Sheaf Toposes for Realizability

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Abstract

We compare realizability models over partial combinatory algebras by embedding them into sheaf toposes. We then use the machinery of Grothendieck toposes and geometric morphisms to study the relationship between realizability models over different partial combinatory algebras. This research is part of the Logic of Types and Computation project at Carnegie Mellon University under the direction of Dana Scott.

1 Introduction

The purpose of this work is to compare realizability models over partial combinatory algebras by embedding them into sheaf toposes. We then use the machinery of Grothendieck toposes and geometric morphisms to study the relationship between realizability models over different partial combinatory algebras. This work is related to Rosolini and Streicher [RS99], where the focus was mainly on the locally cartesian closed structure of realizability models. Here we are also interested in comparison of logical properties.

As a reference on topos theory we use [MM92], and Johnstone and Moerdijk [JM89] as a reference on local map of toposes. Longley's dissertation [Lon94] contains material on partial combinatory algebras, applicative morphisms, and realizability. In Birkedal's dissertation [Bir00] and in

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[ABS99] you can find further information about realizability, and also on local maps of toposes and the corresponding $\sharp$-$\flat$ calculus.

2 Sheaves on Partial Combinatory Algebras

We quickly review the basic definitions about partial combinatory algebras and applicative morphisms. For motivation and examples of PCAs and applicative morphisms, see [Lon94]. For a partial function $f\colon A \rightharpoonup B$ and $x \in A$, the notation $fx\downarrow$ means that fx is defined. When t is a term, $t\downarrow$ means that the value of t, and hence of all of its subterms, is defined.

Definition 2.1 A *partial combinatory algebra (PCA)* is a set $\mathbb{A}$ with a partial application operation $\square \cdot \square\colon \mathbb{A} \times \mathbb{A} \rightharpoonup \mathbb{A}$ and distinguished elements $\mathsf{K}, \mathsf{S} \in \mathbb{A}$ such that, for all $x, y, z \in \mathbb{A}$,

$$\mathsf{K}xy \simeq x\ , \qquad \mathsf{S}xy\downarrow\ , \qquad \mathsf{S}xyz \simeq (xz)(yz)\ ,$$

where $\simeq$ means that if one side is defined then so it the other and they are equal. We usually write xy instead of $x \cdot y$, and associate application to the left. Here we only consider non-trivial PCAs that satisfy $\mathsf{K} \neq \mathsf{S}$.

Example 2.2 Scott's *graph model* $\mathbb{P} = \mathcal{P}\mathbb{N}$, [Sco76], is a model of the untyped λ-calculus, therefore a (total) combinatory algebra. The algebraic lattice of continuous maps $\mathbb{P}^{\mathbb{P}}$ is a retract of the algebraic lattice $\mathbb{P}$. The embedding $\Gamma\colon \mathbb{P}^{\mathbb{P}} \to \mathbb{P}$ is defined by

$$\Gamma f = \left\{ \langle m, n\rangle \;\middle|\; n \in \mathbb{N} \text{ and } m \in f(\mathsf{finset}\, n) \right\}\ ,$$

where $\langle m, n\rangle$ is an effective pairing of natural numbers and $\mathsf{finset}\, n$ is an effective enumeration of finite subsets of $\mathbb{N}$. The set Γf is called the *graph* of f. For a continuous map $f\colon \mathbb{P} \to \mathbb{P}$ the graph Γf uniquely determines f because the value of f at any element $x \in \mathbb{P}$ is the union of values of f at finite subsets of x. The retraction $\Lambda\colon \mathbb{P} \to \mathbb{P}^{\mathbb{P}}$ is defined by

$$(\Lambda x)y = \left\{ m \in \mathbb{N} \;\middle|\; \exists\, n \in \mathbb{N}\,.\,(\langle m, n\rangle \in x \text{ and } \mathsf{finset}\, n \subseteq y) \right\}.$$

Define application on $\mathbb{P}$ by

$$x \cdot y = (\Lambda x)y\ ,$$

and λ-abstraction by

$$\lambda u.\, \phi(u) = \Gamma(\lambda x{:}\mathbb{P}\,.\,\phi(x))\ .$$

Here $\phi(u)$ is an expression with u occurring as a free variable, and involving further λ-abstractions and applications of elements of $\mathbb{P}$. By taking $\mathsf{K} = \lambda xy.\, x$ and $\mathsf{S} = \lambda xyz.\, (xz)(yz)$, $\mathbb{P}$ becomes a total combinatory algebra for the above application operation.

Example 2.3 The *second Kleene Algebra* $\mathbb{B} = \mathbb{N}^{\mathbb{N}}$, or the Baire space, is a partial combinatory algebra. The *continuous partial application* is defined as follows. If $\alpha \in \mathbb{N}^{\mathbb{N}}$, let $\overline{\alpha}n$ be the sequence $[\alpha 0, \dots, \alpha(n-1)]$, encoded reasonably as a natural number. Define the operation $\star$ by

$$\alpha \star \beta = n \iff \exists\, m \in \mathbb{N}\,.\, \big(\alpha(\overline{\beta}m) = n + 1 \land \forall\, k < m\,.\, \alpha(\overline{\beta}k) = 0\big) \ .$$

Now the partial application on $\mathbb{B}$ is defined by

$$(\alpha \mid \beta)n = \alpha \star (n{::}\beta) \ ,$$

where $n{::}\beta$ is the sequence obtained by prepending n to β. We consider $\alpha | \beta$ to be defined only if $(\alpha \mid \beta)n$ is defined for all $n \in \mathbb{N}$. For details about the PCA structure of $\mathbb{B}$, see [KV65].

Definition 2.4 (John Longley) Let $\mathbb{E}$ and $\mathbb{F}$ be PCAs. An *applicative morphism* $\rho \colon \mathbb{E} \xrightarrow{\mathsf{PCA}} \mathbb{F}$ is a total relation $\rho \subseteq \mathbb{E} \times \mathbb{F}$ for which there exists $r \in \mathbb{F}$, such that $rx\!\downarrow$ for all $x \in \mathbb{F}$, and for all $u, v \in \mathbb{E}$, $x, y \in \mathbb{F}$, if $\rho(u, x)$, $\rho(v, y)$ and $uv\!\downarrow$, then $rxy\!\downarrow$ and $\rho(uv, rxy)$. We say that r is a *realizer* for the applicative morphism ρ. The *composition* of applicative morphisms is the usual composition of relations. An applicative morphism is *discrete* when $\rho(u, x)$ and $\rho(v, x)$ implies $u = v$.

Suppose $\rho, \sigma \colon \mathbb{E} \xrightarrow{\mathsf{PCA}} \mathbb{F}$ are applicative morphisms. We say there is an *applicative transformation* from ρ to σ, and write $\rho \preceq \sigma$, if there exists $t \in \mathbb{F}$ such that whenever $\rho(u, x)$ then $\sigma(u, tx)$. When $\rho \preceq \sigma$ and $\sigma \preceq \rho$ we write $\rho \sim \sigma$.

Example 2.5 A discrete applicative morphism $\iota \colon \mathbb{B} \xrightarrow{\mathsf{PCA}} \mathbb{P}$ can be obtained by embedding $\mathbb{B}$ into $\mathbb{P}$ via $\iota a = \big\{\overline{a}n \bigm| n \in \mathbb{N}\big\}$, where $\overline{a}n \in \mathbb{N}$ is the sequence $[a0, \dots, a(n-1)]$ suitably encoded as a natural number. Note that ι is actually a function, i.e., a single-valued total relation. There is also a discrete applicative morphism $\delta \colon \mathbb{P} \xrightarrow{\mathsf{PCA}} \mathbb{B}$, defined by

$$\delta(x, a) \iff x = \big\{n \in \mathbb{N} \bigm| \exists\, k \in \mathbb{N}\,.\, ak = n + 1\big\} \ .$$

In words, a sequence a is a δ-implementation of x if it enumerates the elements of x. The trick with adding 1 ensures that the empty set is accounted for.

Applicative morphisms induce functors between realizability toposes. We only consider *discrete* applicative morphisms here, because they induce functors between categories of modest sets.

Definition 2.6 Let $\mathbb{A}$ be a PCA. A *modest set* $(S, \Vdash_S)$ over $\mathbb{A}$ is a set S with a *realizability relation* $\Vdash_S \subseteq \mathbb{A} \times S$ such that for every $x \in S$ there exists $a \in \mathbb{A}$ such that $a \Vdash_S x$, and for all $a \in \mathbb{A}$, $x, y \in S$,

$$(a \Vdash_S x) \land (a \Vdash_S y) \Longrightarrow x = y \ .$$

A *realized function* $f \colon (S, \Vdash_S) \to (T, \Vdash_T)$ between modest sets is a function $f \colon S \to T$ that is *tracked* by some $a \in \mathbb{A}$, which means that, for all $x \in S$,

$$(b \Vdash_S x) \Longrightarrow ab\!\downarrow \land (ab \Vdash_T fx) \ .$$

The category of modest sets and realized functions is denoted by $\mathsf{Mod}(\mathbb{A})$.

The category of modest sets $\mathsf{Mod}(\mathbb{A})$ is equivalent to the perhaps better known category $\mathsf{PER}(\mathbb{A})$ of partial equivalence relations on $\mathbb{A}$.

Example 2.7 The category $\mathsf{Mod}(\mathbb{P})$ is equivalent to the category of *countably based equilogical spaces* [BBS98]. A countably based equilogical space is a pair $(X, \equiv_X)$ where X is a countably based T_0-space and $\equiv$ is an equivalence relation on X. A morphism $[f] \colon (X, \equiv_X) \to (Y, \equiv_Y)$ is an equivalence class of equivalence preserving continuous maps. Two such maps are considered equivalent when they map equivalent points to equivalent points.

Example 2.8 The category $\mathsf{Mod}(\mathbb{B})$ is the ambient category of *Type Two Effectivity* [Wei00]. It is equivalent to 0-*equilogical spaces* [Bau00], which is the full subcategory on those countably based equilogical spaces whose underlying topological space is 0-dimensional.

Any discrete applicative morphism $\rho \colon \mathbb{E} \xrightarrow{\mathsf{PCA}} \mathbb{F}$ induces a functor

$$\widehat{\rho} \colon \mathsf{Mod}(\mathbb{E}) \longrightarrow \mathsf{Mod}(\mathbb{F}) \ ,$$

defined as follows. A modest set $(S, \Vdash_S)$ over $\mathbb{E}$ is mapped to $\widehat{\rho}(S, \Vdash_S) = (S, \Vdash_{\widehat{\rho}S})$ where $\Vdash_{\widehat{\rho}S}$ is defined by

$$a \Vdash_{\widehat{\rho}S} x \iff \exists\, b \in \mathbb{E} \,.\, ((b \Vdash_S x) \land \rho(b, a)) \ .$$

A realized map $f \colon (S, \Vdash_S) \to (T, \Vdash_T)$ is mapped to the same map $\widehat{\rho}f = f \colon S \to T$. To see that $\widehat{\rho}f$ is realized, let $r \in \mathbb{F}$ be a realizer for ρ, and let $a \in \mathbb{E}$ be a realizer for f. Because ρ is total there exists $b \in \mathbb{F}$ such that $\rho(a, b)$, and rb is a realizer for $\widehat{\rho}f$.

2.1 Sheaves over a PCA

We would like to embed $\mathsf{Mod}(\mathbb{A})$ into a sheaf topos. An obvious choice is the topos of sheaves for a subcanonical Grothendieck topology on $\mathsf{Mod}(\mathbb{A})$, which is generated by suitable regular epimorphic families. There is an equivalent but much simpler description of this topos, which we look at next.

For the site we take the category $\langle\mathbb{A}\rangle$ whose objects are subsets of $\mathbb{A}$, and morphisms are the realized maps between subsets of $\mathbb{A}$. More precisely, if $X, Y \subseteq \mathbb{A}$ then a function $f\colon X \to Y$ is a morphism if there exists $a \in \mathbb{A}$ such that, for all $b \in X$, $ab\downarrow$ and $fb = ab$.

In many cases $\langle\mathbb{A}\rangle$ is equivalent to a well known category. For example, $\langle\mathbb{P}\rangle$ is equivalent to the category $\omega\mathsf{Top}_0$ of countably based T_0-spaces, whereas $\langle\mathbb{B}\rangle$ is equivalent to the category $\mathsf{0Dim}$ of countably based 0-dimensional Hausdorff spaces.

In order to obtain a convenient description of the Grothendieck topology on $\langle\mathbb{A}\rangle$ we need to know precisely which coproducts exist in $\langle\mathbb{A}\rangle$. This is the purpose of the following lemma. Let us call a family $(X_i)_{i\in I}$ *non-trivial* if none of the objects X_i are initial. We are only interested in coproducts of non-trivial families because we can always omit any summands that are initial. The cardinality of a set I is denoted by $|I|$.

Lemma 2.9 *Let $\mathbb{A}$ be a PCA. There exists a regular cardinal number $\mathsf{w}(\mathbb{A})$, called the* weight of $\mathbb{A}$, *such that the coproduct of a non-trivial family $(X_i)_{i\in I}$ exists in $\langle\mathbb{A}\rangle$ if, and only if, $|I| < \mathsf{w}(\mathbb{A})$.*

Proof. We prove the following statement: if a coproduct of a non-trivial family $(X_i)_{i\in I}$ exists in $\langle\mathbb{A}\rangle$, then coproducts of all I-indexed families exist. We then take as the weight of $\mathbb{A}$ the smallest cardinal κ for which there exists a non-trivial κ-indexed family such that its coproduct does not exist in $\langle\mathbb{A}\rangle$.

Let $(X_i)_{i\in I}$ be a non-trivial family and let $C = \coprod_{i\in I} X_i$ be its coproduct in $\langle\mathbb{A}\rangle$, with canonical injections $e_i\colon X_i \to C$ that are realized by $r_i \in \mathbb{A}$, for each $i \in I$. We denote the image of X_i under e_i by $e_i(X_i)$.

First we show that the coproduct of $(1)_{i\in I}$ exists. An object in $\langle A\rangle$ is initial if, and only if, it is the empty set. Because $(X_i)_{i\in I}$ is non-trivial the sets $e_i(X_i)$ are non-empty, hence there exists a choice function $c\colon I \to \mathbb{A}$ such that $c_i \in e_i(X_i)$ for all $i \in I$. The function c is injective because $e_i(X_i) \cap e_j(X_j) = \emptyset$ when $i \neq j$. We show that $S = \{c_i \mid i \in I\}$ is a coproduct of the family $(1)_{i\in I}$ with the i-th canonical injection realized by $\mathsf{K}c_i$. It suffices to find a realizer for an arbitrary function $f\colon S \to \mathbb{A}$. Let $g_i\colon X_i \to \mathbb{A}$ be the morphism realized by $\mathsf{K}(f(c_i))$. Then there exists a unique morphism

$g\colon C \to \mathbb{A}$, realized by some $s \in \mathbb{A}$, such that $g(e_i(x)) = g_i(x) = f(c_i)$ for all $i \in I$ and all $x \in X_i$. In particular, $f(c_i) = g(c_i) = sc_i$ for all $i \in I$, hence s realizes f.

Now let $(Y_i)_{i\in I}$ be an arbitrary I-indexed family. We claim that the set $T = \{\langle c_i, u\rangle \mid i \in I \wedge u \in Y_i\}$ is a coproduct of $(Y_i)_{i\in I}$ with the i-th canonical injection realized by $\lambda^* u.\, \langle c_i, u\rangle$. Let $(g_i\colon Y_i \to Z)_{i\in I}$ be a cocone, where g_i is realized by $t_i \in \mathbb{A}$ for every $i \in I$. We only need to show that the map $\langle c_i, u\rangle \mapsto g_i(u)\colon T \to Z$ is realized. Define $f\colon S \to \mathbb{A}$ by $f(c_i) = t_i$. Because S is a coproduct of 1's f is realized by some $s \in \mathbb{A}$. The map $\langle c_i, u\rangle \mapsto g_i(u)$ is then realized by $\lambda^* w.\, (s(\mathsf{fst}\, w))(\mathsf{snd}\, w)$.

It remains to show that $\mathsf{w}(\mathbb{A})$ is a regular cardinal. It is necessarily infinite because $\langle\mathbb{A}\rangle$ always has all finite coproducts. By [Jec97, Lemma 3.6], it suffices to show that $\sum_{i<\lambda} \kappa_i < \mathsf{w}(\mathbb{A})$ whenever $\lambda < \mathsf{w}(\mathbb{A})$ and $\kappa_i < \mathsf{w}(\mathbb{A})$ for all $i < \lambda$. For every $i < \lambda$, the coproduct $K_i = \coprod_{j<\kappa_i} 1$ exists in $\langle\mathbb{A}\rangle$ because $\kappa_i < \mathsf{w}(\mathbb{A})$. Similarly, the coproduct $L = \coprod_{i<\lambda} K_i$ exists. But L is also a coproduct of $\sum_{i<\lambda} \kappa_i$ many copies of 1, therefore $\sum_{i<\lambda} \kappa_i < \mathsf{w}(\mathbb{A})$. ∎

The name *weight* of a PCA is motivated by the fact that in a topological PCA it often happens that its weight is the successor of its topological weight, which is the cardinality of a minimal base for its topology. For example, the topological weights of $\mathbb{P}$ and $\mathbb{B}$ are $\aleph_0$ and their weights are $\aleph_1$ because $\langle\mathbb{P}\rangle \simeq \omega\mathsf{Top}_0$ and $\langle\mathbb{B}\rangle \simeq \mathsf{0Dim}$ have precisely all countable coproducts. The first Kleene algebra has weight $\aleph_0$. PCAs built as syntactic models of the untyped λ-calculus typically have weight $\aleph_0$ because the corresponding sites have precisely all the finite coproducts.

Corollary 2.10 *Let $\mathbb{A}$ be a PCA. In $\mathsf{Mod}(\mathbb{A})$, the coproduct of a non-trivial family $(X_i)_{i\in I}$ exists if, and only if, $|I| < \mathsf{w}(\mathbb{A})$.*

Proof. The category $\langle\mathbb{A}\rangle$ is the full subcategory of $\mathsf{Mod}(\mathbb{A})$ on the *canonically separated* modest sets, which in turn is equivalent to the full subcategory $\mathsf{Proj}(\mathbb{A})$ on the regular projective modest sets over $\mathbb{A}$. It is easy to check that the inclusion of $\mathsf{Proj}(\mathbb{A})$ into $\mathsf{Mod}(\mathbb{A})$ preserves and reflects all coproducts that exist. By combining this with the fact that every modest set is covered by a regular epi whose domain is regular projective, we get the desired result. ∎

As the Grothendieck topology on $\langle\mathbb{A}\rangle$ we take the *coproduct topology* C which is generated by those families $\{f_i\colon Y_i \to X\}_{i\in I}$ for which $|I| < \mathsf{w}(\mathbb{A})$ and the morphism $[f_i]_{i\in I}\colon \coprod_{i\in I} Y_i \to X$ is an isomorphism.

Definition 2.11 The category of sheaves on $\langle\mathbb{A}\rangle$ for the coproduct topology is denoted by $\mathsf{Sh}(\mathbb{A})$.

Observe that the sheaves on $\langle\mathbb{A}\rangle$ are simply those presheaves P that "preserve products", i.e., $P(\coprod_i Y_i) \cong \prod_i PY_i$.

Theorem 2.12 *The category* $\mathsf{Sh}(\mathbb{A})$ *is equivalent to the category of sheaves* $\mathsf{Sh}(\mathsf{Mod}(\mathbb{A}), R)$ *for the subcanonical Grothendieck topology* R *generated by those families* $\{f_i\colon B_i \to A\}_{i\in I}$ *for which* $|I| < \mathsf{w}(\mathbb{A})$ *and the morphism* $[f_i]_{i\in I}\colon \coprod_{i\in I} B_i \to A$ *is a regular epi.*

Proof. As in the proof of Corollary 2.10, we may replace $\langle\mathbb{A}\rangle$ with the equivalent category $\mathsf{Proj}(\mathbb{A})$ of regular projective modest sets over $\mathbb{A}$.

Let the *jointly-split topology* S on $\mathsf{Proj}(\mathbb{A})$ be generated by those families $\{f_i\colon B_i \to A\}_{i\in I}$ for which $|I| < \mathsf{w}(\mathbb{A})$ and the morphism

$$[f_i]_{i\in I}\colon \textstyle\coprod_{i\in I} B_i \to A$$

splits, i.e., has a right inverse $s\colon A \to \coprod_{i\in I} B_i$. Let us verify that the jointly split families form a basis for Grothendieck topology:

1. Isomorphisms cover: It is obvious that an isomorphism is covering since it is split by its inverse.

2. Stability under pullbacks: Suppose $\{f_i\colon B_i \to A\}_{i\in I}$ covers A. Consider the pullback along $g\colon C \to A$. Since coproducts in $\langle A\rangle$ are stable, we get a pullback diagram

$$\begin{tikzcd}
\coprod_i g^*B_i \arrow[r] \arrow[d, "{[g^*f_i]_i}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \coprod_i B_i \arrow[d, "{[f_i]_i}"'] \\
C \arrow[r, "g"] & A \arrow[u, bend right, "s"']
\end{tikzcd}$$

   The morphism s in the above diagram is the splitting of $[f_i]_i$. We want to show that the left-hand vertical morphism splits, which follows easily from the pullback property of the diagram. Since $[f_i]_i \circ s \circ g = 1_C \circ g$ there exists a unique arrow $t\colon C \to \coprod_i g^*B_i$ such that $1_C = [g^*f_i]_i \circ t$, as required.

3. Transitivity: Suppose $\{f_i\colon B_i \to A\}_{i\in I}$ is a covering family, and for each $i \in I$, the family $\{g_{ij}\colon C_{ij} \to B_i\}_{j\in J_i}$ covers B_i. Then $[f_i]_i$ splits by a morphism s and $[g_{ij}]_j$ splits by a morphism r_i, for each $i \in I$. The

coproduct $C = \coprod_{i \in I} \coprod_{j \in J_i} C_{ij}$ exists because $\sum_{i \in I} |J_i| < \mathsf{w}(\mathbb{A})$. This is so because $\mathsf{w}(\mathbb{A})$ is a regular cardinal, $|I| < \mathsf{w}(\mathbb{A})$ and $|J_i| < \mathsf{w}(\mathbb{A})$ for all $i \in I$. The map $[f_i \circ g_{ij}]_{ij} \colon C \to A$ splits by $\left(\sum_{i \in I} r_i\right) \circ s$.

Next, we show that the jointly split families generate precisely the coproduct topology C. We need to show that a sieve $\{f_i \colon B_i \to A\}_{i \in I}$ contains an S-cover if, and only if, it contains a C-cover. One direction is easy, since every C-cover is obviously an S-cover. For the converse, if $\{f_j \colon B_j \to A\}_{j \in J}$ is jointly split by $s \colon A \to \coprod_{j \in J} B_j$, then we can decompose A into a coproduct $A \cong \coprod_{j \in J} s^* B_j$, as in the pullback diagram

$$\begin{tikzcd} \coprod_{j \in J} s^* B_j \arrow[r] \arrow[d, "1_A"'] \arrow[dr, phantom, "\lrcorner", very near start] & \coprod_{j \in J} B_j \arrow[d, "1"] \\ A \arrow[r, "s"'] & \coprod_{j \in J} B_j \end{tikzcd}$$

Therefore, if a sieve contains a jointly split family $\{f_j \colon B_j \to A\}_{j \in J}$, then it also contains a family whose coproduct is isomorphic to A.

As in the statement of the theorem, let R be the Grothendieck topology on $\mathsf{Mod}(\mathbb{A})$ generated by those families $\{f_i \colon B_i \to A\}_{i \in I}$ for which $|I| < \mathsf{w}(\mathbb{A})$ and $[f_i]_{i \in I} \colon \coprod_{i \in I} B_i \to A$ is a regular epi. To finish the proof, we apply the Comparison Lemma [MM92, Appendix, Corollary 4.3] to $\mathsf{Sh}(\mathsf{Mod}(\mathbb{A}), R)$ and $\mathsf{Sh}(\mathsf{Proj}(\mathbb{A}), S)$. For this we must check three conditions:

1. Topology R is subcanonical: we chose R to be generated by certain regular-epimorphic families.

2. Every object in $\mathsf{Mod}(\mathbb{A})$ is R-covered by objects in $\mathsf{Proj}(\mathbb{A})$: this is equivalent to every modest set being a regular quotient of a regular projective modest set, which is the case.

3. A family $\{f_i \colon B_i \to A\}_{i \in I}$ is S-covering in $\mathsf{Proj}(\mathbb{A})$ if, and only if, it is R-covering in $\mathsf{Mod}(\mathbb{A})$: this holds because a morphism $f \colon B \to A$ in $\mathsf{Proj}(\mathbb{A})$ is split if, and only if, it is a regular epi in $\mathsf{Mod}(\mathbb{A})$. Indeed, if it is split then it is a regular epi by a general category-theoretic argument. Conversely, suppose $f \colon B \to A$ is a regular epi and $A \in \mathsf{Proj}(\mathbb{A})$. Since A is regular projective there exists a right inverse $s \colon B \to A$ of f, hence f is split.

■

Corollary 2.13 *The Yoneda embedding*

$$\mathsf{Mod}(\mathbb{A}) \xrightarrow{\ y\ } \mathsf{Sh}(\mathbb{A})$$

is full and faithful, preserves the locally cartesian closed structure, regular epis, and coproducts. In terms of categorical logic, it preserves and reflects validity of formulas involving full first-order logic, exponentials, dependent types, disjoint sum types, and quotients of $\neg\neg$*-stable equivalence relations. In addition,* y *preserves infinitary disjunctions and coproducts of cardinality less than* $\mathsf{w}(\mathbb{A})$*. In case* $\mathsf{w}(\mathbb{A}) \geq \aleph_1$*,* y *also preserves the natural numbers object.*

More precisely, the functor y is defined as follows. If $I \colon \langle\mathbb{A}\rangle \to \mathsf{Mod}(\mathbb{A})$ is the inclusion, then for $S \in \mathsf{Mod}(\mathbb{A})$, $yS = \mathsf{Hom}(I(\square), S)$ where the hom-set is taken in $\mathsf{Mod}(\mathbb{A})$. We do not have to compose with sheafification because the topology is subcanonical.

Example 2.14 Countably based equilogical spaces embed via the Yoneda embedding into the topos $\mathsf{Sh}(\mathbb{P}) \simeq \mathsf{Sh}(\omega\mathsf{Top}_0, C_\omega)$ of sheaves on the countably based T_0 spaces equipped with the countable coproducts topology.

Example 2.15 Similarly, $\mathsf{Mod}(\mathbb{B})$ embeds into the topos $\mathsf{Sh}(\mathbb{B})$ which is equivalent to the topos $\mathsf{Sh}(\mathsf{0Dim}, C_\omega)$, where $\mathsf{0Dim}$ is the category of countably based 0-dimensional Hausdorff spaces.

2.2 Functors Induced by Applicative Morphisms

Recall that a discrete applicative morphism $\rho \colon \mathbb{E} \xrightarrow{\mathsf{PCA}} \mathbb{F}$ induces a functor $\widehat{\rho} \colon \mathsf{Mod}(\mathbb{E}) \to \mathsf{Mod}(\mathbb{F})$. This functor preserves finite limits and regular epis by [Lon94]. Suppose that in addition $\widehat{\rho}$ preserves those coproducts that $\mathsf{Mod}(\mathbb{E})$ has, and call such a functor $+$*-preserving*. Then $\widehat{\rho}$ preserves covering families, and so (by [MM92, Section VII.7]) induces a geometric morphism $(\rho^*, \rho_*) \colon \mathsf{Sh}(\mathbb{F}) \to \mathsf{Sh}(\mathbb{E})$ between the corresponding sheaf toposes, as in the diagram below.

$$\begin{tikzcd}
\mathsf{Sh}(\mathbb{E}) \arrow[r, shift left, "\rho^*"'] & \mathsf{Sh}(\mathbb{F}) \arrow[l, shift left, "\rho_*"'] \\
\mathsf{Mod}(\mathbb{E}) \arrow[u, "y"] \arrow[r, "\widehat{\rho}"'] & \mathsf{Mod}(\mathbb{F}) \arrow[u, "y"']
\end{tikzcd} \tag{1}$$

The inverse image part ρ^* of the geometric morphism in the above diagram makes the evident square commute up to natural isomorphism.

2.3 Applicative Retractions Induce Local Maps of Toposes

A pair of applicative morphisms $\delta\colon \mathbb{E} \xrightarrow{\mathsf{PCA}} \mathbb{F}$ and $\eta\colon \mathbb{F} \xrightarrow{\mathsf{PCA}} \mathbb{E}$ such that $1_{\mathbb{F}} \preceq \delta \circ \eta$ and $\eta \circ \delta \preceq 1_{\mathbb{E}}$ is called an *applicative adjunction*, written $(\eta \dashv \delta)\colon \mathbb{E} \xrightarrow{\mathsf{PCA}} \mathbb{F}$, see [Lon94]. An applicative adjunction for which $\delta \circ \eta \sim 1_{\mathbb{F}}$ is called an *applicative retraction.*

Example 2.16 As Peter Lietz observed, the morphisms $(\iota \dashv \delta)\colon \mathbb{P} \xrightarrow{\mathsf{PCA}} \mathbb{B}$ of example 2.5 form an applicative retraction. See [Bau00] for details.

Let $(\eta \dashv \delta)\colon \mathbb{E} \xrightarrow{\mathsf{PCA}} \mathbb{F}$ be an applicative retraction of discrete applicative morphisms. Then the induced functors $\widehat{\eta}$ and $\widehat{\delta}$ form an adjoint pair $\widehat{\eta} \dashv \widehat{\delta}$,

$$\begin{tikzcd} \mathsf{Mod}(\mathbb{E}) \arrow[r, shift right, "\widehat{\delta}"'] & \mathsf{Mod}(\mathbb{F}) \arrow[l, shift right, "\widehat{\eta}"'] \end{tikzcd} \quad .$$

In addition, $\widehat{\delta} \circ \widehat{\eta} \cong 1_{\mathsf{Mod}(\mathbb{F})}$. Suppose that $\widehat{\delta}$ is $+$-preserving. Combining this with (1), we get *three* adjoint functors $\eta^* \dashv \eta_* = \delta^* \dashv \delta_*$,

$$\begin{tikzcd} \mathsf{Sh}(\mathbb{E}) \arrow[r, "\eta_*=\delta^*"] & \mathsf{Sh}(\mathbb{F}) \arrow[l, bend right, "\eta^*"'] \arrow[l, bend left, "\delta_*"] \end{tikzcd}$$

where $\delta^* \circ \eta^* \cong 1_{\mathsf{Sh}(\mathbb{F})}$. Thus, a $+$-preserving discrete applicative retraction $(\eta \dashv \delta)\colon \mathbb{E} \xrightarrow{\mathsf{PCA}} \mathbb{F}$ induces a local map of toposes $\mathsf{Sh}(\mathbb{E}) \to \mathsf{Sh}(\mathbb{F})$. Such maps were studied by Birkedal in [Bir99]; there and in [ABS99] a $\sharp$-$\flat$ calculus for the internal logic is obtained, which can be used here to compare realizability in modest sets over $\mathbb{E}$ with that over $\mathbb{F}$.

An applicative retraction does not seem to induce a third adjoint if we use *realizability toposes* $\mathsf{RT}(\mathbb{E})$ and $\mathsf{RT}(\mathbb{F})$ instead of sheaf toposes (although one might consider *relative realizability* [Bir99, ABS99]). On the other hand, many applicative retractions that we have encountered are $+$-preserving, and so they induce local maps of sheaf toposes. A good conceptual explanation of this phenomenon would be desirable.

2.4 A Forcing Semantics for Realizability

The following theorem spells out the Kripke-Joyal semantics in $\mathsf{Sh}(\mathbb{A})$, see [MM92, Section VI.6]. The interpretation of disjunction, negation and existential quantification is simpler than the usual one due to the simple nature of the coproduct topology on the site $\langle\mathbb{A}\rangle$.

Theorem 2.17 *Let $X, Y \in \mathsf{Sh}(\mathbb{A})$, and let*

$$x : X \mid \phi(x) \qquad x : X \mid \psi(x) \qquad x : X, y : Y \mid \rho(x, y)$$

be formulas in the internal language of $\mathsf{Sh}(\mathbb{A})$. Let $A \in \langle\mathbb{A}\rangle$ and $a \in XA$. The Kripke-Joyal forcing relation $\models$ is interpreted as follows:

1. *$A \models \phi(a) \wedge \psi(a)$ if, and only if, $A \models \phi(a)$ and $A \models \psi(a)$.*
2. *$A \models \phi(a) \vee \psi(a)$ if, and only if, there exist $A_1, A_2 \in \langle\mathbb{A}\rangle$ such that $A = A_1 + A_2$, with $A_1 \models \phi(a \cdot \iota_1)$ and $A_2 \models \psi(a \cdot \iota_2)$.*
3. *$A \models \phi(a) \longrightarrow \psi(a)$ if, and only if, for all $f\colon B \to A$ in $\langle\mathbb{A}\rangle$, if $B \models \phi(a \cdot f)$ then $B \models \psi(a \cdot f)$.*
4. *$A \models \neg\phi(a)$ if, and only if, for all $f\colon B \to A$ in $\langle\mathbb{A}\rangle$, if $B \models \phi(a \cdot f)$ then $B = \mathsf{0}$.*
5. *$A \models \forall y \in Y . \rho(a, y)$ if, and only if, for all $f\colon B \to A$ in $\langle\mathbb{A}\rangle$ and all $b \in YB$, one has $B \models \rho(a \cdot f, b)$.*
6. *$A \models \exists y \in Y . \rho(a, y)$ if, and only if, $A = \coprod_{i \in I} A_i$ in $\langle\mathbb{A}\rangle$, $|I| < \mathsf{w}(\mathbb{A})$, and for each $i \in I$ there exist $b_i \in YA_i$ such that $A_i \models \rho(a \cdot \iota_i, b_i)$.*

Proof. We only need to show that the standard interpretations of disjunction, negation, and existential quantification simplify to the forms stated in the theorem. This follows easily from the characterization of the topology via the disjoint sum basis.

Let us first consider disjunction. Suppose $A \models \phi(a) \vee \psi(a)$. Then there exists a family $\{\iota_i\colon A_i \to A\}_{i \in I}$ such that $[\iota_i]_i\colon A_1 + \cdots + A_k \to A$ is an isomorphism and, for every $i \in I$, $A_i \models \phi(a \cdot \iota_i)$ or $A_i \models \psi(a \cdot \iota_i)$. Define the sets J and K by

$$J = \{i \in I \mid\ \models \phi(a \cdot \iota_i)\} \ , \qquad K = I \setminus J \ .$$

Let $A_1' = \coprod_{j \in J} A_j$ and $A_2' = \coprod_{k \in K} A_k$. Then it is clear that $A = A_1' + A_2'$. Let $\kappa_1 = [\iota_j]_{j \in J}\colon A_1 \to A$ and $\kappa_2 = [\iota_k]_{k \in K}\colon A_2 \to A$ be the isomorphisms.

It is now clear that $A_1' \models \phi(a \cdot \kappa_1)$ and $A_2 \models \psi(a \cdot \kappa_2)$, as required. The converse holds, since if $A = A_1 + A_2$, $A_1 \models \phi(a \cdot \iota_1)$ and $A_2 \models \psi(a \cdot \iota_2)$, then $A \models \phi(a) \vee \psi(a)$ because the sum of canonical inclusions $[\iota_1, \iota_2] \colon A_1 + A_2 \to A$ is an isomorphism, thus it covers A.

The interpretation of negation is correct because an object is covered by the empty family $\{\}$ if, and only if, it is the initial object $\mathsf{0}$.

Suppose $A \models \exists\, y \in Y \,.\, \rho(a, y)$. Then there is a family $\{\iota_i \colon A_i \to A\}_{i \in I}$ such that $A = \coprod_{i \in I} A_k$, $\iota_i \colon A_i \to A$ is the canonical inclusion for every $i \in I$, and there exists $b_i \in Y A_i$ such that $A_i \models \rho(a \cdot \iota_i, b_i)$. This proves one direction. The converse is proved easily as well. ∎

If $\mathsf{Mod}(\mathbb{A})$ has countable coproducts a clause involving countable disjunctions can be added. The forcing semantics can be restricted to the modest sets, as long as the formulas are restricted to first-order logic with exponentials, dependent types, subset types, and quotients of $\neg\neg$-stable equivalence relations. It is a consequence of Corollary 2.13 that such a formula is valid in the forcing semantics above if, and only if, it is valid in the realizability interpretation.

2.5 A Transfer Principle for Modest Sets

Suppose $(\eta \dashv \delta) \colon \mathbb{E} \xrightarrow{\mathsf{PCA}} \mathbb{F}$ is an applicative retraction such that $\widehat{\delta}$ is $+$-preserving. The transfer principle from Awodey et. al. [ABS99] can be applied to the induced local map of toposes,

$$
\begin{tikzcd}
\mathsf{Sh}(\mathbb{E}) \arrow[rr, "\eta_* = \delta^*"] & & \mathsf{Sh}(\mathbb{F}) \arrow[ll, bend right, "\eta^*"'] \arrow[ll, bend left, "\delta_*"]
\end{tikzcd} \quad .
$$

We say that a formula θ in the internal language of a topos is *local*[1] if it is built from atomic predicates, including equations, using first-order logic, and if in every subformula of the form $\phi \longrightarrow \psi$, there is no $\forall$ or $\longrightarrow$ in ϕ.

If θ is a local sentence in the internal logic of $\mathsf{Sh}(\mathbb{F})$, we write $\mathsf{Sh}(\mathbb{F}) \models \theta$ when the canonical interpretation of θ is valid in $\mathsf{Sh}(\mathbb{F})$. The sentence θ can also be interpreted in $\mathsf{Sh}(\mathbb{E})$ by mapping the types and relations occurring in θ over to $\mathsf{Sh}(\mathbb{E})$ with η^*. The transfer principle from [ABS99] then tells us that for such a local sentence θ,

$$
\mathsf{Sh}(\mathbb{E}) \models \theta \quad \text{if and only if} \quad \mathsf{Sh}(\mathbb{F}) \models \theta \; .
$$

[1] In [ABS99] such a formula is called “stable”.

It follows that if only types and relations from $\mathsf{Mod}(\mathbb{F})$ occur in θ then the transfer principle restricts to the categories of modest sets:

$$\mathsf{Mod}(\mathbb{E}) \models \theta \quad \text{if and only if} \quad \mathsf{Mod}(\mathbb{F}) \models \theta \; .$$

Here we interpret θ in $\mathsf{Mod}(\mathbb{E})$ by mapping all types and relations that occur in θ over to $\mathsf{Mod}(\mathbb{E})$ by $\widehat{\eta}$. The notation $\mathsf{Mod}(\mathbb{E}) \models \theta$ means that the sentence θ is valid in the standard realizability interpretation, or equivalently, in the forcing semantics as described in Theorem 2.17. The following theorem explains why $\mathsf{Mod}(\mathbb{P})$ and $\mathsf{Mod}(\mathbb{B})$ appear to be very similar, at least as far as simple types are concerned.

Theorem 2.18 *Let* $(\iota \dashv \delta)\colon \mathbb{P} \xrightarrow{\mathsf{PCA}} \mathbb{B}$ *be the applicative retraction from Example 2.16. Let* θ *be a local sentence such that all variables occurring in* θ *have types* $\mathbb{N}$, $\mathbb{N}^{\mathbb{N}}$, *or* $\mathbb{R}$. *Then*

$$\mathsf{Mod}(\mathbb{P}) \models \theta \quad \text{if and only if} \quad \mathsf{Mod}(\mathbb{B}) \models \theta \; ,$$

where $\mathbb{N}$ *is interpreted as the natural numbers object,* $\mathbb{N}^{\mathbb{N}}$ *is interpreted as the obvious exponential, and* $\mathbb{R}$ *is interpreted as the object of Cauchy real numbers.*

Proof. The theorem holds because the functor $\widehat{\iota}\colon \mathsf{Mod}(\mathbb{B}) \to \mathsf{Mod}(\mathbb{P})$ preserves the natural numbers object $\mathbb{N}$, its function space $\mathbb{N}^{\mathbb{N}}$, and the real numbers object $\mathbb{R}$. It is not hard to see that $\widehat{\delta}$ preserves countable coproducts, which are precisely the coproducts that $\mathsf{Mod}(\mathbb{P})$ has. ∎

In Theorem 2.18 we *cannot* allow variables of higher types such as $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$ and $\mathbb{R}^{\mathbb{R}}$ to occur, because it is well known that the following local sentence involving $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$ is valid in $\mathsf{Mod}(\mathbb{B})$ but not in $\mathsf{Mod}(\mathbb{P})$:

$$\forall\, F \in \mathbb{N}^{\mathbb{N}^{\mathbb{N}}} .\, \exists\, \alpha \in \mathbb{N}^{\mathbb{N}} .\, \forall\, \beta \in \mathbb{N}^{\mathbb{N}} .\, F\beta = (\alpha|\beta) \; . \tag{2}$$

The sentence states that every functional $F \in \mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$ has an associate $\alpha \in \mathbb{N}^{\mathbb{N}}$ in the sense of Kleene [Kle59]. Here $\alpha|\beta$ is Kleene's *continuous function application*. The statement $n = (\alpha|\beta)$ is equivalent to

$$\exists\, m \in \mathbb{N} .\, \big(\alpha(\overline{\beta} m) = n + 1 \land \forall\, k \in \mathbb{N} .\, \big(k < m \longrightarrow \alpha(\overline{\beta} k) = 0\big)\big) \; .$$

Similarly, a statement that all functions $f \in \mathbb{R}^{\mathbb{R}}$ are continuous is valid in $\mathsf{Mod}(\mathbb{B})$ but not in $\mathsf{Mod}(\mathbb{P})$. Thus, in a roundabout way, we obtain the following result.

Corollary 2.19 *The functor $\widehat{\iota}\colon \mathsf{Mod}(\mathbb{B}) \to \mathsf{Mod}(\mathbb{P})$ does not preserve exponentials. In particular, $\widehat{\iota}(\mathbb{N}^{\mathbb{N}^{\mathbb{N}}})$ is not isomorphic to the object $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$ in $\mathsf{Mod}(\mathbb{P})$, and $\widehat{\iota}(\mathbb{R}^{\mathbb{R}})$ is not isomorphic to the object $\mathbb{R}^{\mathbb{R}}$ in $\mathsf{Mod}(\mathbb{P})$.*

We can in fact prove Corollary 2.19 directly as follows. Let $X = \mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$ be the object of type 2 functionals in $\mathsf{0Equ}$, which is equivalent to $\mathsf{Mod}(\mathbb{B})$, and let $Y = \mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$ be the object of type 2 functionals in Equ. Both X and Y are equilogical spaces. The space $|X|$ is a Hausdorff space. The space $|Y|$ is the subspace of the total elements of the Scott domain $D = [{\mathbb{N}_\perp}^\omega \to \mathbb{N}_\perp]$. The equivalence relation on $|Y|$ is the consistency relation of D restricted to $|Y|$. Suppose $f\colon |Y| \to |X|$ represented an isomorphism, and let $g\colon |X| \to |Y|$ represent its inverse. Because f is monotone in the specialization order and $|X|$ has a trivial specialization order, $a \equiv_Y b$ implies $fx = fy$. Therefore, $g \circ f\colon |Y| \to |Y|$ is an equivariant retraction. By [Bau00, Proposition 4.1.8], Y is a topological object. By [Bau00, Corollary 4.1.9], this would mean that the topological quotient $|Y|/{\equiv_Y}$ is countably based, but it is not, as is well known.

Another way to see that Y cannot be topological is to observe that Y is an exponential of the Baire space, but the Baire space is not exponentiable in $\omega\mathsf{Top}_0$, and in particular $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$ is not a topological object.

Finally, we remark that statement (2) is of course valid in $\mathsf{Mod}(\mathbb{P})$ if $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$ is replaced by $\widehat{\iota}(\mathbb{N}^{\mathbb{N}^{\mathbb{N}}})$. But then it becomes a simple truism, since it can be shown that in $\mathsf{Mod}(\mathbb{P})$ the space $\widehat{\iota}(\mathbb{N}^{\mathbb{N}^{\mathbb{N}}})$ is just the set of those functionals that have an associate:

$$\widehat{\iota}(\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}) = \left\{ F \in \mathbb{N}^{\mathbb{N}^{\mathbb{N}}} \mid \exists\, \alpha \in \mathbb{N}^{\mathbb{N}} \,.\, \forall\, \beta \in \mathbb{N}^{\mathbb{N}} \,.\, F\beta = (\alpha|\beta) \right\} .$$

Corollary 2.19 should be contrasted with the fact that there is an epi-mono $\widehat{\iota}(\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}) \to \mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$ that is *not* iso but nevertheless induces a natural bijection between the global points of $\widehat{\iota}(\mathbb{N}^{\mathbb{N}^{\mathbb{N}}})$ and the global points of $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$. In this sense, the finite types over $\mathbb{N}$ in $\mathsf{Mod}(\mathbb{P})$ and $\mathsf{Mod}(\mathbb{B})$ are equivalent as far as the *cartesian closed structure* is concerned, but here we see that they have different *logical* properties.

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