$.

The requirement that and admissible representation $\delta \colon \mathbb{B} \rightharpoonup X$ be a quotient map implies that X is a sequential space, since it is a quotient of the sequential space $\mathsf{dom}(\delta)$. It is easy to show that any two admissible representations are isomorphic in $\mathsf{Rep}(\mathbb{B})$. An obvious question to ask is which sequential spaces have admissible representations.

Definition 3.6 Let AdmSeq be the full subcategory of Seq on those sequential T_0-spaces that have admissible representations. [3]

[3] It is believed that the T_0 requirement is inessential for the results proved here, but that

Schröder [22] has characterized AdmSeq as follows.

Definition 3.7 [Schröder [22]] A *pseudobase* for a space X is a family $\mathcal{B}$ of subsets of X such that whenever $\langle x_n \rangle_{n\in\mathbb{N}} \to_{\mathcal{O}(X)} x_\infty$ and $x_\infty \in U \in \mathcal{O}(X)$ then there exists $B \in \mathcal{B}$ such that $x_\infty \in B \subseteq U$ and $\langle x_n \rangle_{n\in\mathbb{N}}$ is eventually in B.

Theorem 3.8 (Schröder [22]) *A sequential T_0-space has an admissible representation if, and only if, it has a countable pseudobase.*

From Schröder's proof of Theorem 3.8 we get a specific admissible representation δ for a T_0-space X with a countable pseudobase $\{B_k \mid k \in \mathbb{N}\}$, defined by

$$\begin{aligned}&\delta(\alpha) = x \iff \\ &\qquad \forall\, k \in \mathbb{N}\,.\,(x \in B_{\alpha k}) \land \forall\, U \in \mathcal{O}(X)\,.\,(x \in U \Longrightarrow \exists\, k \in \mathbb{N}\,.\, B_{\alpha k} \subseteq U)\ .\end{aligned}$$

The above formula says that α is a δ-representation of x when α enumerates (indices of) a sequence of pseudobasic open neighborhoods of x that get arbitrarily small. In case X is a T_0-space with a countable base $\{U_k \mid k \in \mathbb{N}\}$, we may use an equivalent but simpler admissible representation δ', defined by

$$\delta'(\alpha) = x \iff \{U_{\alpha k} \mid k \in \mathbb{N}\} = \{U_n \mid n \in \mathbb{N} \land x \in U_n\}\ .$$

The above formula says that α is a δ'-representation of x when it enumerates the basic open neighborhoods of x.

If $X \in \mathsf{AdmSeq}$ then its admissible representation is determined up to isomorphism in $\mathsf{Rep}(\mathbb{B})$. Therefore, AdmSeq is equivalent to the full subcategory of $\mathsf{Rep}(\mathbb{B})$ on the admissible representations, so that AdmSeq can be thought of as a subcategory of $\mathsf{Rep}(\mathbb{B})$. The following result by Schröder [22] tells us that the inclusion of AdmSeq into $\mathsf{Rep}(\mathbb{B})$ preserves the cartesian closed structure.

Theorem 3.9 (Schröder [22]) *Let (X, δ_X) and (Y, δ_Y) be admissible representations for sequential T_0-spaces X and Y. Then the product $(X, \delta_X) \times (Y, \delta_Y)$ formed in $\mathsf{Rep}(\mathbb{B})$ is an admissible representation of the product $X \times Y$ formed in Seq, and similarly the exponential $(Y, \delta_Y)^{(X,\delta_X)}$ formed in $\mathsf{Rep}(\mathbb{B})$ is an admissible representation for the exponential Y^X formed in Seq.*

4 $\mathsf{Rep}(\mathbb{B})$ as a subcategory of Equ

In this section we describe $\mathsf{Rep}(\mathbb{B})$ as a full subcategory of equilogical spaces. We then study the properties of the inclusion $\mathsf{Rep}(\mathbb{B}) \to \mathsf{Equ}$.

has not been checked yet.

Definition 4.1 A 0-*equilogical space* is an equilogical space whose underlying topological space is 0-dimensional. The category $\mathsf{0Equ}$ is the full subcategory of Equ on 0-equilogical spaces.

Thus $\mathsf{0Equ}$ is formed just like Equ, where we use $\mathsf{0Dim}$ instead of $\omega\mathsf{Top}_0$.

Theorem 4.2 *The categories* $\mathsf{0Equ}$, $\mathsf{Rep}(\mathbb{B})$, *and* $\mathsf{PER}(\mathbb{B})$ *are equivalent.*

Proof. We show that $\mathsf{0Equ}$ and $\mathsf{PER}(\mathbb{B})$ are equivalent, since we already know that $\mathsf{PER}(\mathbb{B})$ and $\mathsf{Rep}(\mathbb{B})$ are equivalent. By Embedding Theorem 3.1 for $\mathbb{B}$, a countably based T_0-space is 0-dimensional if, and only if, it embeds in $\mathbb{B}$. Thus every 0-equilogical space is isomorphic to one whose underlying topological space is a subspace of $\mathbb{B}$. This make it clear that equivalence relations on 0-dimensional countably based T_0-spaces correspond to partial equivalence relations on $\mathbb{B}$. Morphisms work out, too, since by the Extension Theorem for $\mathbb{B}$ 3.4 every partial continuous map on $\mathbb{B}$ can be extended to a realized one. $\square$

The inclusion functor $I\colon \mathsf{0Equ} \to \mathsf{Equ}$ has a right adjoint $D\colon \mathsf{Equ} \to \mathsf{0Equ}$, which is defined as follows. For every countably based T_0-space X there exists an admissible representation $\delta_X\colon \mathbb{B} \rightharpoonup X$. The subspace $X_0 = \mathsf{dom}(\delta) \subseteq \mathbb{B}$ is a countably based 0-dimensional Hausdorff space. Now if $X = (|X|, \equiv_X)$ is an equilogical space, let $DX = (X_0, \equiv_{DX})$ where $a \equiv_{DX} b$ if, and only if, $\delta_X a \equiv_X \delta_X b$. If $f\colon X \to Y$ is a morphism in Equ, tracked by $g\colon |X| \to |Y|$, then Df is the morphism tracked by a continuous map $h\colon X_0 \to Y_0$ that tracks $g\colon X \to Y$, as shown in the following commutative diagram:

$$\begin{tikzcd}
X_0 \arrow[r, "h"] \arrow[d, "\delta_X"'] & Y_0 \arrow[d, "\delta_Y"] \\
X \arrow[r, "g"'] & Y
\end{tikzcd}$$

Such a map h exists because δ_X and δ_Y were chosen to be admissible representations. The main properties of the adjoints $I \dashv D$ are summarized in the following theorem.

Theorem 4.3

(i) *Functors I and D are a section and a retraction, i.e., $D \circ I$ is naturally equivalent to* 1_{0Equ}.

(ii) *I is full and faithful and preserves countable colimits and limits (which are precisely all the limits and colimits that exist in* Equ*).*

(iii) *D is faithful and preserves countable limits and colimits (which are precisely all the limits and colimits that exist in* $\mathsf{0Equ}$*).*

(iv) *D is not full, but its restriction to* EPQ_0 *is full.*

Proof. (i) This follows by a general category-theoretic argument from the fact that I is full and faithful, cf. the dual of [11, Proposition 3.4.1].

(ii) It is obvious that I is full and faithful since it is just the inclusion functor of a full subcategory. It preserves colimits because it is a left adjoint, and it preserves limits because the inclusion $\mathsf{0Dim} \to \omega\mathsf{Top}_0$ does.

(iii) It is obvious that D is faithful, and it preserves limits because it is a right adjoint. That D preserves finite colimits can be verified explicitly, and it also follows from [17, Proposition 2.5.11]. That D preserves countable coproducts holds because a countable coproduct of admissible representations is again an admissible representation.

(iv) If D were full then by [11, Proposition 3.4.3] it would follow that the counit of the adjunction $\eta\colon I \circ D \to 1_{\mathsf{Equ}}$ is a natural isomorphism, which obviously is not the case. For example, $\eta_{\mathbb{R}}$ is not a natural isomorphism, where $\mathbb{R}$ are the real numbers equipped with the Euclidean topology, because every morphism $\mathbb{R} \to I(D\mathbb{R})$ is constant, as it must be tracked by a continuous map from $\mathbb{R}$ into the 0-dimensional Hausdorff space $|I(D\mathbb{R})|$. However, when D is restricted to EPQ_0 then we can show that it is full as follows. Suppose $X, Y \in \mathsf{EPQ}_0$, and let $r_X : X_0 \to |X|$ and $r_Y : Y_0 \to |Y|$ be admissible representations. Suppose $f\colon DX \to DY$ is a morphism tracked by a continuous map $g\colon X_0 \to Y_0$. The situation is shown in the following diagram:

$$\begin{tikzcd}
X_0 \arrow[r, "g"] \arrow[d, "r_X"'] & Y_0 \arrow[d, "r_Y"] \\
|X| \arrow[r, "h"] \arrow[d, "q_X"'] & |Y| \arrow[d, "q_Y"] \\
\|X\| \arrow[r, "f"] & \|Y\|
\end{tikzcd}$$

Because q_Y is ω-projecting, f is tracked by an arrow $h\colon |X| \to |Y|$ so that the lower square commutes. Therefore f is a morphism in Equ, hence $Df = f$. □

Remark 4.4 Since I and D both preserve all limits and colimits that exist, one wonders whether they have any further adjoints.[4] This does not seem to be the case. One might try embedding the categories Equ and $\mathsf{Rep}(\mathbb{B})$ into larger categories and extending I and D, in hope that the "missing" adjoint can be obtained that way. This idea was worked out in [2] for a general applicative retraction $I \dashv D$ between PER models. The PER models were embedded into suitable toposes of sheaves over PCAs. The adjunction $I \dashv D$ then extends to an adjunction at the level of toposes, with a further right

[4] Note that Equ and $\mathsf{0Equ}$ are only *countably* complete and cocomplete so that we cannot directly apply the Adjoint Functor Theorem.

adjoint. This makes it possible to apply the logical transfer principle from [3] to show that a certain class of first-order sentences is valid in the internal logic of Equ if, and only if, it is valid in the internal logic of $\mathsf{Rep}(\mathbb{B})$.

The next question to ask is whether I and D preserve any exponentials.

Theorem 4.5

(i) *Functor D restricted to EPQ_0 preserves exponentials.*

(ii) *If $X, Y \in \mathsf{0Equ}$ and there exists in $\omega\mathsf{Top}_0$ a 0-dimensional weak exponential of $|X|$ and $|Y|$, then I preserves the exponential Y^X.*

(iii) *Functor I preserves the natural numbers object $\mathbb{N}$, the exponentials $\mathbb{N}^\mathbb{N}$ and $2^\mathbb{N}$, and the object $\mathbb{R}_c$ of Cauchy reals.*

(iv) *Functor I does* not *preserve exponentials in general. In particular, it does not preserve $\mathbb{N}^{\mathbb{N}^\mathbb{N}}$.*

Proof. (i) This follows from results obtained in Section 5, and so we postpone the proof until then. It can be found on page 16.

(ii) If $W \in \mathsf{0Dim}$ is a weak exponential of X and Y in $\omega\mathsf{Top}_0$, then it is also a weak exponential of X and Y in $\mathsf{0Dim}$. Therefore, the construction of Y^X from W in Equ, as described in Section 2 coincides with the one in $\mathsf{0Equ}$.

(iii) The Baire space $\mathbb{N}^\mathbb{N}$ and the Cantor space $2^\mathbb{N}$ both satisfy the condition from (ii). The real numbers object $\mathbb{R}_c$ is a regular quotient of $\mathbb{N} \times 2^\mathbb{N}$ [4, Proposition 5.5.3], and the left adjoint I preserves it because it preserves $\mathbb{N}$, $2^\mathbb{N}$, products, and coequalizers.

(iv) Let $X = \mathbb{N}^{\mathbb{N}^\mathbb{N}}$ in $\mathsf{0Equ}$, and let $Y = \mathbb{N}^{\mathbb{N}^\mathbb{N}}$ in Equ. The space $|X|$ is a Hausdorff space. The space $|Y|$ is the subspace of the total elements of the Scott domain $D_Y = [{\mathbb{N}_\perp}^\omega \to \mathbb{N}_\perp]$. The equivalence relation on $|Y|$ is the consistency relation of D_Y restricted to $|Y|$. Suppose $f \colon |Y| \to |X|$ represented an isomorphism, and let $g \colon |X| \to |Y|$ represent its inverse. Because f is monotone in the specialization order and $|X|$ has a trivial specialization order, $a \equiv_Y b$ implies $fx = fy$. Therefore, $g \circ f \colon |Y| \to |Y|$ is an equivariant retraction. By [4, Proposition 4.1.8], Y is a topological object. By [4, Corollary 4.1.9], this would mean that the topological quotient $\|Y\|$ is countably based, but it is not, as is well known. Another way to see that Y cannot be topological is to observe that Y is an exponential of the Baire space, but the Baire space is not exponentiable in $\omega\mathsf{Top}_0$, and in particular $\mathbb{N}^{\mathbb{N}^\mathbb{N}}$ is not a topological object in Equ. □

Remark 4.6 In [2] we used a logical transfer principle between Equ and $\mathsf{Rep}(\mathbb{B})$ to prove that I does not preserve ${\mathbb{R}_c}^{\mathbb{R}_c}$ either.

As already mentioned in the introduction, we could obtain the results of this section by applying Longley's theory of applicative adjunctions between applicative morphisms of partial combinatory algebras [17]. Lietz [16] used this approach to compare the realizability toposes $\mathsf{RT}(\mathcal{P}\mathbb{N})$ and $\mathsf{RT}(\mathbb{B})$.

5 A Common Subcategory of Equ and $\mathsf{Rep}(\mathbb{B})$

In Sections 2 and 3 we saw that sequential spaces contain cartesian closed subcategories PQ_0 and AdmSeq which are also cartesian closed subcategories of Equ and $\mathsf{Rep}(\mathbb{B})$, respectively. In this section we prove that PQ_0 and AdmSeq are the same category.

Lemma 5.1 *Suppose $\mathcal{B} = \{B_i \mid i \in \mathbb{N}\}$ is a countable pseudobase for a countably based T_0-space Y. Let X be a first-countable space and $f \colon X \to Y$ a continuous map. For every $x \in X$ and every neighborhood V of fx there exists a neighborhood U of x and $i \in \mathbb{N}$ such that $fx \in f(U) \subseteq B_i \subseteq V$.*

Proof. Note that the elements of the pseudobase do not have to be open sets, so this is not just a trivial consequence of continuity of f. We prove the lemma by contradiction. Suppose there were $x \in X$ and a neighborhood V of fx such that for every neighborhood U of x and for every $i \in \mathbb{N}$, if $B_i \subseteq V$ then $f_*(U) \not\subseteq B_i$. Let $U_0 \supseteq U_1 \supseteq \cdots$ be a descending countable neighborhood system for x. Let $p \colon \mathbb{N} \to \mathbb{N}$ be a surjective map that attains each value infinitely often, that is for all $k, j \in \mathbb{N}$ there exists $i \geq k$ such that $pi = j$. For every $i \in \mathbb{N}$, if $B_{pi} \subseteq V$ then $f_*(U_i) \not\subseteq B_{pi}$. Therefore, for every $i \in \mathbb{N}$ there exists $x_i \in U_i$ such that if $B_{pi} \subseteq V$ then $fx_i \notin B_{pi}$. The sequence $\langle x_n \rangle_{n \in \mathbb{N}}$ converges to x, hence $\langle fx_n \rangle_{n \in \mathbb{N}}$ converges to fx. Because $\mathcal{B}$ is a pseudobase there exists $j \in \mathbb{N}$ such that $B_j \subseteq V$ and $\langle fx_n \rangle_{n \in \mathbb{N}}$ is eventually in B_j, say from the k-th term onwards. There exists $i \geq k$ such that $pi = j$. Now we get $fx_i \in B_{pi} \subseteq V$, which is a contradiction. □

Theorem 5.2 PQ_0 *and* AdmSeq *are the same category.*

Proof. It was independently observed by Schröder that PQ_0 is a full subcategory of AdmSeq, which is the easier of the two inclusions. The proof goes as follows. Suppose $q \colon X \to Y$ is an ω-projecting quotient map. We need to show that Y is a sequential space with an admissible representation. It is sequential because it is a quotient of a sequential space. There exists an admissible representation $\delta_X \colon \mathbb{B} \rightharpoonup X$. Let $\delta_Y = q \circ \delta_X$. Suppose $f \colon \mathbb{B} \rightharpoonup Y$ is a continuous partial map. Because q is ω-projecting f lifts though X, and because δ_X is an admissible representation, it further lifts through $\mathbb{B}$.

It remains to prove the converse, namely that if a sequential T_0-space X has an admissible representation then there exists an ω-projecting quotient $q \colon Y \to X$. Since X has an admissible representation it has a countable pseudobase $\mathcal{B} = \{B_i \mid i \in \mathbb{N}\}$, by Theorem 3.8. The powerset $\mathcal{P}\mathbb{N}$ ordered by inclusion is an algebraic lattice. We equip it with the Scott topology, which is generated by the subbasic open sets $\uparrow n = \{a \in \mathcal{P}\mathbb{N} \mid n \in a\}$, $n \in \mathbb{N}$. Let $q \colon \mathcal{P}\mathbb{N} \rightharpoonup X$ be a partial map defined by

$$qa = x \iff (\forall\, n \in a \,.\, x \in B_n) \land \forall\, U \in \mathcal{O}(X) \,.\, (x \in U \Longrightarrow \exists\, n \in a \,.\, B_n \subseteq U) \ .$$

The map q is well defined because $qa = x$ and $qa = y$ implies that x and y share the same neighborhoods, so they are the same point of the T_0-space X. Furthermore, q is surjective because $\mathcal{B}$ is a pseudobase. To see that p is continuous, suppose $pa = x$ and $x \in U \in \mathcal{O}(X)$. There exists $n \in \mathbb{N}$ such that $x \in B_n \subseteq U$. If $n \in b \in \mathsf{dom}(p)$ then $pb \in B_n \subseteq U$. Therefore, $a \in {\uparrow}n$ and $p_*({\uparrow}n) \subseteq B_n \subseteq U$, which means that p is continuous. Let $Y = \mathsf{dom}(p)$.

Let us show that $q \colon Y \to X$ is ω-projecting. Suppose $f \colon Z \to X$ is a continuous map and $Z \in \omega\mathsf{Top}_0$. Define a map $g \colon Z \to \mathcal{P}\mathbb{N}$ by

$$gz = \left\{ n \in \mathbb{N} \mid \exists\, U \in \mathcal{O}(Z)\,.\,(z \in U \wedge f_*(U) \subseteq B_n) \right\} \ .$$

The map g is continuous almost by definition. Indeed, if $gz \in {\uparrow}n$ then there exists a neighborhood U of z such that $f_*(U) \subseteq B_n$, but then $g_*(U) \in {\uparrow}n$. To finish the proof we need to show that $fz = p(gz)$ for all $z \in Z$. If $n \in gz$ then $fz \in B_n$ because there exists $U \in \mathcal{O}(Z)$ such that $z \in U$ and $f_*(U) \subseteq B_n$. If $fz \in V \in \mathcal{O}(X)$ then by Lemma 5.1 there exists $U \in \mathcal{O}(Z)$ and $n \in \mathbb{N}$ such that $z \in U$ and $f_*(U) \subseteq B_n \subseteq U$. Hence, $n \in gz$. This proves that $fz = p(gz)$. □

Remark 5.3 Matthias Schröder has showed recently that if a sequential T_0-space X arises as a topological quotient of a subspace of $\mathbb{B}$, then X has an admissible representation. This result implies Theorem 5.2, and also gives a very nice characterization of EPQ_0: it is precisely the category of all T_0-spaces that are topological quotients of countably based T_0-spaces.

The relationships between the categories are summarized by the following diagram:

$$\begin{tikzcd}
 & \mathsf{Seq} & \mathsf{Equ} \simeq \mathsf{PER}(\mathcal{P}\mathbb{N}) \arrow[dd, shift left, "D"] \arrow[dd, phantom, "\dashv"] \\
\omega\mathsf{Top}_0 \arrow[r] & \mathsf{PQ}_0 = \mathsf{AdmSeq} \arrow[u] \arrow[ur] \arrow[dr] & \\
 & & \mathsf{0Equ} \simeq \mathsf{Rep}(\mathbb{B}) \simeq \mathsf{PER}(\mathbb{B}) \arrow[uu, shift left, "I"]
\end{tikzcd} \tag{1}$$

The unlabeled arrows are full and faithful inclusions, preserve countable limits, and countable coproducts. The inclusion $\omega\mathsf{Top}_0 \to \mathsf{PQ}_0$ preserves all exponentials that happen to exist in $\omega\mathsf{Top}_0$, and the other three unlabeled inclusions preserve cartesian closed structure. The right-hand triangle involving the two inclusions and the coreflection D commutes up to natural isomorphism (and the one involving the inclusion I does not).

We still owe the proof of Theorem 4.5(i), namely, that D restricted to EPQ_0 preserves exponentials. But this is now obvious, since the right-hand triangle involving D commutes.

6 Transfer Results between Equ and $\mathsf{Rep}(\mathbb{B})$

The correspondence (1) explains why domain-theoretic computational models agree so well with computational models studied by TTE—as long as we only build spaces by taking products, coproducts, exponentials, and regular subspaces, starting from countably based T_0-spaces, we remain in PQ_0, the common cartesian closed core of equilogical spaces and TTE.

As a first example of a transfer result, we translate a characterization of Kleene-Kreisel countable functionals [12] from Equ to $\mathsf{Rep}(\mathbb{B})$. In [6] we proved that the iterated exponentials $\mathbb{N}$, $\mathbb{N}^{\mathbb{N}}$, $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$, ... of the natural numbers object $\mathbb{N}$ in Equ are precisely the Kleene-Kreisel countable functionals. Because $\mathbb{N}$ is the natural numbers object in $\mathsf{Rep}(\mathbb{B})$ as well, and it belongs to PQ_0, the same hierarchy appears in $\mathsf{Rep}(\mathbb{B})$.

Proposition 6.1 *In $\mathsf{Rep}(\mathbb{B})$, the hierarchy of exponentials $\mathbb{N}$, $\mathbb{N}^{\mathbb{N}}$, $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$, ... , built from the natural numbers object $\mathbb{N}$, corresponds to the Kleene-Kreisel countable functionals.*

As a second example, we consider transfer between the *internal logics* of Equ and $\mathsf{Rep}(\mathbb{B})$. Because Equ and $\mathsf{Rep}(\mathbb{B})$ are equivalent to realizability models $\mathsf{PER}(\mathcal{P}\mathbb{N})$ and $\mathsf{PER}(\mathbb{B})$, respectively, they admit a realizability interpretation of first-order intuitionistic logic. This has been worked out in detail in [4]. It is often advantageous to work in the internal logic, because it lets us argue abstractly and conceptually about objects and morphisms. We never have to mention explicitly the realizers of morphisms or the underlying topological spaces, which makes arguments more perspicuous. Every map that can be defined in the internal logic is automatically realized (and computable, if we work with the computable versions of the realizability models).

Suppose we want to use internal logic to construct a particular map $f\colon X \to Y$ where $X, Y \in \mathsf{PQ}_0$. For example, we might want to define the definite integration operator $I\colon \mathbb{R}^{[0,1]} \to \mathbb{R}$,

$$If = \int_0^1 f(x)\,dx\ .$$

It may happen that X and Y are much more amenable to the internal logic of $\mathsf{Rep}(\mathbb{B})$ than to the internal logic of Equ, or vice versa. In such a case we can pick whichever internal logic is better and work in it, because if a map $f\colon X \to Y$ is definable in one internal logic, then it exists as a morphism in both Equ and $\mathsf{Rep}(\mathbb{B})$.

Let us see how this applies in the case of definite integration. The real numbers $\mathbb{R}$ are much better behaved in $\mathsf{Rep}(\mathbb{B})$ than in Equ, because $\mathbb{R}$ can be characterized in the internal logic of $\mathsf{Rep}(\mathbb{B})$ as *the Cauchy complete Archimedean field*, which gives us all the properties of $\mathbb{R}$ we could wish for. On the other hand, in the internal logic of Equ, $\mathbb{R}$ does not seem to be characterizable at all, and it does not even satisfy the Archimedean axiom

$$\forall x \in \mathbb{R} . \exists n \in \mathbb{N} . x < n ,$$

because in Equ there is no *continuous* choice map $c \colon \mathbb{R} \to \mathbb{N}$ that would satisfy $x < cx$ for all $x \in \mathbb{R}$.[5] This makes it impractical to argue about $\mathbb{R}$ in the internal logic of Equ. The situation with the space $\mathbb{R}^{[0,1]}$ of continuous real function on the unit interval is similar—it is much better behaved in the internal logic of $\mathsf{Rep}(\mathbb{B})$ than in the internal logic of Equ. In particular, in $\mathsf{Rep}(\mathbb{B})$ the statement "every map $f \colon [0,1] \to \mathbb{R}$ is uniformly continuous" is valid, whereas it is not valid in the internal logic of Equ. This makes it clear that the internal logic of $\mathsf{Rep}(\mathbb{B})$ is the better choice. Indeed, in the internal logic of $\mathsf{Rep}(\mathbb{B})$ definite integral may be defined in the usual way as a limit of Riemann sums. The convergence of Riemann sums can then be proved constructively because $\mathsf{Rep}(\mathbb{B})$ "believes" that all maps from $[0,1]$ to $\mathbb{R}$ are uniformly continuous. Once we have constructed the definite integral operator $I \colon \mathbb{R}^{[0,1]} \to \mathbb{R}$ in $\mathsf{Rep}(\mathbb{B})$, we can transfer it to Equ via PQ_0.

7 Conclusion

Let me conclude by commenting on the following comparison of domain theory and TTE from Weihrauch's recently published book on computable analysis [27, Section 9.8, p. 267]:

> "The domain approach developed so far is consistent with TTE. Roughly speaking, a domain (for the real numbers) contains approximate objects as well as precise objects which are treated in separate sets in TTE. A computable domain function must map also all approximate objects reasonably. In many cases, constructing a domain which corresponds to given representation still is a difficult task. Concepts for handling multi-valued functions and for computational complexity have not yet been developed for the domain approach. The elegant handling of higher type functions in domain theory can be simulated in TTE by means of function space representations $[\delta \to \delta']$ (Definition 3.3.13). To date, there seems to be no convincing reason to learn domain theory as a prerequisite for computable analysis."

The present paper provides a precise mathematical comparison of TTE and the domain approach, as exemplified by equilogical spaces. The correspondence (1) gives us a clear picture about the relationships between the domain approach and TTE. Overall, it supports the claim that these two approaches are consistent, at least as far as computability on PQ_0 is concerned.

Indeed, domains are built from the approximate as well as the precise objects, and I join Weihrauch in pointing out that it is a good idea to distin-

[5] The Archimedean axiom is valid in $\mathsf{Rep}(\mathbb{B})$ because there is a continuous choice map $|D\mathbb{R}| \to \mathbb{N}$ such that $[a] < ca$ for all $a \in |D\mathbb{R}|$, where $[a]$ the real number represented by the realizer a. The point is that ca may depend on the realizer a.

guish the precise objects from the approximate ones. In domain theory this is most easily done by taking seriously domains *with totality*, or more generally PERs on domains, which leads to the notion of equilogical spaces and domain representations, which were studied by Blanck [10].

I hope that the adjoint functors I and D between Equ and $\mathsf{Rep}(\mathbb{B})$ will ease the task of constructing a domain which corresponds to a given representation.

Power-domains are the domain-theoretic models of non-deterministic computation, and I believe they could be used to model multi-valued functions.

In this paper we did not consider the computational complexity or even computability in Equ and $\mathsf{Rep}(\mathbb{B})$. In [4] the inclusion $\mathsf{Rep}(\mathbb{B}) \to \mathsf{Equ}$ and its coreflection are constructed for the computable versions of equilogical spaces and TTE, from which we may conclude that computability in domain theory is essentially the same as in TTE.

By Theorem 4.5, the higher type function spaces in equilogical spaces do not generally agree with the corresponding function space representations in TTE. However, the two approaches to higher types do agree on an important class of spaces, namely the category PQ_0, which contains all countably based T_0-spaces, therefore also all countably based continuous and algebraic domains. Higher types seem not to catch a lot of interest in the TTE community. This may be because the descriptions of higher types in terms of representations can get quite unwieldy and are hard to work with. The theory of cartesian closed categories and the internal logic of $\mathsf{Rep}(\mathbb{B})$ ought to be helpful here, as they allows us to talk about the higher types abstractly, without having to refer to their representations all the time. After all, higher types cannot be ignored in computable analysis: real numbers are a quotient of type 1, integration and differentiation operators have type 2, solving a differential equation is a type 3 process, and still higher types are reached when we study spaces of distributions and operators on Hilbert spaces.

Finally, is there a convincing reason to learn domain theory as a prerequisite for computable analysis? By Theorem 4.2, $\mathsf{Rep}(\mathbb{B})$ is a full subcategory of Equ. This may suggest the view that the domain approach is more general than TTE. At any rate, they are *not* competing approaches. They fit with each other very well, and each has its advantages: domain theory handles higher types more elegantly and is more general than TTE, whereas TTE provides a more convenient internal logic and handles questions about computational complexity better. So why not learn both, and a bit of category theory, realizability, and constructive logic on top?

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