J. Symbolic Computation (1998) **11**, 1–000

Multibasic and Mixed Hypergeometric Gosper-Type Algorithms

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(Received 17 September 1997)

Gosper's summation algorithm finds a hypergeometric closed form of an indefinite sum of hypergeometric terms, if such a closed form exists. We extend his algorithm to the case when the terms are simultaneously hypergeometric and multibasic hypergeometric. We also provide algorithms for finding polynomial as well as hypergeometric solutions of recurrences in the mixed case. We do not require the bases to be transcendental, but only that $q_1^{k_1} \cdots q_m^{k_m} \neq 1$ unless $k_1 = \cdots = k_m = 0$. Finally, we generalize the concept of greatest factorial factorization to the mixed hypergeometric case.

1. Introduction and notation

Let $\mathbb{F}$ be a field of characteristic zero and $\langle t_n \rangle_{n=0}^{\infty}$ a sequence of elements from $\mathbb{F}$ which is eventually non-zero. Call t_n:

1. *hypergeometric*, if there are polynomials $p_1, p_2 \in \mathbb{F}[x]$ such that $p_1(n)t_{n+1} = p_2(n)t_n$ for all n;
2. *q-hypergeometric* or *basic hypergeometric*, if there are polynomials $p_1, p_2 \in \mathbb{F}[x]$ such that $p_1(q^n)t_{n+1} = p_2(q^n)t_n$ for all n, where $q \in \mathbb{F} \setminus \{0\}$ is the *base*;
3. *multibasic hypergeometric*, if there are polynomials $p_1, p_2 \in \mathbb{F}[y_1, \ldots, y_m]$ such that $p_1(q_1^n, \ldots, q_m^n)t_{n+1} = p_2(q_1^n, \ldots, q_m^n)t_n$ for all n, where $q_1, \ldots, q_m \in \mathbb{F} \setminus \{0\}$ are the *bases*;
4. *mixed hypergeometric*, if there are polynomials $p_1, p_2 \in \mathbb{F}[x, y_1, \ldots, y_m]$ such that $p_1(n, q_1^n, \ldots, q_m^n)t_{n+1} = p_2(n, q_1^n, \ldots, q_m^n)t_n$ for all n.

The well-known *Gosper's algorithm* (Gosper, 1977; Gosper, 1978) finds hypergeometric solutions f_n of the nonhomogeneous first-order recurrence

$$f_{n+1} - f_n = t_n$$

where t_n is a given hypergeometric sequence. Besides its obvious use for indefinite hypergeometric summation, it also plays a crucial role in the algorithms for definite hypergeometric summation, construction of annihilating recurrences, and automated verification of identities (Zeilberger, 1990a; Zeilberger, 1991; Wilf and Zeilberger, 1992). Therefore it is not surprising that analogous algorithms have been designed for many other settings,

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e.g., integration of hyperexponential functions (Almkvist and Zeilberger, 1990), basic (Zeilberger, 1990b; Koornwinder, 1993; Paule and Riese, 1997) and bibasic (Riese, 1996) hypergeometric summation. We generalize Gosper's algorithm, as well as some related ones, to the mixed hypergeometric case.

The algebraic setting of the paper (with the exception of Section 8) is the rational-function field $\mathbb{F}(x, \mathbf{y})$ where $\mathbb{F}$ is an arbitrary field of characteristic zero, together with an $\mathbb{F}$-automorphism $\mathbf{E}$ which acts by $\mathbf{E}x = x + 1$ and $\mathbf{E}y_i = q_i y_i$. This is discussed in detail in Section 2. Some auxiliary algorithms used later as subroutines are sketched in Section 3, while in Sections 4 and 5 the necessary ingredients for Gosper's algorithm are developed. Although there only first-order recurrences are checked for polynomial solutions, we provide in Section 4 algorithm `MixedPoly`† which finds all polynomial solutions of a parametric nonhomogeneous polynomial-coefficient recurrence of any order. A mixed hypergeometric canonical form of rational functions is described in Section 5. After these preparations, we present in Section 6 an analogue of Gosper's algorithm for the mixed hypergeometric case. Our algorithm `MixedGosper`‡ is a common generalization of the algorithms presented in (Gosper, 1978; Zeilberger, 1990b; Riese, 1996). When specialized to the bibasic case, it essentially agrees with the algorithm given by Riese (1996). However, looking at his case analysis in the computation of multiplicities γ and δ (Riese, 1996, pp. 7–8), it is not immediately clear how to extend that to the multibasic case. In Section 7 we provide algorithm `MixedHyper` which finds all mixed hypergeometric solutions of a homogeneous polynomial-coefficient recurrence of any order. This is a common generalization of the algorithms presented in (Petkovšek, 1992) and (Abramov *et al.*, 1998). In Section 8 we extend the concept of greatest factorial factorization due to Paule (1995) to an arbitrary automorphism σ of the multivariate polynomial ring.

Notation. The set of integers is denoted by $\mathbb{Z}$, the set of nonnegative integers by $\mathbb{N}_0$, and the field of rational numbers by $\mathbb{Q}$.

If $n, m \in \mathbb{N}_0$ and $\mathbf{a} = (a_1, a_2, \ldots, a_m)$, $\mathbf{b} = (b_1, b_2, \ldots, b_m)$ are m-tuples of elements of a ring, we write $\mathbf{ab}$ for the componentwise product $(a_1 b_1, a_2 b_2, \ldots, a_m b_m)$, and $\mathbf{a}^n$ for the componentwise power $(a_1^n, a_2^n, \ldots, a_m^n)$. If $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_m) \in \mathbb{N}_0^m$ then we write $\mathbf{a}^{\boldsymbol{\alpha}}$ for the power product $a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_m^{\alpha_m}$.

We say that two multivariate polynomials over a field are *coprime* if they do not have a non-constant common factor. When a and b are coprime, we write $a \perp b$. When $\mathcal{S}$ is a set of polynomials and $a \perp b$ for all $b \in \mathcal{S}$, we write $a \perp \mathcal{S}$.

2. Algebraic preliminaries

Let $\mathbb{F}$ be a field of characteristic zero. Let $q_1, \ldots, q_m \in \mathbb{F} \setminus \{0\}$, and suppose that for any integers $k_1, \ldots, k_m \in \mathbb{Z}$,

$$q_1^{k_1} q_2^{k_2} \cdots q_m^{k_m} = 1 \implies k_1 = k_2 = \cdots = k_m = 0. \tag{2.1}$$

This generalizes the condition that q is not a root of unity in the q-hypergeometric case (Abramov *et al.*, 1998). For example, if $\mathbb{F} = \mathbb{R}$, $q_1 = \sqrt[3]{2}$ and $q_2 = \sqrt[5]{2}$, then $q_1^3 q_2^{-5} = 1$

† available at `http://www.cis.upenn.edu/~wilf/AeqB.html` in the *Mathematica* package `gosper.m` as `MixedPoly`

‡ available at `http://www.cis.upenn.edu/~wilf/AeqB.html` in the *Mathematica* package `gosper.m` as `GosperSum`

and we should have chosen $q = \sqrt[15]{2}$ in the first place. On the other hand, $q_1 = 2$ and $q_2 = 3$ would be a legitimate choice in this case. We call q_i's the *bases*, and write $\mathbf{q} = (q_1, q_2, \ldots, q_m)$.

Let $\mathbf{y} = (y_1, y_2, \ldots, y_m)$ be an m-tuple of variables, $\mathbb{F}[x, \mathbf{y}]$ the ring of polynomials over $\mathbb{F}$ in x and $\mathbf{y}$, and $\mathbb{F}(x, \mathbf{y})$ the corresponding rational function field. We define an $\mathbb{F}$-automorphism $\mathbf{E}$ of $\mathbb{F}(x, \mathbf{y})$ (i.e., $\mathbf{E}$ is a field automorphism of $\mathbb{F}(x, \mathbf{y})$ which fixes each element of $\mathbb{F} \subseteq \mathbb{F}(x, \mathbf{y})$) by stipulating further that $\mathbf{E}x = x + 1$ and $\mathbf{E}y_k = q_k y_k$ for $k = 1, \ldots, m$. Then $\mathbb{F}(x, \mathbf{y})$ together with $\mathbf{E}$ is a difference field and $\mathbb{F}[x, \mathbf{y}]$ is a difference subring of $\mathbb{F}(x, \mathbf{y})$ (see (Cohn, 1965) for the relevant definitions).

Let $\mathcal{M}$ be the set of power products in $y_1, y_2, \ldots, y_m$:

$$\mathcal{M} = \{y_1^{k_1} y_2^{k_2} \cdots y_m^{k_m} \mid k_i \in \mathbb{N}_0 \text{ for } i = 1, \ldots, m\}.$$

If $u = y_1^{k_1} y_2^{k_2} \cdots y_m^{k_m} \in \mathcal{M}$, we write $u(\mathbf{q})$ for the corresponding power product of the bases $q_1^{k_1} q_2^{k_2} \cdots q_m^{k_m}$. Note that $\mathbf{E}u = u(\mathbf{q})u$ for all $u \in \mathcal{M}$.

As a multiplicative monoid, $\mathcal{M}$ is obviously isomorphic to $\mathbb{N}_0^m$, the direct product of m copies of the additive monoid $\mathbb{N}_0$. We denote by $\preceq$ an *admissible term order* in $\mathbb{N}_0^m$, which is a total order satisfying

1. $\mathbf{0} \preceq \boldsymbol{\alpha}$,
2. $\boldsymbol{\alpha} \preceq \boldsymbol{\beta} \;\Rightarrow\; \boldsymbol{\alpha} + \boldsymbol{\gamma} \preceq \boldsymbol{\beta} + \boldsymbol{\gamma}$

for all $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \in \mathbb{N}_0^m$. An example of an admissible term order is the *lexicographic order* $\preceq_{\text{lex}}$, with $\boldsymbol{\alpha} \prec_{\text{lex}} \boldsymbol{\beta}$ when $\boldsymbol{\alpha} \neq \boldsymbol{\beta}$ and $\alpha_k < \beta_k$ where $k = \min\{i;\ \alpha_i \neq \beta_i\}$.

DEFINITION 2.1. *Let $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}_0^m$. Then we write*

$$\boldsymbol{\alpha} \subseteq \boldsymbol{\beta}$$

whenever $\alpha_i \leq \beta_i$ for all i between 1 and m.

Clearly, $(\mathbb{N}_0^m, \subseteq)$ is a partial order isomorphic to $(\mathcal{M}, |)$ where $|$ denotes divisibility of power products, and is contained in any admissible term order:

$$\boldsymbol{\alpha} \subseteq \boldsymbol{\beta} \;\Rightarrow\; \boldsymbol{\alpha} \preceq \boldsymbol{\beta}, \quad \text{for all } \boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}_0^m.$$

We adjoin to $\mathbb{N}_0^m$ an absorbing bottom element†, $\perp$, such that for all $\boldsymbol{\alpha} \in \mathbb{N}_0^m$

$$\perp \prec \boldsymbol{\alpha},$$
$$\perp + \boldsymbol{\alpha} = \boldsymbol{\alpha} + \perp = \perp.$$

DEFINITION 2.2. *Let $p \in \mathbb{F}[x, \mathbf{y}]$. Write*

$$p(x, \mathbf{y}) = \sum_{\boldsymbol{\alpha} \in \mathbb{N}_0^m} p_{\boldsymbol{\alpha}}(x) \mathbf{y}^{\boldsymbol{\alpha}} = \sum_{i \in \mathbb{N}_0} c_i(\mathbf{y}) x^i \tag{2.2}$$

where only finitely many $p_{\boldsymbol{\alpha}} \in \mathbb{F}[x]$ and $c_i \in \mathbb{F}[\mathbf{y}]$ are non-zero.

1. *We define the* multidegree *of p in $\mathbf{y}$ as*

$$\deg_{\mathbf{y}} p(x, \mathbf{y}) = \begin{cases} \max_{\preceq}\{\boldsymbol{\alpha} \in \mathbb{N}_0^m;\ p_{\boldsymbol{\alpha}} \neq 0\}, & p \neq 0, \\ \perp, & p = 0. \end{cases}$$

† not to be confused with the notation for coprime polynomials introduced at the end of Section 1

2. *Similarly,*

$$\operatorname{mindeg}_{\mathbf{y}} p(x, \mathbf{y}) = \begin{cases} \min_{\preceq}\{\boldsymbol{\alpha} \in \mathbb{N}_0^m;\ p_{\boldsymbol{\alpha}} \neq 0\}, & p \neq 0, \\ \perp, & p = 0. \end{cases}$$

3. *We write* $[\mathbf{y}^{\boldsymbol{\alpha}}]\,p(x, \mathbf{y})$ *for* $p_{\boldsymbol{\alpha}}(x)$ *and* $[x^i]\,p(x, \mathbf{y})$ *for* $c_i(\mathbf{y})$ *in (2.2).*
4. *When* $\deg_{\mathbf{y}} p \prec \boldsymbol{\alpha}$ *we write* $p = o(\mathbf{y}^{\boldsymbol{\alpha}})$.
5. *Let* $\boldsymbol{\delta} = deg_{\mathbf{y}} p$. *We call* p mixed monic *when* $[\mathbf{y}^{\boldsymbol{\delta}}]\,p(x, \mathbf{y})$ *is monic as a univariate polynomial in* x.

Note that the concepts of multidegree and mixed monicity are relative to the chosen term order $\preceq$. By convention, $\gcd(a, b)$ always denotes a mixed monic greatest common divisor of $a, b \in \mathbb{F}[x, \mathbf{y}]$.

We need the following well-known result from the theory of linear recurrent sequences.

LEMMA 2.1. *Let* $\mathbb{F}$ *be a field of characteristic zero and* $r_1, \ldots, r_k \in \mathbb{F} \setminus \{0\}$, *with* $r_i \neq r_j$ *for* $i \neq j$. *Let* $d_1, d_2, \ldots, d_k \in \mathbb{N}_0$ *and* $d = d_1 + d_2 + \cdots + d_k$. *Then the* d *functions* $g_{ij}\colon \mathbb{N}_0 \to \mathbb{F}$, *defined by* $g_{ij}(n) = n^j r_i^n$, *for* $i = 1, \ldots, k$, $j = 0, \ldots, d_i - 1$, *are linearly independent in the vector space* $\mathbb{N}_0 \to \mathbb{F}$ *over* $\mathbb{F}$.

For a proof, see, e.g., (Stanley, 1986, Thm. 4.1.1).

The main object of our interest is the ring of sequences $\mathbb{N}_0 \to \mathbb{F}$. To simplify notation, we denote the sequence $\langle 0, 1, 2, \ldots \rangle$ by n, and $\langle 1, q_i, q_i^2, \ldots \rangle$ by q_i^n, for $i = 1, 2, \ldots, m$. We write $\mathbb{F}[n, \mathbf{q}^n]$ for the subring of $\mathbb{N}_0 \to \mathbb{F}$ generated by n, $q_1^n, \ldots, q_m^n$ and the constant sequences, and call its elements the *polynomial sequences*. This is justified by the following theorem.

THEOREM 2.1. *Let* $\Phi\colon\ \mathbb{F}[x, \mathbf{y}] \to \mathbb{F}[n, \mathbf{q}^n]$ *be the ring homomorphism mapping* $x \mapsto n$ *and* $y_i \mapsto q_i^n$. *Then* Φ *is an isomorphism between the ring of polynomials* $\mathbb{F}[x, \mathbf{y}]$ *and the ring of polynomial sequences* $\mathbb{F}[n, \mathbf{q}^n]$.

PROOF. It is obvious that Φ is an epimorphism. We show that it is a monomorphism. Let $f \in \mathbb{F}[x, \mathbf{y}]$. Write f as

$$f = \sum_{i=1}^{k} p_i u_i,$$

where $p_1, \ldots, p_k \in \mathbb{F}[x]$, $u_1, \ldots, u_k \in \mathcal{M}$, and $u_i \neq u_j$ for $i \neq j$. Suppose $\Phi f = 0$:

$$0 = \Phi f = \sum_{i=1}^{k} p_i(n) u_i(\mathbf{q})^n.$$

Because $q_1, \ldots, q_m$ satisfy condition (2.1), $u_i(\mathbf{q}) \neq u_j(\mathbf{q})$ for $i \neq j$. The result now follows from Lemma 2.1. □

As a consequence, $\mathbb{F}[n, \mathbf{q}^n]$ is an integral domain, and its field of fractions $\mathbb{F}(n, \mathbf{q}^n)$ (whose elements we call the *rational sequences*) is isomorphic to the rational function field $\mathbb{F}(x, \mathbf{y})$. The map $\Phi\colon\ \mathbb{F}[x, \mathbf{y}] \to \mathbb{F}[n, \mathbf{q}^n]$ defined in Theorem 2.1 can be naturally extended to a map from $\mathbb{F}(x, \mathbf{y})$ to $\mathbb{F}(n, \mathbf{q}^n)$.

We define a homomorphism $\mathbf{S}$ on $\mathbb{N}_0 \to \mathbb{F}$ by setting $(\mathbf{S}a)(n) = a(n+1)$ for all

$a\colon \mathbb{N}_0 \to \mathbb{F}$. This makes $\mathbb{F}(n, \mathbf{q}^n)$ into a difference field and $\mathbb{F}[n, \mathbf{q}^n]$ into a difference subring of $\mathbb{F}(n, \mathbf{q}^n)$. As $\Phi \circ \mathbf{E} = \mathbf{S} \circ \Phi$, we see that Φ extended to $\mathbb{F}(x, \mathbf{y})$ is a difference isomorphism of the two fields $\mathbb{F}(x, \mathbf{y})$ and $\mathbb{F}(n, \mathbf{q}^n)$, as well as of the two rings $\mathbb{F}[x, \mathbf{y}]$ and $\mathbb{F}[n, \mathbf{q}^n]$. This allows us to work in $\mathbb{F}(x, \mathbf{y})$ resp. $\mathbb{F}[x, \mathbf{y}]$ instead of in $\mathbb{F}(n, \mathbf{q}^n)$ resp. $\mathbb{F}[n, \mathbf{q}^n]$ whenever suitable.

We conclude this section by two simple technical lemmas.

LEMMA 2.2. *Let $p \in \mathbb{F}[x, \mathbf{y}] \setminus \{0\}$, $k \in \mathbb{Z} \setminus \{0\}$, and $a \in \mathbb{F}$. Then $\mathbf{E}^k p = ap$ if and only if $p = ru$ for some $r \in \mathbb{F}$, $u \in \mathcal{M}$, and $a = u(\mathbf{q})^k$.*

PROOF. Sufficiency is obvious. Suppose $\mathbf{E}^k p = ap$. Write p as

$$p(x, \mathbf{y}) = \sum_{i=1}^{n} p_i(x)\, u_i(\mathbf{y}),$$

where $u_1, \ldots, u_n \in \mathcal{M}$ are pairwise different and $p_1, \ldots, p_n \in \mathbb{F}[x] \setminus \{0\}$. It follows that

$$\sum_{i=1}^{n} a\, p_i(x)\, u_i(\mathbf{y}) = ap = \mathbf{E}^k p = \sum_{i=1}^{n} p_i(x+k)\, u_i(\mathbf{q})^k\, u_i(\mathbf{y}).$$

Hence, for $i = 1, \ldots, n$

$$a\, p_i(x) = u_i(\mathbf{q})^k\, p_i(x+k).$$

By comparing the leading coefficients in the above equation, we conclude that $a = u_i(\mathbf{q})^k$ for $i = 1, \ldots, n$. However, if it were the case that $u_i(\mathbf{q})^k = u_j(\mathbf{q})^k$ for some $i \neq j$, condition (2.1) would be violated. It follows that $n = 1$, and $p(x, \mathbf{y}) = r(x)\, u(\mathbf{y})$ for some $r \in \mathbb{F}[x] \setminus \{0\}$ and $u \in \mathcal{M}$. From $\mathbf{E}^k p = ap$ we get $r(x+k) = r(x)$, which is only possible if r is a constant. $\square$

DEFINITION 2.3. *For $1 \leq i \leq m$, we denote by $\boldsymbol{\pi}_i$ the endomorphism of $\mathbb{F}[x, \mathbf{y}]$ which substitutes 0 for y_i.*

LEMMA 2.3. *The endomorphisms $\boldsymbol{\pi}_i$, $1 \leq i \leq m$, commute with $\mathbf{E}$ and $\mathbf{E}^{-1}$.*

PROOF. Let $\mathcal{Y}_i = \{y_1, y_2, \ldots, y_m\} \setminus \{y_i\}$ and $p \in \mathbb{F}[x, \mathbf{y}]$. Consider p to be a polynomial in $\mathbb{F}[x, \mathcal{Y}_i][y_i]$. It is easy to check that $\mathbf{E}\boldsymbol{\pi}_i p = \boldsymbol{\pi}_i \mathbf{E} p$ and $\mathbf{E}^{-1}\boldsymbol{\pi}_i p = \boldsymbol{\pi}_i \mathbf{E}^{-1} p$. $\square$

3. Algorithmic preliminaries

The main algorithmic subproblems that we encounter are the following:

1. `(disp)` For polynomials $a, b \in \mathbb{F}[x, \mathbf{y}] \setminus \{0\}$ such that $a, b \perp \mathcal{M}$, compute the *dispersion set*

$$D(a, b) = \{n \in \mathbb{N}_0;\ a \not\perp \mathbf{E}^n b\}$$

   containing all nonnegative integers n such that a and $\mathbf{E}^n b$ have a non-constant common divisor.
2. `(introot)` Find the set of all nonnegative integer roots n of $P(n, \mathbf{q}^n) = 0$ where $P \in \mathbb{F}[x, \mathbf{y}] \setminus \{0\}$.

3. (`qmon`) Given $a \in \mathbb{F} \setminus \{0\}$, find integers $k_1, \ldots, k_m$ (if any) such that $a = q_1^{k_1} \cdots q_m^{k_m}$. Note that by (2.1) such integers are unique.

In Section 3.1 we reduce `disp` to `introot`. In Sections 3.2 and 3.3 we show how to solve `introot` in two important special cases when $\mathbb{F} = \mathbb{Q}(q_1, \ldots, q_m)$ is a purely transcendental extension of $\mathbb{Q}$, and when $\mathbb{F} = \mathbb{Q}$ (and hence $q_1, \ldots, q_m \in \mathbb{Q}$), respectively. We do not elaborate on `qmon`, because in the two special cases of transcendental resp. rational bases it is rather obvious how to solve it.

3.1. Computing the dispersion set

Define polynomials $R_1, R_2, \ldots, R_m, R \in \mathbb{F}[x, \mathbf{y}][\xi, \boldsymbol{\eta}]$ as polynomial resultants

$$\begin{aligned} R_i(\xi, \boldsymbol{\eta}) &= \mathrm{Res}_{y_i}(a(x, \mathbf{y}), b(x + \xi, \boldsymbol{\eta}\mathbf{y})) \quad (1 \le i \le m), \\ R(\xi, \boldsymbol{\eta}) &= \mathrm{Res}_x(a(x, \mathbf{y}), b(x + \xi, \boldsymbol{\eta}\mathbf{y})). \end{aligned}$$

Here ξ is a variable and $\boldsymbol{\eta} = (\eta_1, \eta_2, \ldots, \eta_m)$ is an m-tuple of variables. Let

$$P(\xi, \boldsymbol{\eta}) = R(\xi, \boldsymbol{\eta}) \prod_{i=1}^{m} R_i(\xi, \boldsymbol{\eta}). \tag{3.1}$$

The following lemma leads to an algorithm for computing $D(a, b)$:

Lemma 3.1. $D(a, b) = \{n \in \mathbb{N}_0;\ P(n, \mathbf{q}^n) = 0\}$.

Proof. For $n \in \mathbb{N}_0$, let $\phi_n : \mathbb{F}[x, \mathbf{y}, \xi, \boldsymbol{\eta}] \to \mathbb{F}[x, \mathbf{y}]$ be the evaluation homomorphism which substitutes n for ξ and $\mathbf{q}^n$ for $\boldsymbol{\eta}$. It is easy to see that for any non-zero polynomial $p \in \mathbb{F}[x, \mathbf{y}]$, the homomorphic image $\phi_n(p(x+\xi, \boldsymbol{\eta}\mathbf{y})) = p(x+n, \mathbf{q}^n\mathbf{y}) = \mathbf{E}^n p(x, \mathbf{y})$ is non-zero. Therefore, by the Homomorphism Lemma for resultants (see, e.g., (Mishra, 1993, Lemma 7.3.1)),

$$\begin{aligned} R_i(n, \mathbf{q}^n) &= \phi_n(R_i(\xi, \boldsymbol{\eta})) = \mathrm{Res}_{y_i}(\phi_n(a(x, \mathbf{y})), \phi_n(b(x + \xi, \boldsymbol{\eta}\mathbf{y}))) = \mathrm{Res}_{y_i}(a, \mathbf{E}^n b) \\ &\qquad (1 \le i \le m), \\ R(n, \mathbf{q}^n) &= \phi_n(R(\xi, \boldsymbol{\eta})) = \mathrm{Res}_x(\phi_n(a(x, \mathbf{y})), \phi_n(b(x + \xi, \boldsymbol{\eta}\mathbf{y}))) = \mathrm{Res}_x(a, \mathbf{E}^n b). \end{aligned}$$

Thus we have the following chain of equivalences:

$$\begin{aligned} n \in D(a, b) &\iff \text{one of } \deg_x \gcd(a, \mathbf{E}^n b), \deg_{y_i} \gcd(a, \mathbf{E}^n b) \text{ is positive} \\ &\iff \text{one of } \mathrm{Res}_x(a, \mathbf{E}^n b), \mathrm{Res}_{y_i}(a, \mathbf{E}^n b) \text{ vanishes} \\ &\iff \text{one of } R(n, \mathbf{q}^n), R_i(n, \mathbf{q}^n) \text{ vanishes} \\ &\iff R(n, \mathbf{q}^n) \prod_{i=1}^{m} R_i(n, \mathbf{q}^n) = 0 \\ &\iff P(n, \mathbf{q}^n) = 0. \end{aligned} \tag{3.2}$$

The second equivalence above follows from the well-known properties of polynomial resultants. □

Next we show how to find integral solutions n of equation (3.2) in two special cases.

3.2. Transcendental bases

Let $\mathbb{F} = \mathbb{Q}(q_1, \ldots, q_m)$ where $q_1, \ldots, q_m$ are algebraically independent over $\mathbb{Q}$. Let $p \in \mathbb{F}[x, \mathbf{y}] \setminus \{0\}$. We look for $n \in \mathbb{N}_0$ such that

$$p(n, q_1^n, \ldots, q_m^n) = 0. \tag{3.3}$$

We present a recursive algorithm for finding an upper bound for n. Once the bound is known, all integers between zero and the bound can be checked.

In equation (3.3), the coefficients are elements of $\mathbb{F}$, which are rational functions of $q_1, \ldots, q_m$. We can clear the denominators and obtain an equation in which q_i occur polynomially:

$$r(n, q_1, \ldots, q_m, q_1^n, \ldots, q_m^n) = 0, \tag{3.4}$$

where $r \in \mathbb{Q}[x, z_1, \ldots, z_m, y_1, \ldots, y_m] \setminus \{0\}$. We show how to reduce recursively the problem of finding an upper bound for solutions of (3.4). Consider all terms of r with lowest degree of y_m, and let that degree be j. Among these terms, consider the one with the lowest degree of z_m, and let d be that degree. The term has the form $s z_m^d y_m^j$ for some $s \in \mathbb{Q}[x, z_1, \ldots, z_{m-1}, y_1, \ldots, y_{m-1}] \setminus \{0\}$. Let M be an upper bound on natural solutions of equation

$$s(n, q_1, \ldots, q_{m-1}, q_1^n, \ldots, q_{m-1}^n) = 0, \tag{3.5}$$

which we can get recursively. Then $\max(M, d)$ is an upper bound for solutions of (3.4). Suppose $n > \max(M, d)$. Then n is not a solution of (3.5), and the lowest power of q_m that occurs in (3.4) is $d + nj$. Since this power occurs only in the term $s(n) q_m^d q_m^{nj}$, the term does not cancel, and n is not a solution of (3.4).

The base case of the recursion is an equation $r(n) = 0$, where $r \in \mathbb{Q}[x] \setminus \{0\}$. This can be handled easily, since any natural solution of this equation must divide the constant term (after we have cleared the denominators).

3.3. Rational bases

Suppose $q_1, \ldots, q_m \in \mathbb{Q}$. Let $p \in \mathbb{Q}[x, \mathbf{y}] \setminus \{0\}$. Again we consider the problem of finding $n \in \mathbb{N}_0$ such that

$$p(n, q_1^n, \ldots, q_m^n) = 0. \tag{3.6}$$

Write p as

$$p = \sum_{i=1}^{k} p_i u_i,$$

where $p_1, \ldots, p_k \in \mathbb{Q}[x] \setminus \{0\}$, $u_1, \ldots, u_k \in \mathcal{M}$, and $u_i \neq u_j$ for $i \neq j$. Equation (3.6) can be written as

$$\sum_{i=1}^{k} p_i(n) u_i(\mathbf{q})^n = 0. \tag{3.7}$$

Because bases $q_1, \ldots, q_m$ satisfy condition (2.1), $|u_i(\mathbf{q})| \neq |u_j(\mathbf{q})|$ for $i \neq j$. Let $s_i = u_i(\mathbf{q})$ for $i = 1, \ldots, k$. We may assume that $|s_1| < |s_2| < \cdots < |s_k|$. Suppose $p_k(x) = a_d x^d +$

$a_{d-1}x^{d-1} + \cdots + a_0$. Equation (3.7) is equivalent to

$$a_d n^d s_k^n + \sum_{i=0}^{d-1} a_i n^i s_k^n + \sum_{i=1}^{k-1} p_i(n)\, s_i^n = 0. \tag{3.8}$$

The first term in (3.8) dominates the sum of the others. We only need a lower bound on n, such that the absolute value of the first term is greater than the absolute value of the sum of the other terms. Then we can check all integers between zero and the lower bound.

Let $\operatorname{dom}(a, b, k)$ be a function which gives an integer lower bound, such that for all $n \geq \operatorname{dom}(a, b, k)$ it is true that $a^n > bn^k$. Here $a > 1$, $b > 0$ and $k \in \mathbb{Z}$.

Let $\delta = 1/(d + k)$. For $i = 0, \ldots, d-1$, define

$$M_i = \left\lceil \left| \frac{a_i}{a_d \delta} \right|^{\frac{1}{d-i}} \right\rceil .$$

Let K_i be the maximum absolute value of the coefficients of p_i. For $i = 1, \ldots, k-1$, define

$$N_i = \operatorname{dom}\left(\left| \frac{s_k}{s_i} \right|, \left| \frac{2K_i}{\delta a_d} \right|, \deg p_i - d \right) .$$

Let $N = \max(2, M_0, \ldots, M_{d-1}, N_1, \ldots, N_{k-1})$. The choice of M_i ensures that

$$|\delta a_d n^d s_k^n| > |a_i n^i s_k^n|$$

for all $n \geq N$. The choice of K_i ensures that $|p_i(n)| < 2K_i n^{\deg p_i}$ for all $n \geq 2$. Therefore,

$$|\delta a_d n^d s_k^n| > |p_i(n) s_i^n|$$

for all $n > N$. This means that equation (3.6) does not have any solutions larger than N. We can find all solutions of (3.6) by checking all integers between 0 and N.

4. Polynomial solutions

In this section we present an algorithm for finding all polynomial solutions $f \in \mathbb{F}[x, \mathbf{y}]$ of parametric nonhomogeneous equations of the form

$$\mathbf{L}f = g + \sum_{j=1}^{s} \lambda_j h_j \tag{4.1}$$

where

$$\mathbf{L} = \sum_{i=0}^{\rho} p_i \mathbf{E}^i \tag{4.2}$$

is a linear recurrence operator with polynomial coefficients $p_i \in \mathbb{F}[x, \mathbf{y}]$, λ_j are free parameters (ranging over $\mathbb{F}$) to be determined, and $g, h_j \in \mathbb{F}[x, \mathbf{y}]$ are given polynomials. More precisely, the problem is to compute a basis of the affine space $\mathbf{L}_p^{-1}(g)$ where $\mathbf{L}_p : \mathbb{F}[x, \mathbf{y}] \oplus \mathbb{F}^s \to \mathbb{F}[x, \mathbf{y}]$ and $\mathbf{L}_p : (f, \boldsymbol{\lambda}) \mapsto \mathbf{L}f - \sum_{j=1}^{s} \lambda_j h_j$ for $f \in \mathbb{F}[x, \mathbf{y}]$ and $\boldsymbol{\lambda} \in \mathbb{F}^s$. Thus by a *solution* of (4.1) we mean a pair $(f, \boldsymbol{\lambda})$ with $f \in \mathbb{F}[x, \mathbf{y}]$ and $\boldsymbol{\lambda} \in \mathbb{F}^s$ such that (4.1) is satisfied.

As a special case, (4.1) includes nonhomogeneous equations without parameters (when all $h_j = 0$) as well as homogeneous equations (when also $g = 0$). The ability to solve parametric nonhomogeneous equations is crucial if one wants to apply Zeilberger's Creative Telescoping algorithm (Zeilberger, 1991) in the mixed hypergeometric case. Another reason for allowing linear parameters in the equation is the nature of our algorithm which finds the terms of the solution one by one, introducing new free parameters into the right-hand side at each step.

Let $f(x, \mathbf{y})$ be a polynomial solution of (4.1). Write

$$\begin{aligned}
\boldsymbol{\alpha} &= \max_{0 \le i \le \rho} \deg_{\mathbf{y}} p_i, && (4.3)\\
p_{i,\boldsymbol{\alpha}}(x) &= [\mathbf{y}^{\boldsymbol{\alpha}}]\, p_i, && (4.4)\\
d &= \max_{0 \le i \le \rho} \deg_x p_{i,\boldsymbol{\alpha}}(x), && (4.5)\\
p_{i,\boldsymbol{\alpha},d} &= [x^d]\, p_{i,\boldsymbol{\alpha}}(x), && (4.6)\\
\mathrm{rhs}(\boldsymbol{\lambda}) &= g + \sum_{j=1}^{s} \lambda_j h_j, && (4.7)\\
\boldsymbol{\varphi} &= \deg_{\mathbf{y}} f(x, \mathbf{y}), && (4.8)\\
t(x) &= [\mathbf{y}^{\boldsymbol{\varphi}}]\, f(x, \mathbf{y}), && (4.9)\\
\boldsymbol{\beta} &= \deg_{\mathbf{y}} \mathrm{rhs}(\boldsymbol{\lambda}), && (4.10)\\
r_{\boldsymbol{\beta}} &= [\mathbf{y}^{\boldsymbol{\beta}}]\, \mathrm{rhs}(\boldsymbol{\lambda}), && (4.11)
\end{aligned}$$

where $t \in \mathbb{F}[x] \setminus \{0\}$, $p_{i,\boldsymbol{\alpha}} \in \mathbb{F}[x]$ and $p_{i,\boldsymbol{\alpha},d} \in \mathbb{F}$. In (4.10) we regard λ_j's as variables over $\mathbb{F}(x)$, and $\mathrm{rhs}(\boldsymbol{\lambda})$ as belonging to $\mathbb{F}(x, \boldsymbol{\lambda})[\mathbf{y}]$. This means that after the parameters λ_j are given specific values $\lambda'_j \in \mathbb{F}$, the multidegree of $\mathrm{rhs}(\boldsymbol{\lambda}') = g + \sum_{j=1}^{s} \lambda'_j h_j$ in $\mathbf{y}$ can be lower than $\boldsymbol{\beta}$.

LEMMA 4.1. *Let* $\mathbf{L}, \boldsymbol{\alpha}, p_{i,\boldsymbol{\alpha},d}$ *and* $\boldsymbol{\varphi}$ *be as given in (4.2)–(4.8). If* $\deg_{\mathbf{y}} \mathbf{L} f \prec \boldsymbol{\alpha} + \boldsymbol{\varphi}$, *then* $\boldsymbol{\varphi} = \deg_{\mathbf{y}} f$ *satisfies* $P(\mathbf{q}^{\boldsymbol{\varphi}}) = 0$ *where*

$$P(x) = \sum_{i=0}^{\rho} p_{i,\boldsymbol{\alpha},d} x^i \tag{4.12}$$

is the characteristic polynomial *of* $\mathbf{L}$.

PROOF. From (4.9), $\mathbf{E}^i f = t(x+i)\mathbf{q}^{i\boldsymbol{\varphi}}\mathbf{y}^{\boldsymbol{\varphi}} + o(\mathbf{y}^{\boldsymbol{\varphi}})$, so $\mathbf{L} f = T(x)\mathbf{y}^{\boldsymbol{\alpha}+\boldsymbol{\varphi}} + o(\mathbf{y}^{\boldsymbol{\alpha}+\boldsymbol{\varphi}})$ where

$$T(x) = \sum_{i=0}^{\rho} p_{i,\boldsymbol{\alpha}}(x)\mathbf{q}^{i\boldsymbol{\varphi}} t(x+i). \tag{4.13}$$

If $\deg_{\mathbf{y}} \mathbf{L} f \prec \boldsymbol{\alpha} + \boldsymbol{\varphi}$ then $T = 0$. This is an ordinary recurrence relation with nonzero polynomial solution $t(x)$. As the coefficient of $x^{d+\deg_x t}$ in $T(x)$ must vanish, $\sum_{i=0}^{\rho} p_{i,\boldsymbol{\alpha},d}\mathbf{q}^{i\boldsymbol{\varphi}} = 0$ as claimed. □

Let $\mathcal{R}$ denote the set of exponents of those roots of the characteristic polynomial (4.12) (if any) which are power products of the bases:

$$\mathcal{R} = \{\boldsymbol{\sigma} \in \mathbb{N}_0^m;\ P(\mathbf{q}^{\boldsymbol{\sigma}}) = 0\}. \tag{4.14}$$

When $\mathcal{R}$ is empty we take $\max \mathcal{R} = \perp$. The following lemma gives rise to an algorithm for finding all polynomial solutions of equation (4.1).

LEMMA 4.2. *Let $(f, \boldsymbol{\lambda}')$ be a solution of (4.1) with $f \in \mathbb{F}[x, \mathbf{y}]$ and $\boldsymbol{\lambda}' \in \mathbb{F}^s$.*

1. *If $\boldsymbol{\alpha} + \max \mathcal{R} \succ \boldsymbol{\beta}$ then $\deg_{\mathbf{y}} f \preceq \max \mathcal{R}$.*
2. *Let $\boldsymbol{\alpha} + \max \mathcal{R} \preceq \boldsymbol{\beta}$.*
   (a) *If $\boldsymbol{\alpha} \subseteq \boldsymbol{\beta}$ then $\deg_{\mathbf{y}} f \preceq \boldsymbol{\beta} - \boldsymbol{\alpha}$.*
   (b) *If $\boldsymbol{\alpha} \not\subseteq \boldsymbol{\beta}$ then $\deg_{\mathbf{y}} rhs(\boldsymbol{\lambda}') \prec \boldsymbol{\beta}$.*

PROOF. Let $\varphi = \deg_{\mathbf{y}} f$. Let T be as in (4.13).

1. $\boldsymbol{\alpha} + \max \mathcal{R} \succ \boldsymbol{\beta}$
   If $T = 0$ then $\deg_{\mathbf{y}} \mathbf{L} f \prec \boldsymbol{\alpha} + \varphi$ and $\varphi \in \mathcal{R}$, by Lemma 4.1. If $T \neq 0$ then $\deg_{\mathbf{y}} \mathbf{L} f = \boldsymbol{\alpha} + \varphi$. As $\deg_{\mathbf{y}} \mathrm{rhs}(\boldsymbol{\lambda}') \preceq \boldsymbol{\beta}$, it follows that $\boldsymbol{\alpha} + \varphi \preceq \boldsymbol{\beta} \prec \boldsymbol{\alpha} + \max \mathcal{R}$, so $\varphi \prec \max \mathcal{R}$. In either case, $\varphi \preceq \max \mathcal{R}$ as claimed.
2. $\boldsymbol{\alpha} + \max \mathcal{R} \preceq \boldsymbol{\beta}$
   (a) $\boldsymbol{\alpha} \subseteq \boldsymbol{\beta}$
   If $T = 0$ then $\deg_{\mathbf{y}} \mathbf{L} f \prec \boldsymbol{\alpha} + \varphi$ and $\varphi \in \mathcal{R}$ by Lemma 4.1, so $\varphi \preceq \max \mathcal{R}$ and therefore $\boldsymbol{\alpha} + \varphi \preceq \boldsymbol{\beta}$. If $T \neq 0$ then $\deg_{\mathbf{y}} \mathbf{L} f = \boldsymbol{\alpha} + \varphi$. As $\deg_{\mathbf{y}} \mathrm{rhs}(\boldsymbol{\lambda}') \preceq \boldsymbol{\beta}$, it follows that $\boldsymbol{\alpha} + \varphi \preceq \boldsymbol{\beta}$. In either case, $\varphi \preceq \boldsymbol{\beta} - \boldsymbol{\alpha}$ as claimed.
   (b) $\boldsymbol{\alpha} \not\subseteq \boldsymbol{\beta}$
   Assume that $\deg_{\mathbf{y}} \mathrm{rhs}(\boldsymbol{\lambda}') = \boldsymbol{\beta}$. If $T = 0$ then $\deg_{\mathbf{y}} \mathbf{L} f \prec \boldsymbol{\alpha} + \varphi$ and $\varphi \in \mathcal{R}$ by Lemma 4.1, so $\deg_{\mathbf{y}} \mathbf{L} f \prec \boldsymbol{\alpha} + \varphi \preceq \boldsymbol{\alpha} + \max \mathcal{R} \preceq \boldsymbol{\beta} = \deg_{\mathbf{y}} \mathrm{rhs}(\boldsymbol{\lambda}')$, a contradiction. If $T \neq 0$ then $\boldsymbol{\alpha} + \varphi = \boldsymbol{\beta}$ which implies that $\boldsymbol{\alpha} \subseteq \boldsymbol{\beta}$, contrary to the assumption. Both cases lead to contradiction, so $\deg_{\mathbf{y}} \mathrm{rhs}(\boldsymbol{\lambda}') \prec \boldsymbol{\beta}$ as claimed.

□

Based on Lemma 4.2, we can find the general solution $(f, \boldsymbol{\lambda})$ of equation (4.1) as follows: First compute the set $\mathcal{R}$ as given in (4.14). Then distinguish three cases:

1. $\boldsymbol{\alpha} + \max \mathcal{R} \succ \boldsymbol{\beta}$
   Set $\varphi = \max \mathcal{R}$ and look for f in the form
   $$f = t(x)\mathbf{y}^{\varphi} + f_1 \tag{4.15}$$
   where $f_1 = o(\mathbf{y}^{\varphi})$. To find $t(x)$, apply the algorithm of (Abramov *et al.*, 1995) to $T = 0$ (an ordinary homogeneous recurrence relation). Then remove $\max \mathcal{R}$ from $\mathcal{R}$ and find f_1 recursively by solving
   $$\mathbf{L} f_1 = \mathrm{rhs}(\boldsymbol{\lambda}) - \mathbf{L}\left(t(x)\mathbf{y}^{\varphi}\right). \tag{4.16}$$
2. $\boldsymbol{\alpha} + \max \mathcal{R} \preceq \boldsymbol{\beta}$
   (a) $\boldsymbol{\alpha} \subseteq \boldsymbol{\beta}$
   Set $\varphi = \boldsymbol{\beta} - \boldsymbol{\alpha}$ and look for f in the form (4.15). To find $t(x)$, apply the algorithm of (Abramov *et al.*, 1995) to $T = r_{\boldsymbol{\beta}}$ (an ordinary parametric non-homogeneous recurrence relation). Then remove $\max \mathcal{R}$ from $\mathcal{R}$ (only in case that $\boldsymbol{\alpha} + \max \mathcal{R} = \boldsymbol{\beta}$), and find f_1 recursively by solving (4.16).

(b) $\boldsymbol{\alpha} \not\subseteq \boldsymbol{\beta}$
Let $\boldsymbol{\lambda} = \boldsymbol{\lambda}'$ be the solution of the system of linear algebraic equations for the free parameters $\boldsymbol{\lambda}$ obtained by equating the coefficients of powers of x in $r_{\boldsymbol{\beta}}$ to zero. Then find f recursively by solving $\mathbf{L}f = \mathrm{rhs}(\boldsymbol{\lambda}')$.

Remarks:

1. Note that in steps 1 and 2(a), $t(x)$ can contain new free parameters which are then joined with the existing ones. This explains the need for allowing parameters in the right-hand side of the equation.
2. In step 2(b), the number of free parameters will drop by the rank of the linear system to be solved.
3. If the ordinary recurrence in steps 1 or 2(a) has no polynomial solution, or the linear system in step 2(b) is unsolvable, then the original parametric recurrence has no polynomial solution, and the algorithm terminates unsuccessfully.
4. At each step, either the cardinality of the set $\mathcal{R}$ drops, or else it stays the same but the multidegree $\boldsymbol{\beta} = \deg_{\mathbf{y}} \mathrm{rhs}(\boldsymbol{\lambda})$ decreases in the admissible term order $\prec$. It follows that the pair $(\mathrm{card}\mathcal{R}, \boldsymbol{\beta})$ decreases in the lexicographic ordering of $\mathbb{N}_0 \times \mathbb{N}_0^m$ which uses $<$ in the first component and $\prec$ in the second component. As every admissible term order is a well-order, this assures termination of the algorithm.
5. Unless the algorithm terminates unsuccessfully, eventually $\mathcal{R}$ becomes empty and $\mathrm{rhs}(\boldsymbol{\lambda})$ becomes 0. Then the only polynomial solution of (4.1) is $f = 0$.

An iterative version of this tail-recursive algorithm (called `MixedPoly`) is given in Appendix A.

5. A canonical form for rational functions

Let $r \in \mathbb{F}(x, \mathbf{y}) \setminus \{0\}$. Write r as

$$r = \frac{u}{v} \cdot \frac{a_0}{b_0},$$

where $u, v \in \mathcal{M}$, $a_0, b_0 \in \mathbb{F}[x, \mathbf{y}]$, $a_0 b_0 \perp \mathcal{M}$, $ua_0 \perp vb_0$, and b_0 is mixed monic (Def. 2.2).

There are finitely many values $h \in \mathbb{N}_0$ such that $a_0 \not\perp \mathbf{E}^h b_0$. These values are the elements of the dispersion set $D(a_0, b_0)$ which can be found as described in Section 3. So let $D(a_0, b_0) = \{h_1, h_2, \ldots, h_N\}$ where $0 \le h_1 < h_2 < \cdots < h_N$.

Lemma 5.1. *Consider the algorithm* `CanonicalForm` *in Appendix B. Define $h_{N+1} = \infty$, and let $0 \le k \le i, j \le N$, $h \in \mathbb{N}_0$ and $h < h_{k+1}$. Then $a_i \perp \mathbf{E}^h b_j$.*

Proof. Let $S = \{h_1, \ldots, h_N\}$. Suppose $h \notin S$. Since $a_i \mid a_0$ and $b_j \mid b_0$ and $a_0 \perp \mathbf{E}^h b_0$, it follows that $a_i \perp \mathbf{E}^h b_j$.

To prove the lemma for $h \in S$, we use induction on k. When $k = 0$, there is nothing to prove because there is no $h \in S$ such that $h < h_1$. Assume that the lemma holds for all $h \in S$, $h < h_k$. We show that it holds for $h = h_k$. Since $a_i \mid a_k$ and $b_j \mid b_k$, it follows that $\gcd(a_i, \mathbf{E}^{h_k} b_j)$ divides $\gcd(a_k, \mathbf{E}^{h_k} b_k)$. Furthermore,

$$\gcd(a_k, \mathbf{E}^{h_k} b_k) = \gcd\left(\frac{a_{k-1}}{s_k}, \frac{\mathbf{E}^{h_k} b_{k-1}}{s_k}\right) = 1$$

by the definition of a_k, b_k and s_k in algorithm `CanonicalForm`. This completes the proof. □

THEOREM 5.1. *Let $r \in \mathbb{F}(x, \mathbf{y}) \setminus \{0\}$. There exist polynomials $a, b, c \in \mathbb{F}[x, \mathbf{y}] \setminus \{0\}$ such that*

1. *b, c are mixed monic,*
2. *$c \perp \mathcal{M}$,*
3. *$a \perp \mathbf{E}^k b$ for all $k \in \mathbb{N}_0$,*
4. *$a \perp c$,*
5. *$b \perp \mathbf{E}c$, and*

$$r = \frac{a}{b} \cdot \frac{\mathbf{E}c}{c}. \tag{5.1}$$

PROOF. Let a, b, c be constructed by the algorithm `CanonicalForm` from Appendix B. Conditions 1. and 2. are satisfied by construction, and condition 3. follows from Lemma 5.1 by taking $i = j = k = N$. Identity (5.1) is verified directly,

$$\begin{aligned} \frac{a}{b} \cdot \frac{\mathbf{E}c}{c} &= \frac{u \cdot a_N}{v \cdot b_N} \cdot \prod_{i=1}^{N} \prod_{j=1}^{h_i} \frac{\mathbf{E}^{-j+1} s_i}{\mathbf{E}^{-j} s_i} = \\ &= \frac{u \cdot a_0}{\prod_{i=1}^{N} s_i} \cdot \frac{\prod_{i=1}^{N} \mathbf{E}^{-h_i} s_i}{v \cdot b_0} \cdot \prod_{i=1}^{N} \frac{s_i}{\mathbf{E}^{-h_i} s_i} = \frac{u \cdot a_0}{v \cdot b_0} = r. \end{aligned}$$

Proof of 4.: Suppose $a \not\perp c$. Then also $a_N \not\perp \mathbf{E}^{-j} s_i$ for some i and j such that $1 \leq i \leq N$ and $1 \leq j \leq h_i$. By definition $\mathbf{E}^{h_i - j} b_{i-1} = \mathbf{E}^{h_i - j} b_i \cdot \mathbf{E}^{-j} s_i$, so it follows that $a_N \not\perp \mathbf{E}^{h_i - j} b_{i-1}$. Since $h_i - j < h_i$, this contradicts Lemma 5.1.

Proof of 5.: Suppose $b \not\perp \mathbf{E}c$. Then also $b_N \not\perp \mathbf{E}^{-j} s_i$ for some i and j such that $1 \leq i \leq N$ and $0 \leq j \leq h_i - 1$. By definition $\mathbf{E}^{-j} a_{i-1} = \mathbf{E}^{-j} a_i \cdot \mathbf{E}^{-j} s_i$, so it follows that $a_{i-1} \not\perp \mathbf{E}^j b_N$. Since $j < h_i$, this contradicts Lemma 5.1. □

LEMMA 5.2. *Let $a, b, c, A, B, C \in \mathbb{F}[x, \mathbf{y}] \setminus \{0\}$ be polynomials such that $a \perp c$, $b \perp \mathbf{E}c$, $c \perp \mathcal{M}$, and $A \perp \mathbf{E}^k B$ for all $k \in \mathbb{N}_0$. If*

$$\frac{a}{b} \cdot \frac{\mathbf{E}c}{c} = \frac{A}{B} \cdot \frac{\mathbf{E}C}{C}, \tag{5.2}$$

then c divides C.

PROOF. Let $g = \gcd(c, C)$, $d = c/g$, and $D = C/g$. Then $d \perp D$, $a \perp d$, and $b \perp \mathbf{E}d$. Clear denominators in (5.2) and cancel $g \cdot \mathbf{E}g$ on both sides. The result

$$A \cdot b \cdot d \cdot \mathbf{E}D = a \cdot B \cdot D \cdot \mathbf{E}d$$

shows that $d \mid B \cdot \mathbf{E}d$ and $\mathbf{E}d \mid A \cdot d$. Using these two relations repeatedly, we find that

$$\begin{aligned} d &\mid B \cdot \mathbf{E}B \cdots \mathbf{E}^{k-1} B \cdot \mathbf{E}^k d \\ d &\mid \mathbf{E}^{-1} A \cdot \mathbf{E}^{-2} A \cdots \mathbf{E}^{-k} A \cdot \mathbf{E}^{-k} d \end{aligned}$$

for all $k \in \mathbb{N}_0$. Because $d \perp \mathcal{M}$, and $\mathbb{F}$ has characteristic zero, $d \perp \mathbf{E}^k d$ and $d \perp \mathbf{E}^{-k} d$ for large enough k. It follows that d divides both $B \cdot \mathbf{E}B \cdots \mathbf{E}^{k-1} B$ and $\mathbf{E}^{-1} A \cdot \mathbf{E}^{-2} A \cdots \mathbf{E}^{-k} A$

for large enough k. But these two polynomials are coprime by assumption, so d must be a constant. Hence, c divides C. □

THEOREM 5.2. *Let $r \in \mathbb{F}[x, \mathbf{y}] \setminus \{0\}$. Then the factorization of r described in Theorem 5.1 is unique.*

PROOF. Suppose that a, b, c and A, B, C are two factorizations of r, as described in Theorem 5.1. Then

$$r = \frac{a}{b} \cdot \frac{\mathbf{E}c}{c} = \frac{A}{B} \cdot \frac{\mathbf{E}C}{C}.$$

By Lemma 5.2, c divides C, and vice versa. As c and C are mixed monic they must be equal, hence $a/b = A/B$. As $a \perp b$, $A \perp B$, and b, B are mixed monic, it follows that $b = B$ and $a = A$ as well. □

The factorization of non-zero rational functions described in Theorem 5.1 is thus a canonical form. We introduce special notation for it.

DEFINITION 5.1. *Let $r \in \mathbb{F}(x, \mathbf{y}) \setminus \{0\}$ be a non-zero rational function. We write*

$$(a, b, c) = \mathrm{C.f.}(r)$$

to denote the unique polynomials $a, b, c \in \mathbb{F}[x, \mathbf{y}]$ which satisfy the conditions of Theorem 5.1.

THEOREM 5.3. *Let $a, b \in \mathbb{F}[x, \mathbf{y}] \setminus \{0\}$, and $(A, B, C) = \mathrm{C.f.}(b/a)$. The homogeneous first-order recurrence*

$$a \cdot \mathbf{E}f - bf = 0 \tag{5.3}$$

has a non-zero polynomial solution $f \in \mathbb{F}[x, \mathbf{y}]$ if and only if $A/B = u(\mathbf{q})$ for some $u \in \mathcal{M}$. In that case, $f = \lambda \cdot u \cdot C$ for some $\lambda \in \mathbb{F} \setminus \{0\}$.

PROOF. Suppose (5.3) has a non-zero solution $f \in \mathbb{F}[x, \mathbf{y}]$. Write $f = \lambda \cdot u \cdot g$ where $\lambda \in \mathbb{F} \setminus \{0\}$, $u \in \mathcal{M}$ and $g \perp \mathcal{M}$ is mixed monic. Then $\mathrm{C.f.}(\mathbf{E}f/f) = (u(\mathbf{q}), 1, g)$. Since from (5.3)

$$\frac{\mathbf{E}f}{f} = \frac{b}{a} = \frac{A}{B} \cdot \frac{\mathbf{E}C}{C},$$

$(A, B, C) = \mathrm{C.f.}(\mathbf{E}f/f)$ as well. By Theorem 5.2 it follows that $A = u(\mathbf{q})$, $B = 1$ and $C = g$, so $A/B = u(\mathbf{q})$ and $f = \lambda \cdot u \cdot C$.

Conversely, if $A/B = u(\mathbf{q})$ for some $u \in \mathcal{M}$, then $f = u \cdot C$ is a non-zero solution of (5.3). □

We remark that our canonical form differs from the Paule/Riese/Strehl form (PRS, for short) described in (Paule and Strehl, 1995; Paule and Riese, 1997) for the basic and in (Riese, 1996) for the bibasic case, in the following three respects:

1. In the PRS form the monomial factors of the numerator and denominator of r are listed separately while in our form they are included with A resp. B.

2. In the PRS form all polynomials either have unit constant terms or else are primitive and the overall constant factor is listed separately, while in our form B and C have unit leading coefficients in the chosen term order and the overall constant factor is included with A.
3. In the PRS form the polynomial corresponding to our B is given by a constant multiple of $\mathbf{E}^{-1}B$.

6. Mixed Gosper's algorithm

Let $S_n = \sum_{k=0}^{n-1} t_k$. Clearly substituting S_n for s_n satisfies the first-order recurrence

$$s_{n+1} - s_n = t_n. \tag{6.1}$$

Conversely, any solution s_n of (6.1) differs from S_n only by an additive constant – more precisely, $S_n = s_n - s_0$. Therefore we consider the following problem:

Given a sequence t_n, decide if equation (6.1) has a mixed hypergeometric solution s_n, and if so, find it.

Let s_n and t_n satisfy (6.1), with

$$\frac{s_{n+1}}{s_n} =: T_n \in \mathbb{F}(n, \mathbf{q}^n).$$

Then the two quotients

$$r_n := \frac{t_{n+1}}{t_n} = \frac{s_{n+2} - s_{n+1}}{s_{n+1} - s_n} = \frac{T_{n+1} - 1}{1 - 1/T_n}$$

and

$$R_n := \frac{s_n}{t_n} = \frac{s_n}{s_{n+1} - s_n} = \frac{1}{T_n - 1}$$

both belong to $\mathbb{F}(n, \mathbf{q}^n)$. So t_n must be mixed hypergeometric itself, and s_n is a rational multiple of t_n: $s_n = R_n t_n$. Using this, (6.1) yields a recurrence for the unknown rational sequence R_n,

$$r_n R_{n+1} - R_n = 1. \tag{6.2}$$

By Theorem 2.1, equation (6.2) is equivalent to

$$r \cdot \mathbf{E}R - R = 1, \tag{6.3}$$

where $r, R \in \mathbb{F}(x, \mathbf{y})$ are Φ^{-1}-isomorphic images of r_n and R_n, respectively.

Next we show how to find rational solutions $R \in \mathbb{F}(x, \mathbf{y})$ of equation (6.3). The following theorem provides a multiple of the denominator and a divisor of the numerator of R. The missing factor in the numerator can then be found using algorithm `MixedPoly` of Section 4.

Definition 6.1. *Let $r \in \mathbb{F}(x, \mathbf{y}) \setminus \{0\}$ be a non-zero rational function, and $(a, b, c) =$ C.f.(r). For $1 \le i \le m$ define* exponents *$e_i(r)$ as follows: If $\boldsymbol{\pi}_i(a)\boldsymbol{\pi}_i(b) \neq 0$, let $(a_i, b_i, c_i) =$ C.f.$(\boldsymbol{\pi}_i(b)/\boldsymbol{\pi}_i(a))$. If there are $v, w \in \mathcal{M}$ such that $v \perp w$ and $a_i/b_i = v(\mathbf{q})/w(\mathbf{q})$, then $e_i(r) = \deg_{y_i} w$. If not, or if $\boldsymbol{\pi}_i(a)\boldsymbol{\pi}_i(b) = 0$, then $e_i(r) = 0$.*

THEOREM 6.1. *Let $R = f/(ug)$ be a rational solution of (6.3) with $f, g \in \mathbb{F}[x, \mathbf{y}]$, $u \in \mathcal{M}$, $g \perp \mathcal{M}$, and $f \perp ug$. Then*

1. *$g \mid c$ where $(a, b, c) = \mathrm{C.f.}(r)$,*
2. *$\deg_{y_i} u \le e_i(r)$,*
3. *$\mathbf{E}^{-1}b \mid f$.*

PROOF. 1. From (6.3),

$$r = \frac{R+1}{\mathbf{E}R} = \frac{(f+ug)u(\mathbf{q})}{\mathbf{E}f} \cdot \frac{\mathbf{E}g}{g}. \tag{6.4}$$

On the other hand, $(a, b, c) = \mathrm{C.f.}(r)$, so

$$r = \frac{a}{b} \cdot \frac{\mathbf{E}c}{c}. \tag{6.5}$$

As $\mathbf{E}f \perp \mathbf{E}g$, $g \perp (f + ug)u(\mathbf{q})$, $g \perp \mathcal{M}$, and $a \perp \mathbf{E}^k b$ for all $k \in \mathbb{N}_0$, it follows by Lemma 5.2 that g divides c.

2. Write $F = fc/g \in \mathbb{F}[x, \mathbf{y}]$. Then $R = f/(ug) = F/(uc)$. Combining this with (6.4) and (6.5), we find that

$$(F + uc) \cdot u(\mathbf{q}) \cdot b = a \cdot \mathbf{E}F. \tag{6.6}$$

Now assume that $y_i \mid u$. Then applying $\boldsymbol{\pi}_i$ (see Def. 2.3) to Eqn. (6.6) and rearranging yields

$$\boldsymbol{\pi}_i(a) \cdot \mathbf{E}\boldsymbol{\pi}_i(F) - u(\mathbf{q}) \cdot \boldsymbol{\pi}_i(b) \cdot \boldsymbol{\pi}_i(F) = 0. \tag{6.7}$$

Because $F \mid fc$, $f \perp u$ and $c \perp \mathcal{M}$, it follows that $y_i \not\mid F$ and $\boldsymbol{\pi}_i(F) \neq 0$. Assume that $\boldsymbol{\pi}_i(a) = 0$. Then $y_i \mid a$ and from (6.6), $y_i \mid b \cdot F$. But $y_i \not\mid b$ because $a \perp b$, so $y_i \mid F$. This contradiction shows that $\boldsymbol{\pi}_i(a) \neq 0$. In an analogous way we conclude that $\boldsymbol{\pi}_i(b) \neq 0$.

Let $(a_i, b_i, c_i) = \mathrm{C.f.}(\boldsymbol{\pi}_i(b)/\boldsymbol{\pi}_i(a))$. Then $((u(\mathbf{q}) \cdot a_i, b_i, c_i) = \mathrm{C.f.}(u(\mathbf{q}) \cdot \boldsymbol{\pi}_i(b)/\boldsymbol{\pi}_i(a))$. Since equation (6.7) has a non-zero polynomial solution $\boldsymbol{\pi}_i(F)$, it follows by Theorem 5.3 that there is $u_1 \in \mathcal{M}$ such that $u(\mathbf{q}) \cdot a_i/b_i = u_1(\mathbf{q})$, and that $\boldsymbol{\pi}_i(F) = \lambda u_1 c_i$ for some $\lambda \in \mathbb{F}$. Then $a_i/b_i = u_1(\mathbf{q})/u(\mathbf{q})$ is a quotient of two monomials. Write $u_1 = v \cdot t$ and $u = w \cdot t$ where $t, v, w \in \mathcal{M}$ and $v \perp w$. By Definiton 6.1, $e_i(r) = \deg_{y_i} w$. As $t \mid u_1 \mid \boldsymbol{\pi}_i(F)$ it follows that $t \perp y_i$, so

$$\deg_{y_i} u = \deg_{y_i} w = e_i(r).$$

We have shown that $\deg_{y_i} u$ is either 0 or $e_i(r)$, so in either case $\deg_{y_i} u \le e_i(r)$.

3. From (6.6) it follows that $b \mid a \cdot \mathbf{E}F$. As $a \perp b$, we have that $b \mid \mathbf{E}F \mid \mathbf{E}f\mathbf{E}c$. But $b \perp \mathbf{E}c$, so $\mathbf{E}^{-1}b \mid f$.

□

From Theorem 6.1 it follows that we can look for R in the form

$$R = \frac{\mathbf{E}^{-1}b \cdot p}{u \cdot c} \tag{6.8}$$

where $(a, b, c) = \mathrm{C.f.}(r)$ and $u = \prod_{i=1}^m y_i^{e_i(r)}$ are known while $p \in \mathbb{F}[x, \mathbf{y}]$ is an unknown polynomial. Inserting (6.8) and (6.5) into (6.3) yields

$$a \cdot \mathbf{E}p - u(\mathbf{q}) \cdot \mathbf{E}^{-1}b \cdot p = u(\mathbf{q})u \cdot c, \tag{6.9}$$

an nonhomogeneous first-order linear recurrence relation with polynomial coefficients satisfied by p. Algorithm `MixedPoly` of Section 4 can now be applied to find a polynomial solution p of Eqn. (6.9). The full algorithm is given in Appendix C.

We conclude this section by giving some examples of sums which can be evaluated automatically by `MixedGosper`. We write $(a;q)_n = \prod_{i=0}^{n-1}(1-aq^i)$.

Several bibasic examples can be found in (Gasper and Rahman, 1990) and also in (Riese, 1996). An indefinite multibasic summation formula (too big to reproduce it here) was proved by Singh (1996). The formula contains an *arbitrary* number, k, of bases. Such formulæ cannot be proved by our algorithm. However, any specialization of this formula in which k is replaced by a specific natural number can be, at least in principle, not only proved, but also derived by `MixedGosper`. Singh (1996) shows that several well-known basic and bibasic summation formulæ can be obtained as specializations of this k-basic master formula.

The following two examples are due to Gosper (1998).

EXAMPLE 6.1. *In this tribasic example* $\mathbb{F} = \mathbb{Q}(a,b,c,p,q,r)$ *where* a,b,c *are parameters and* p,q,r *are the bases. Using* `MixedGosper` *one finds that*

$$\sum_{k=0}^{n} \frac{(-a)^k p^{\binom{k}{2}}(1-abp^kq^k)(1-acp^kr^k)(b;q)_k(c;r)_k}{(ap;p)_k(abcpqr;pqr)_k} \quad =$$

$$(a-1)(abc-1) + \frac{(-a)^n ap^{\binom{n+1}{2}}(b;q)_{n+1}(c;r)_{n+1}}{(ap;p)_n(abcpqr;pqr)_n}.$$

EXAMPLE 6.2. *In this quadbasic example* $\mathbb{F} = \mathbb{Q}(a,b,c,d,p,q,r,s)$ *where* a,b,c,d *are parameters and* p,q,r,s *are the bases. Let*

$$f(k) = \frac{(\frac{a}{c};\frac{p}{r})_k(ac;pr)_k(\frac{d}{b};\frac{s}{q})_k(bd;qs)_k b^k}{a^k p^{\binom{k+1}{2}}(\frac{bq}{cr};\frac{q}{r})_k(bcqr;qr)_k(\frac{ds}{ap};\frac{s}{p})_k(adps;ps)_k}.$$

Then, using `MixedGosper`*, one obtains*

$$\sum_{k=0}^{n} q^{\binom{k+1}{2}}(1-\frac{ap^k}{bq^k})(1-abp^kq^k)(1-\frac{ds^k}{cr^k})(1-cdr^ks^k)f(k) \quad =$$

$$(b-c)(1-\frac{1}{bc})(a-d)(ad-1)+\frac{q^{\binom{n}{2}}(cr^n-ap^n)(acp^nr^n-1)(bq^n-ds^n)(bdq^ns^n-1)f(n)}{bcr^n}.$$

EXAMPLE 6.3. *For a simple mixed hypegeometric example, define*

$$\varphi_{q,n}(a,b,c) = \prod_{i=0}^{n-1}(a+bi+cq^i).$$

Then, using `MixedGosper`*, one obtains*

$$\sum_{k=0}^{n}(bk+cq^k)\varphi_{q,k}(1,b,c) = \varphi_{q,n+1}(1,b,c) - 1. \tag{6.10}$$

As $\varphi_{q,n}(1,1,0) = n!$ *and* $\varphi_{q,n}(1,0,-a) = (a;q)_n$*, both well-known summation formulæ*

$$\sum_{k=0}^{n} k\,k! = (n+1)! - 1, \qquad \sum_{k=0}^{n} (a;q)_k q^k = \frac{1-(a;q)_{n+1}}{a},$$

turn out to be special cases of (6.10). – It is not hard to imagine how more complex mixed hypergeometric formulæ could be built using φ *or similar functions.*

7. Mixed hypergeometric solutions

In this section we derive algorithm `MixedHyper` for finding all mixed hypergeometric solutions f of $\mathbf{L}f = 0$ where $\mathbf{L}$ is as in (4.2). Let $\mathbf{E}f = rf$ where $r \in \mathbb{F}(x, \mathbf{y})$, then $\mathbf{E}^i f = \prod_{j=0}^{i-1} \left(\mathbf{E}^j r\right) f$. The crucial idea is to look for r in the canonical form described in Theorem 5.1. More precisely, we use a slightly modified canonical form

$$r = z \frac{a}{b} \frac{\mathbf{E}c}{c} \tag{7.1}$$

where $z \in \mathbb{F}\backslash\{0\}$, $a \in \mathbb{F}[x, \mathbf{y}]$ is mixed monic, and a, b, c satisfy conditions 1–5 of Theorem 5.1. After inserting (7.1) into $\mathbf{L}f = 0$, clearing denominators and cancelling f we obtain

$$\sum_{i=0}^{\rho} z^i P_i \cdot \mathbf{E}^i c = 0 \tag{7.2}$$

where

$$P_i = p_i \prod_{j=0}^{i-1} \mathbf{E}^j a \prod_{j=i}^{\rho-1} \mathbf{E}^j b.$$

Since all terms in (7.2) except for $i = 0$ contain a as an explicit factor, it follows that a divides $p_0 c \prod_{j=0}^{\rho-1} \mathbf{E}^j b$. Because of properties 3. and 4. of the canonical form, a divides p_0. Similarly, all terms in (7.2) except for $i = \rho$ contain $\mathbf{E}^{\rho-1} b$ as an explicit factor, therefore $\mathbf{E}^{\rho-1} b$ divides $z^\rho p_\rho \mathbf{E}^\rho c \prod_{j=0}^{\rho-1} \mathbf{E}^j a$. Because of properties 3. and 5. of the canonical form, $\mathbf{E}^{\rho-1} b$ divides p_ρ. Thus we have a finite number of choices for a and b: they are mixed monic factors of p_0 and $\mathbf{E}^{1-\rho} p_\rho$, respectively.

For each choice of a and b, equation (7.2) is a linear recurrence with polynomial coefficients satisfied by the unknown polynomial $c(x, \mathbf{y})$. However, $z \in \mathbb{F} \setminus \{0\}$ is also unknown. To find $z(\mathbf{E}c/c)$, write $\boldsymbol{\alpha} = \min_{0 \le i \le \rho} \operatorname{mindeg}_{\mathbf{y}} P_i$, $p_{i,\boldsymbol{\alpha}}(x) = [\mathbf{y}^{\boldsymbol{\alpha}}]\, P_i$, $d = \max_{0 \le i \le \rho} \deg_x p_{i,\boldsymbol{\alpha}}(x)$, $p_{i,\boldsymbol{\alpha},d} = [x^d]\, p_{i,\boldsymbol{\alpha}}(x)$, and $\boldsymbol{\varphi} = \operatorname{mindeg}_{\mathbf{y}} c(x, \mathbf{y})$. Looking at the coefficient of $\mathbf{y}^{\boldsymbol{\alpha}+\boldsymbol{\varphi}}$ in (7.2), we find that $P(z\mathbf{q}^{\boldsymbol{\varphi}}) = 0$ where

$$P(x) = \sum_{i=0}^{\rho} p_{i,\boldsymbol{\alpha},d}\, x^i. \tag{7.3}$$

Write $z\mathbf{q}^{\boldsymbol{\varphi}} = \tau$ where τ is a root of P. Then $z = \tau \mathbf{q}^{-\boldsymbol{\varphi}}$, hence (7.2) can be rewritten as

$$\sum_{i=0}^{\rho} \tau^i \mathbf{q}^{-i\boldsymbol{\varphi}} P_i \cdot \mathbf{E}^i c = 0.$$

Dividing by $\mathbf{y}^{\boldsymbol{\varphi}}$ we obtain finally

$$\sum_{i=0}^{\rho} \tau^i P_i \cdot \mathbf{E}^i (c/\mathbf{y}^{\boldsymbol{\varphi}}) = 0.$$

Suppose that we know $\bar{c} = c/\mathbf{y}^{\boldsymbol{\varphi}}$. Then

$$\tau\frac{\mathbf{E}\bar{c}}{\bar{c}} = \tau\mathbf{q}^{-\boldsymbol{\varphi}}\frac{\mathbf{E}c}{c} = z\frac{\mathbf{E}c}{c}$$

which is just what we are looking for. It remains to see how to find all Laurent polynomial solutions $\bar{c} \in \mathbb{F}[x, \mathbf{y}, \mathbf{y}^{-1}]$ of

$$\sum_{i=0}^{\rho} \tau^i P_i \cdot \mathbf{E}^i(\bar{c}) = 0. \tag{7.4}$$

Obviously, if we know lower bounds b_j for the degrees of y_j in $\bar{c}$, then we can substitute $\bar{c} = \bar{d}y_1^{b_1}\cdots y_m^{b_m}$ in (7.4) and use `MixedPoly` to find polynomial solutions $\bar{d}$ of the resulting equation. Observe that any lexicographic order $\preceq$ on $\mathbb{Z}^m$ is total (though not well-founded) and satisfies $\boldsymbol{\alpha} \preceq \boldsymbol{\beta} \ \Rightarrow \ \boldsymbol{\alpha}+\boldsymbol{\gamma} \preceq \boldsymbol{\beta}+\boldsymbol{\gamma}$ for all $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \in \mathbb{Z}^m$. To obtain b_j, we order the terms lexicographically with y_j as the first variable, and write once again $\boldsymbol{\alpha}_j = \min_{0\le i\le\rho} \operatorname{mindeg}_{\mathbf{y}} P_i$, $p_{i,\boldsymbol{\alpha}_j}(x) = [\mathbf{y}^{\boldsymbol{\alpha}_j}]\tau^i P_i$, $d_j = \max_{0\le i\le\rho} \deg_x p_{i,\boldsymbol{\alpha}_j}(x)$, $p_{i,\boldsymbol{\alpha}_j,d_j} = [x^{d_j}]\,p_{i,\boldsymbol{\alpha}_j}(x)$, and $\boldsymbol{\varphi}_j = \operatorname{mindeg}_{\mathbf{y}}\bar{c}(x,\mathbf{y})$. Then $P^{(j)}(\mathbf{q}^{\boldsymbol{\varphi}_j}) = 0$ where

$$P^{(j)}(x) = \sum_{i=0}^{\rho} p_{i,\boldsymbol{\alpha}_j,d_j} x^i.$$

The bound b_j can now be read off as the j-th component of $\boldsymbol{\varphi}_j$.

In summary, we find the factors of $r = \tau(a/b)(\mathbf{E}\bar{c}/\bar{c})$ as follows:

1. a is a mixed monic factor of p_0,
2. b is a mixed monic factor of $\mathbf{E}^{1-\rho}p_\rho$,
3. τ is a root of polynomial $P(x)$ defined in (7.3),
4. $\bar{c}$ is a non-zero Laurent polynomial solution of (7.4).

Checking each admissible triple a, b, τ for Laurent polynomial solutions $\bar{c}$ of recurrence (7.4) constitutes algorithm `MixedHyper` which is given in appendix D.

8. Greatest factorial factorization

The concept of *greatest factorial factorization* of polynomials (GFF, for short) which is an analogue of the well-known square-free factorization (SFF), plays a fundamental role in symbolic summation. It has been introduced by Paule (1995) for the hypergeometric case, and subsequently extended to the basic (Paule and Strehl, 1995; Paule and Riese, 1997) as well as bibasic cases (Riese, 1996). Here we sketch an extension of the GFF concept to an arbitrary polynomial ring with an automorphism σ, including the hypergeometric, basic, bibasic, multibasic, and mixed hypergeometric GFF, as well as SFF, as special cases.

Let $\mathbb{F}$ be a field of characteristic zero and $\mathbb{F}[\mathbf{x}] = \mathbb{F}[x_1, x_2, \ldots, x_n]$ the ring of n-variate polynomials over $\mathbb{F}$. For $p, q \in \mathbb{F}[\mathbf{x}]$, we write $p \sim q$ if there is an $a \in \mathbb{F} \setminus \{0\}$ such that $p = aq$. Such p and q are called *associated*.

Let σ be an $\mathbb{F}$-automorphism of $\mathbb{F}[\mathbf{x}]$ (i.e., a ring automorphism of $\mathbb{F}[\mathbf{x}]$ which fixes each element of $\mathbb{F} \subseteq \mathbb{F}[\mathbf{x}]$). To specify σ it suffices to give the n polynomials $\sigma x_1, \ldots, \sigma x_n$. Note that σ preserves irreducibility of polynomials, and so for any irreducible $p \in \mathbb{F}[\mathbf{x}] \setminus \mathbb{F}$, either $\sigma p \sim p$ or $\sigma p \perp p$.

In analogy to Moenck (1977) we write

$$[p]_{\sigma}^{\underline{k}} = \prod_{i=0}^{k-1} \sigma^{-i} p$$

for the *k-th falling σ-factorial* of p.

DEFINITION 8.1. *Let $p \in \mathbb{F}[\mathbf{x}] \setminus \{0\}$. Then*

$$\sigma\text{-span}(p) = \max\{k \in \mathbb{N}_0;\ [q]_{\sigma}^{\underline{k}} \text{ divides } p, \text{ for some } q \in \mathbb{F}[\mathbf{x}] \setminus \mathbb{F}\}.$$

Note that $0 \leq \sigma\text{-span}(p) \leq \max_{1 \leq i \leq n} deg_{x_i} p$.

DEFINITION 8.2. *Let $p \in \mathbb{F}[\mathbf{x}] \setminus \{0\}$. A list of polynomials $\langle p_1, \ldots, p_k \rangle$ from $\mathbb{F}[\mathbf{x}]$ where $k \geq 0$ is a σ-GFF of p if the following conditions are satisfied:*

(σ-GFF1) $p \sim [p_1]_{\sigma}^{\underline{1}} \cdots [p_k]_{\sigma}^{\underline{k}}$,

(σ-GFF2) *if $k > 0$ then $p_k \in \mathbb{F}[\mathbf{x}] \setminus \mathbb{F}$,*

(σ-GFF3) *for all $1 \leq i \leq k$, either $p_i = 1$, or $\sigma\text{-span}\left([p_1]_{\sigma}^{\underline{1}} \cdots [p_i]_{\sigma}^{\underline{i}}\right) = i$.*

It is clear that every polynomial $p \in \mathbb{F}[\mathbf{x}] \setminus \{0\}$ has a σ-GFF. To find it, factor p into non-constant irreducible factors, combine those which compose falling σ-factorials of length $k = \sigma\text{-span}(p)$ into $[p_k]_{\sigma}^{\underline{k}}$, then repeat this procedure with $p/[p_k]_{\sigma}^{\underline{k}}$. Note that the σ-GFF of a constant polynomial $p \in \mathbb{F} \setminus \{0\}$ is $\langle\,\rangle$, the empty list.

EXAMPLE 8.1. *1. When σ is the identity automorphism, σ-GFF agrees with SFF.*

2. *When the polynomial ring is $\mathbb{F}[x]$ and $\sigma x = x + 1$, σ-GFF agrees, up to a constant normalization factor, with GFF as defined in (Paule, 1995).*
3. *Let $\mathbb{F} = \mathbb{K}(q)$ where $\mathbb{K}$ is a field of characteristic zero and q is transcendental over $\mathbb{K}$. When the polynomial ring is $\mathbb{F}[x]$ and $\sigma x = qx$, σ-GFF agrees (on polynomials which are not divisible by x), up to a constant normalization factor, with qGFF as defined in (Paule and Riese, 1997).*
4. *Let $\mathbb{F} = \mathbb{K}(p, q)$ where $\mathbb{K}$ is a field of characteristic zero and p, q are algebraically independent over $\mathbb{K}$. When the polynomial ring is $\mathbb{F}[x, y]$ and $\sigma x = qx$, $\sigma y = py$, σ-GFF agrees (on polynomials divisible by neither x nor y), up to a constant normalization factor, with $GFF_{p,q}$ as defined in (Riese, 1996).*
5. *When the polynomial ring is $\mathbb{F}[x, \mathbf{y}]$, $\sigma = \mathbf{E}$ as defined in Section 2, and $q_1, \ldots, q_m$ satisfy (2.1), σ-GFF provides GFF for the mixed hypergeometric case.*

Note that cases 2, 3 and 4 are all contained in case 5.

As shown by the following example, σ-GFFs are in general not unique.

EXAMPLE 8.2. *Let σ be the $\mathbb{F}$-automorphism of $\mathbb{F}[x]$ defined by $\sigma x = -x$. Obviously both $\langle 1, 1 + x \rangle$ and $\langle 1, 1 - x \rangle$ are σ-GFFs of $1 - x^2$ in this case.*

In order to have unique σ-GFF's (up to associated factors), we restrict our attention to a subclass of automorphisms σ.

Definition 8.3. *An $\mathbb{F}$-automorphism σ of $\mathbb{F}[\mathbf{x}]$ is* aperiodic *if for every irreducible $p \in \mathbb{F}[\mathbf{x}] \setminus \mathbb{F}$ either*

a) *$\sigma p \sim p$, or*

b) *$\sigma^m p \perp p$, for all $m \in \mathbb{Z} \setminus \{0\}$.*

Example 8.3. *1. The identity automorphism is aperiodic because in this case $\sigma p \sim p$ for all $p \in \mathbb{F}[\mathbf{x}]$.*

2. Let $q \in \mathbb{F}$ be a primitive m-th root of unity. Then the $\mathbb{F}$-automorphism σ of $\mathbb{F}[x]$ defined by $\sigma x = qx$ is not aperiodic because, e.g., $\sigma(x+1) = qx + 1 \not\sim x + 1$ while $\sigma^m(x+1) = q^m x + 1 \sim x + 1$.

3. If $q_1, \ldots, q_m \in \mathbb{F}$ satisfy (2.1) it follows from Lemma 2.2 that the $\mathbb{F}$-automorphism $\sigma = \mathbf{E}$ of $\mathbb{F}[x]$ as defined in Section 2 is aperiodic. This includes cases 2 – 5 of Example 8.1.

Lemma 8.1. *Assume that σ is an aperiodic $\mathbb{F}$-automorphism of $\mathbb{F}[\mathbf{x}]$. Let $q \in \mathbb{F}[\mathbf{x}]$, and let $p_1, \ldots, p_{k-1} \in \mathbb{F}[\mathbf{x}] \setminus \{0\}$, $p_k \in \mathbb{F}[\mathbf{x}] \setminus \mathbb{F}$ satisfy*

(D1) *$\sigma p_j, \sigma^{-j} p_j \perp [p_i]_\sigma^{\underline{i}}$, for any $1 \le i < j \le k$,*

(D2) *$r\sigma^j r$ does not divide p_i, for any $r \in \mathbb{F}[\mathbf{x}] \setminus \mathbb{F}$ and $1 \le j \le i \le k$.*

If $[q]_\sigma^{\underline{k}}$ divides $[p_1]_\sigma^{\underline{1}} \cdots [p_k]_\sigma^{\underline{k}}$, then q divides p_k.

Proof. Assume first that q is irreducible. As σ is aperiodic, we distinguish two cases.

a) $\sigma q \sim q$

In this case $[q]_\sigma^{\underline{k}} \sim q^k$ and so $q^k \mid [p_1]_\sigma^{\underline{1}} \cdots [p_k]_\sigma^{\underline{k}}$. Let $i \le k$ be minimal such that $q \mid p_i$. Assume that $i < k$. If there is $j > i$ such that $q \mid p_j$ then $\sigma p_j \not\perp [p_i]_\sigma^{\underline{i}}$, contrary to (D1). Otherwise $q^k \mid [p_i]_\sigma^{\underline{i}}$ and hence $q^2 \mid p_i$, contrary to (D2). So in case a) we must have $i = k$. It follows that $q \mid p_k$.

b) $\sigma^m q \perp q$, for all $m \in \mathbb{Z} \setminus \{0\}$

Let $j \le k$ be maximal such that $[q]_\sigma^{\underline{k}} \not\perp [p_j]_\sigma^{\underline{j}}$. Then there are u and v, $0 \le u < k$ and $0 \le v < j$, such that $\sigma^{-u} q \mid \sigma^{-v} p_j$. Hence $\sigma^{v-u} q \mid p_j$. We are going to prove that $u = v$ and $j = k$, by distinguishing two subcases.

b1) $u > v$

In this case $\sigma^{v-u+1} \mid [q]_\sigma^{\underline{k}} \mid [p_1]_\sigma^{\underline{1}} \cdots [p_j]_\sigma^{\underline{j}}$. If $\sigma^{v-u+1} \mid [p_l]_\sigma^{\underline{l}}$ for some $l < j$ then $\sigma p_j \not\perp [p_l]_\sigma^{\underline{l}}$, contrary to (D1). Otherwise $\sigma^{v-u+1} \mid [p_j]_\sigma^{\underline{j}}$ and so $\sigma^{v-u+1} \mid \sigma^{-w} p_j$ for some w such that $0 \le w < j$. Hence $r, \sigma^{w+1} r \mid p_j$ where $r = \sigma^{v-u} q$. As $1 \le w+1 \le j$, it follows from b) that $r \perp \sigma^{w+1} r$ whence $r\sigma^{w+1} r \mid p_j$, contrary to (D2). So this case is not possible.

b2) $u \le v$

Unless $u = v$ and $j = k$, we have $\sigma^{v-u-j} \mid [q]_\sigma^{\underline{k}} \mid [p_1]_\sigma^{\underline{1}} \cdots [p_j]_\sigma^{\underline{j}}$. If $\sigma^{v-u-j} \mid [p_l]_\sigma^{\underline{l}}$ for some $l < j$ then $\sigma^{-j} p_j \not\perp [p_l]_\sigma^{\underline{l}}$, contrary to (D1). Otherwise $\sigma^{v-u-j} \mid [p_j]_\sigma^{\underline{j}}$ and so $\sigma^{v-u-j} \mid \sigma^{-w} p_j$ for some w such that $0 \le w < j$. Hence $r, \sigma^{j-w} r \mid p_j$ where $r = \sigma^{w+v-u-j} q$. As $1 \le j - w \le j$, it follows from b) that $r \perp \sigma^{j-w} r$ whence $r\sigma^{j-w} r \mid p_j$, contrary to (D2). So in case b) we must have $u = v$ and $j = k$. It follows that $q \mid p_k$.

Finally, if q is reducible write $q = q_1 \cdots q_m$ where $m > 1$ and the q_i's are irreducible. As

$[q_m]_\sigma^{\underline{k}} \mid [q]_\sigma^{\underline{k}} \mid [p_1]_\sigma^{\underline{1}} \cdots [p_k]_\sigma^{\underline{k}}$ and q_m is irreducible, we already know that $q_m \mid p_k$. Therefore

$$[q_1 \cdots q_{m-1}]_\sigma^{\underline{k}} \mid [p_1]_\sigma^{\underline{1}} \cdots [p_{k-1}]_\sigma^{\underline{k-1}} [\frac{p_k}{q_m}]_\sigma^{\underline{k}}.$$

Clearly, replacing p_k by p_k/q_m invalidates neither (D1) nor (D2). Hence, by induction on m, $q_1 \cdots q_{m-1} \mid p_k/q_m$. It follows that $q = q_1 \cdots q_m \mid p_k$. □

We can now give a characterization of σ-GFFs which is akin to the definition of GFF in (Paule, 1995).

Corollary 8.1. *Assume that σ is an aperiodic $\mathbb{F}$-automorphism of $\mathbb{F}[\mathbf{x}]$. Let $p = [p_1]_\sigma^{\underline{1}} \cdots [p_k]_\sigma^{\underline{k}}$ where $p_1, \ldots, p_{k-1} \in \mathbb{F}[\mathbf{x}] \setminus \{0\}$ and $p_k \in \mathbb{F}[\mathbf{x}] \setminus \mathbb{F}$. Then $\langle p_1, \ldots, p_k \rangle$ is a σ-GFF for p if and only if $p_1, \ldots, p_k$ satisfy conditions (D1) and (D2) of Lemma 8.1.*

Proof. $(\Rightarrow)$ Let $\langle p_1, \ldots, p_k \rangle$ be a σ-GFF for p.

(D1) Assume that $q \mid [p_i]_\sigma^{\underline{i}}$ where $q \in \mathbb{F}[\mathbf{x}] \setminus \mathbb{F}$ is irreducible and $1 \le i < j \le k$. If $q \mid \sigma p_j$ then $[\sigma^{-1} q]_\sigma^{\underline{j}} \mid [p_j]_\sigma^{\underline{j}}$, so $[q]_\sigma^{\underline{j+1}} \mid [p_i]_\sigma^{\underline{i}} [p_j]_\sigma^{\underline{j}}$. If $q \mid \sigma^{-j} p_j$ then $[\sigma^j q]_\sigma^{\underline{j}} \mid [p_j]_\sigma^{\underline{j}}$, so $[\sigma^j q]_\sigma^{\underline{j+1}} \mid [p_i]_\sigma^{\underline{i}} [p_j]_\sigma^{\underline{j}}$. In either case $\sigma\text{-span}\left([p_1]_\sigma^{\underline{1}} \cdots [p_j]_\sigma^{\underline{j}}\right) \ge j+1$, contrary to ($\sigma$-GFF3).

(D2) Assume that $q\,\sigma^j q \mid p_i$ for some irreducible $q \in \mathbb{F}[\mathbf{x}] \setminus \mathbb{F}$ and $1 \le j \le i \le k$. Then $[q]_\sigma^{\underline{i}} [\sigma^j q]_\sigma^{\underline{i}} \mid [p_i]_\sigma^{\underline{i}}$. But $\sigma q \mid [\sigma^j q]_\sigma^{\underline{i}}$, so $[\sigma q]_\sigma^{\underline{i+1}} \mid [p_i]_\sigma^{\underline{i}}$. Hence $\sigma\text{-span}\left([p_1]_\sigma^{\underline{1}} \cdots [p_i]_\sigma^{\underline{i}}\right) \ge i+1$, contrary to ($\sigma$-GFF3).

$(\Leftarrow)$ Let $p_1, \ldots, p_k$ satisfy conditions (D1) and (D2) of Lemma 8.1. To prove that they satisfy (σ-GFF3) as well, assume that $p_i \in \mathbb{F}[\mathbf{x}] \setminus \mathbb{F}$ and $\sigma\text{-span}\left([p_1]_\sigma^{\underline{1}} \cdots [p_i]_\sigma^{\underline{i}}\right) = s > i$, for some i such that $1 \le i \le k$. Then there is an irreducible $q \in \mathbb{F}[\mathbf{x}] \setminus \mathbb{F}$ such that $[q]_\sigma^{\underline{s}} \mid [p_1]_\sigma^{\underline{1}} \cdots [p_i]_\sigma^{\underline{i}}$. Hence $[q]_\sigma^{\underline{i}} \mid [p_1]_\sigma^{\underline{1}} \cdots [p_i]_\sigma^{\underline{i}}$ and $[\sigma^{-1} q]_\sigma^{\underline{i}} \mid [p_1]_\sigma^{\underline{1}} \cdots [p_i]_\sigma^{\underline{i}}$. By Lemma 8.1, $q \mid p_i$ and $\sigma^{-1} q \mid p_i$. We distinguish two cases.

a) $\sigma q \sim q$

If there is $j < i$ such that $q \mid p_j$ then $\sigma p_i \not\perp [p_j]_\sigma^{\underline{j}}$, contrary to (D1). Otherwise $q^s \mid [p_i]_\sigma^{\underline{i}}$. As $s > i$, it follows that $q^2 \mid p_i$, contrary to (D2).

b) $\sigma^m q \perp q$, for all $m \in \mathbb{Z} \setminus \{0\}$

In this case $q \perp \sigma^{-1} q$ and so $q\,\sigma^{-1} q \mid p_i$, contrary to (D2). □

Example 8.4. *Let $q \in \mathbb{F}$ be a primitive third root of unity, and let σ be the $\mathbb{F}$-automorphism of $\mathbb{F}[x]$ defined by $\sigma x = qx$. The polynomials $p_1 = 1+x$, $p_2 = 1$, $p_3 = 1+qx$ satisfy conditions (D1) and (D2) of Lemma 8.1. However, $\langle p_1, p_2, p_3 \rangle$ is not a σ-GFF of $p = [p_1]_\sigma^{\underline{1}} [p_2]_\sigma^{\underline{2}} [p_3]_\sigma^{\underline{3}}$ because $p = [1+x]_\sigma^{\underline{4}}$ and $\sigma\text{-span}(p) = 4$. This example shows that Corollary 8.1 can fail when σ is not aperiodic.*

The next corollary shows uniqueness of σ-GFFs (up to associated factors).

Corollary 8.2. *Assume that σ is an aperiodic $\mathbb{F}$-automorphism of $\mathbb{F}[\mathbf{x}]$. If $\langle p_1, \ldots, p_k \rangle$ and $\langle q_1, \ldots, q_l \rangle$ are σ-GFFs for the same $p \in \mathbb{F}[\mathbf{x}]$, then $k = l$ and $p_i \sim q_i$ for $1 \le i \le k$.*

Proof. By (σ-GFF3), $k = \sigma\text{-span}(p) = l$.

We prove the rest of the corollary by induction on k.

If $k = 0$ the assertion holds vacuously. Let $k > 0$. By Corollary 8.1, the p_i's as well as the q_i's satisfy conditions (D1) and (D2) of Lemma 8.1. Therefore $p_k \mid q_k$ and $q_k \mid p_k$, hence $p_k \sim q_k$ and $[p_1]_\sigma^{\underline{1}} \cdots [p_{k-1}]_\sigma^{\underline{k-1}} \sim [q_1]_\sigma^{\underline{1}} \cdots [q_{k-1}]_\sigma^{\underline{k-1}}$. Let $r = [p_1]_\sigma^{\underline{1}} \cdots [p_{k-1}]_\sigma^{\underline{k-1}}$ and $m = \sigma\text{-span}(r)$. Then $\langle p_1, \ldots, p_m \rangle$ and $\langle q_1, \ldots, q_m \rangle$ are σ-GFFs for r, and $p_i = q_i = 1$ for $m < i < k$. By the inductive hypothesis, $p_i \sim q_i$ for $1 \leq i \leq m$. □

Finally, the formulæ

$$
\begin{aligned}
\gcd(p, \sigma p) &\sim [p_1]_\sigma^{\underline{0}} \cdots [p_k]_\sigma^{\underline{k-1}} \gcd(p_1, \sigma p_1) \cdots \gcd(\sigma^{-k+1} p_k, \sigma p_k), \\
\frac{p}{\gcd(p, \sigma^{-1} p)} &\sim \frac{p_1 p_2 \cdots p_k}{\gcd(p_1, \sigma^{-1} p_1) \cdots \gcd(p_k, \sigma^{-k} p_k)}, \\
\frac{p}{\gcd(p, \sigma p)} &\sim \frac{p_1 \sigma^{-1} p_2 \cdots \sigma^{-k+1} p_k}{\gcd(p_1, \sigma p_1) \cdots \gcd(\sigma^{-k+1} p_k, \sigma p_k)}.
\end{aligned}
$$

can be proved in much the same way as the corresponding formulæ in (Paule, 1995).

We remark that analogously to the hypergeometric, basic and bibasic cases, σ-GFF could be used to derive and explain Gosper's algorithm in the mixed hypergeometric case.

9. Concluding remarks

We have shown how to compute the hypergeometric canonical form of rational functions, how to perform Gosper's algorithm, and how to find polynomial as well as hypergeometric solutions of recurrences, all in the mixed hypergeometric case. We have also indicated how to extend the concept of GFF to this case.

It remains to provide mixed hypergeometric generalizations of algorithms for finding rational solutions of recurrences (Abramov, 1995; van Hoeij, 1998a) and of algorithms for factorization of the corresponding operators (Bronstein and Petkovšek, 1996). The more efficient algorithm of van Hoeij for finding hypergeometric solutions (van Hoeij, 1998b) should also admit of a generalization to the mixed hypergeometric case.

Acknowledgements

The authors wish to thank Bill Gosper for providing the interesting tribasic and quadbasic examples just in the nick of time. They also thank an anonymous referee for very careful reading of a preliminary version as well as for providing additional references to the relevant literature.

A. Algorithm `MixedPoly`

`INPUT:` $p_0, \ldots, p_\rho, g, h_1, \ldots, h_s \in \mathbb{F}[x, \mathbf{y}]$, $p_0, p_\rho \neq 0$

`OUTPUT:` general solution $(f, \boldsymbol{\lambda}) \in \mathbb{F}[x, \mathbf{y}] \times \mathbb{F}^s$ of $\mathbf{L} f = g + \sum_{j=1}^s \lambda_j h_j$ where $\mathbf{L} = \sum_{i=0}^\rho p_i \cdot \mathbf{E}^i$

`CALLING SEQUENCE:` `MixedPoly`(eqn, unknown, params)

`EXTERNAL ALGORITHMS USED:`

`Poly`$(e, t, \boldsymbol{\lambda})$ returns general solution $(t, \boldsymbol{\lambda})$ of the parametric nonhomogeneous ordinary recurrence e (see (Abramov *et al.*, 1995))
`LinSolve`$(e, x, \boldsymbol{\lambda})$ returns general solution $\boldsymbol{\lambda}$ of the linear algebraic equations resulting from equating the coefficients of like powers of x on both sides of the polynomial equation e

$\boldsymbol{\alpha} := \max_{0\le i\le\rho} \deg_{\mathbf{y}} p_i$
$p_{i,\boldsymbol{\alpha}}(x) := [\mathbf{y}^{\boldsymbol{\alpha}}]\, p_i$
$d := \max_{0\le i\le\rho} \deg_x p_{i,\boldsymbol{\alpha}}(x)$
$p_{i,\boldsymbol{\alpha},d} := [x^d]\, p_{i,\boldsymbol{\alpha}}(x)$
$P(x) := \sum_{i=0}^{\rho} p_{i,\boldsymbol{\alpha},d} x^i$
$\mathcal{R} := \{\boldsymbol{\sigma} \in \mathbb{N}_0^m;\ P(\mathbf{q}^{\boldsymbol{\sigma}}) = 0\}$
$\mathrm{rhs} := g + \sum_{j=1}^{s} \lambda_j h_j$
$f := 0$
`while` $\mathcal{R} \ne \emptyset$ `or` $\mathrm{rhs} \ne 0$ `do`
  `if` $\mathcal{R} \ne \emptyset$ `then` $\boldsymbol{\mu} := \max_{\prec} \mathcal{R}$ `else` $\boldsymbol{\mu} := \perp$
  `if` $\mathrm{rhs} \ne 0$ `then` $\boldsymbol{\beta} := \deg_{\mathbf{y}} \mathrm{rhs}$ `else` $\boldsymbol{\beta} := \perp$
  `if` $\boldsymbol{\alpha} + \boldsymbol{\mu} \succ \boldsymbol{\beta}$ `then`
    $\boldsymbol{\varphi} := \boldsymbol{\mu}$
    $(t', \boldsymbol{\lambda}') :=$ `Poly`$(\sum_{i=0}^{\rho} p_{i,\boldsymbol{\alpha}}(x)\mathbf{q}^{i\boldsymbol{\varphi}} t(x+i) = 0, t, \boldsymbol{\lambda})$
    $f := f + t'\mathbf{y}^{\boldsymbol{\varphi}}$
    $\mathrm{rhs} := \mathrm{rhs}|_{\boldsymbol{\lambda}\leftarrow\boldsymbol{\lambda}'} - \mathbf{L}\,(t'\mathbf{y}^{\boldsymbol{\varphi}})$
    $\mathcal{R} := \mathcal{R} \setminus \{\boldsymbol{\varphi}\}$
  `end`
  `else if` $\boldsymbol{\alpha} \subseteq \boldsymbol{\beta}$ `then`
    $\boldsymbol{\varphi} := \boldsymbol{\beta} - \boldsymbol{\alpha}$
    $(t', \boldsymbol{\lambda}') :=$ `Poly`$(\sum_{i=0}^{\rho} p_{i,\boldsymbol{\alpha}}(x)\mathbf{q}^{i\boldsymbol{\varphi}} t(x+i) = [\mathbf{y}^{\boldsymbol{\beta}}]\,\mathrm{rhs}, t, \boldsymbol{\lambda})$
    $f := f + t'\mathbf{y}^{\boldsymbol{\varphi}}$
    $\mathrm{rhs} := \mathrm{rhs}|_{\boldsymbol{\lambda}\leftarrow\boldsymbol{\lambda}'} - \mathbf{L}\,(t'\mathbf{y}^{\boldsymbol{\varphi}})$
    `if` $\boldsymbol{\varphi} = \boldsymbol{\mu}$ `then` $\mathcal{R} := \mathcal{R} \setminus \{\boldsymbol{\varphi}\}$
  `end`
  `else`
    $\boldsymbol{\lambda}' :=$ `LinSolve`$([\mathbf{y}^{\boldsymbol{\beta}}]\,\mathrm{rhs} = 0, x, \boldsymbol{\lambda})$
    $\mathrm{rhs} := \mathrm{rhs}|_{\boldsymbol{\lambda}\leftarrow\boldsymbol{\lambda}'}$
  `end`
`end`
`return` f.

NB: If either `Poly` or `LinSolve` fails then `MixedPoly` fails as well.

B. Algorithm `CanonicalForm`

`INPUT:` $r \in \mathbb{F}(x, \mathbf{y}) \setminus \{0\}$
`OUTPUT:` canonical form of r
`EXTERNAL ALGORITHMS USED:`
  `Resultant`(a, b, x) returns the resultant of polynomials a, b w.r.t. x
  `GCD`(a, b) returns the mixed monic gcd of polynomials a, b

```
let r = (u/v) · (a_0/b_0) where a_0 b_0 ⊥ M, a_0 u ⊥ b_0 v, u,v ∈ M, and b_0 is
mixed monic
P(ξ,η) := Resultant(a(x,y), b(x+ξ, ηy), x)
for i = 1,...,m do
    P(ξ,η) := P(ξ,η)·Resultant(a(x,y), b(x+ξ, ηy), y_i)
end
let h_1 < h_2 < ··· < h_N be the roots h ∈ N_0 of p(h) = P(h, q^h)
c_0 := 1
for i = 1,...,N do
    s_i := GCD(a_{i-1}, E^{h_i} b_{i-1})
    a_i := a_{i-1}/s_i
    b_i := b_{i-1}/E^{-h_i} s_i
    c_i := c_{i-1} ∏_{j=1}^{h_i} E^{-j} s_i
end
a := u · a_N
b := v · b_N
c := c_N
return (a,b,c).
```

C. Algorithm MixedGosper

```
INPUT: mixed hypergeometric sequence t_n
OUTPUT: mixed hypergeometric sequence s_n such that s_{n+1} - s_n = t_n, if it
exists

r := Φ(t_{n+1}/t_n) (see Theorem 2.1)
(a,b,c) := CanonicalForm(r)
u := 1
for i = 1 to m do
    if π_i(ab) = 0
    then
        e_i := 0
    else
        (a_i,b_i,c_i) := CanonicalForm(π_i(b/a))
        if ∃v,w ∈ M : (v ⊥ w and a_i/b_i = v(q)/w(q))
        then
            e_i := deg_{y_i} w
        else e_i := 0
    end
    u := u · y_i^{e_i}
end
(p,⟨⟩) := MixedPoly(a · Ep − u(q) · E^{-1}b · p = u(q)u · c, p, ⟨⟩)
R := (p · E^{-1}b)/(u · c)
return Φ^{-1}(R) · t_n.
```

NB: If MixedPoly fails then MixedGosper fails as well.

D. Algorithm `MixedHyper`

INPUT: $p_0, \dots, p_\rho \in \mathbb{F}[x, \mathbf{y}]$, $p_0, p_\rho \neq 0$
OUTPUT: mixed hypergeometric solution f of $\sum_{i=0}^{\rho} p_i \cdot \mathbf{E}^i f = 0$, if it exists

for all mixed monic factors a of p_0 and b of $\mathbf{E}^{1-\rho} p_\rho$ do
  for $i = 1$ to ρ do $P_i := p_i \prod_{j=0}^{i-1} \mathbf{E}^j a \prod_{j=i}^{\rho-1} \mathbf{E}^j b$
  $\boldsymbol{\alpha} := \min_{0 \le i \le \rho} \mathrm{mindeg}_{\mathbf{y}} P_i$
  for $i = 0$ to ρ do $p_{i,\boldsymbol{\alpha}} := [\mathbf{y}^{\boldsymbol{\alpha}}] P_i$
  $d := \max_{0 \le i \le \rho} \deg_x p_{i,\boldsymbol{\alpha}}$
  for $i = 0$ to ρ do $p_{i,\boldsymbol{\alpha},d} := [x^d] p_{i,\boldsymbol{\alpha}}$
  for all τ such that $\sum_{i=1}^{\rho} p_{i,\boldsymbol{\alpha},d} \tau^i = 0$ do
    for $j = 1$ to m do
      order terms lexicographically with y_j first
      $\boldsymbol{\alpha}_j := \min_{0 \le i \le \rho} \mathrm{mindeg}_{\mathbf{y}} P_i$
      for $i = 0$ to ρ do $p_{i,\boldsymbol{\alpha}_j} := [\mathbf{y}^{\boldsymbol{\alpha}_j}] P_i$
      $d_j := \max_{0 \le i \le \rho} \deg_x p_{i,\boldsymbol{\alpha}_j}$
      for $i = 0$ to ρ do $p_{i,\boldsymbol{\alpha}_j,d_j} := [x^d] p_{i,\boldsymbol{\alpha}_j}$
      let $\sigma_1, \dots, \sigma_s$ be the roots of $\sum_{i=0}^{\rho} p_{i,\boldsymbol{\alpha}_j,d_j} \sigma^i = 0$
      let b_j be the minimum of the j-th components of $\sigma_1, \dots, \sigma_s$
    $u := y_1^{b_1} \cdots y_m^{b_m}$
    $(\bar{d}, \langle\,\rangle) :=$ MixedPoly$(\sum_{i=0}^{\rho} \tau^i u(\mathbf{q})^i P_i \cdot \mathbf{E}^i \bar{d} = 0, \bar{d}, \langle\,\rangle)$
    if $\bar{d} \neq 0$ then
      $r := \tau u(\mathbf{q}) \frac{a}{b} \frac{\mathbf{E}\bar{d}}{\bar{d}}$
      return a non-zero solution f of $\mathbf{E} f = r f$ and stop
    end
  end
end
stop.

NB: If the original term order is lexicographic with y_1 as the first variable, then in the innermost loop j runs from 2 to m and we set $b_1 = 0$.

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