Not not to be or not to be?

Andrej Bauer

Institute for Mathematics, Physics, and Mechanics
University of Ljubljana
Slovenia

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What is wrong with these theorems?

Theorem: Three points are either collinear or not.

But testing for collinearity is numerically unstable.

Theorem: A non-constant polynomial has a complex root.

But classical proofs say nothing about how to compute a root.

Theorem: Most functions are everywhere discontinuous.

But such functions are irrelevant for computer science.

Branches of Math tailored for Comp. Science

- Theory of computability
- Computational complexity & algorithms
- Numerical analysis
- Domain theory
- Cryptography
- Queueing theory
- Finite model theory
- Machine learning
- Type theory
- ...

Did you ever ask yourself . . .

. . . whether mathematics itself

can be tailored for computer science?

One way to do this is *realizability theory*.

Overview

1. Building a Realizability World
2. Life in a Realizability World
3. Practical Considerations

How do we create a world of mathematics?

Use category theory, of course.

Two steps:

1. Find a category that describes our view of the world.
   category = objects + morphisms
2. Apply tools of categorical logic to study it.
   Take categorical logic course in the Philosophy Dept.

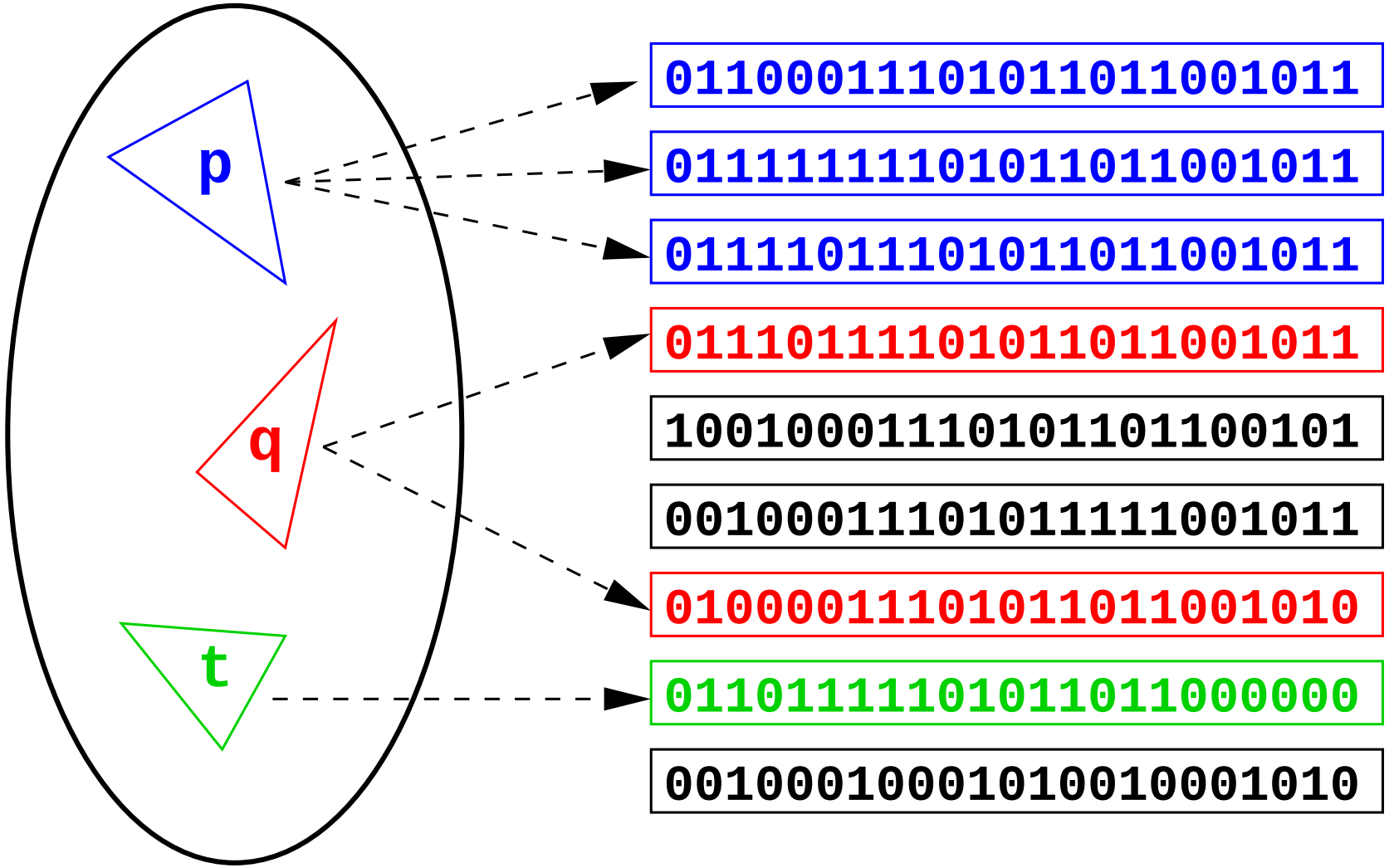
Computational Views of the World

- Everything is made of Turing machines.
- Everything is made of ML programs.
- Everything is made of Scott domains.
- The *relative* view:
  1. Which data can be *represented*?
  2. How do we *compute* with data?

Relative Computability

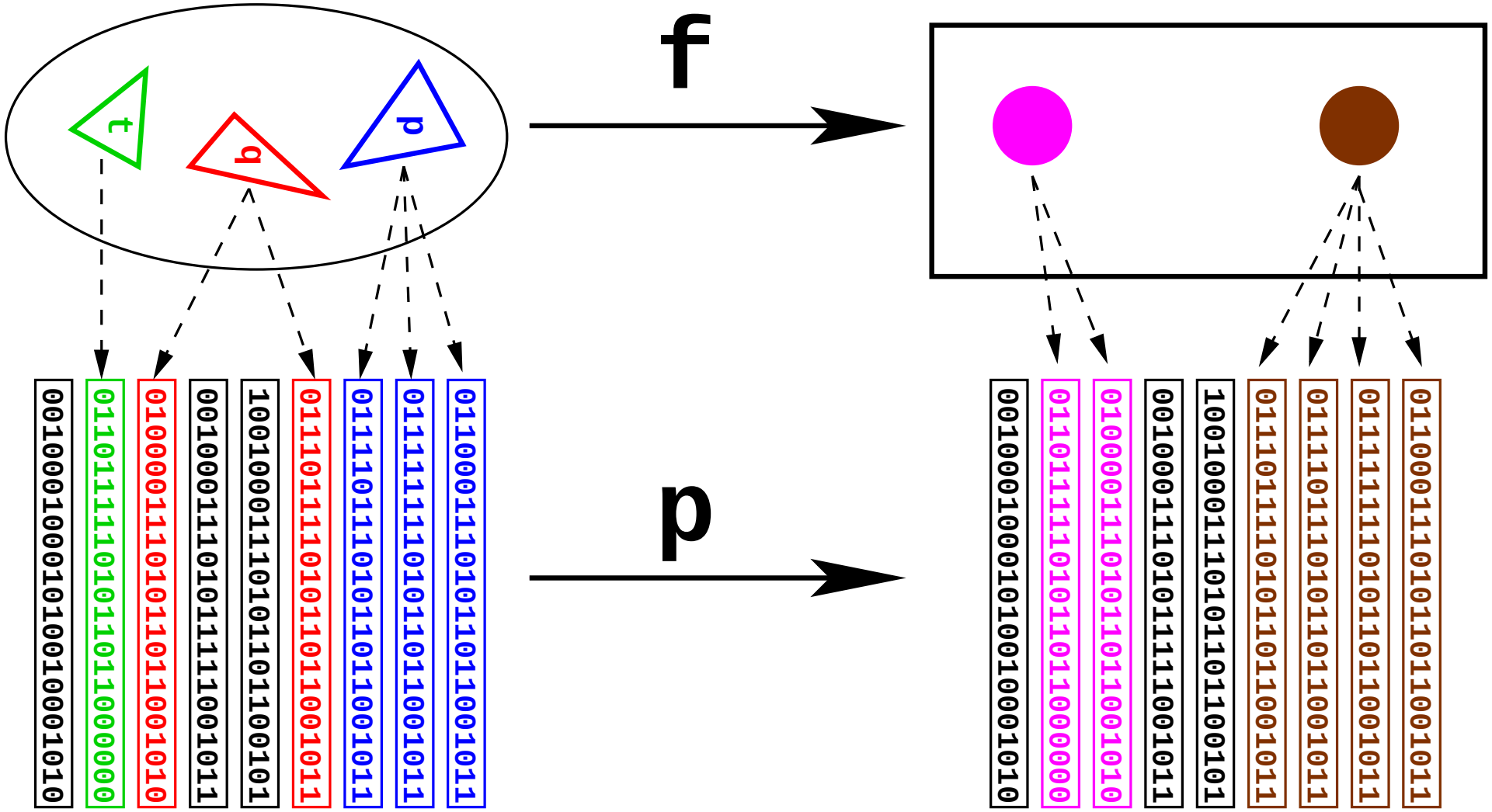
- Data: contents of infinite RAM
  - *all* configurations possible, also non-computable
  - other sources of data (input streams) can be added
- Computation: (finite) program
  - any chosen general programming language
  - different language features give different worlds

Modest Sets



$c \Vdash x$ “c realizes (represents) x”

Realized Functions



If $c \Vdash x$ then $p(c) \Vdash f(x)$.

Computation-aware Sets and Functions

Foundation of classical mathematics:

sets & functions

Foundation of *computation-aware* mathematics:

modest sets & realized functions

Category Theory at Work

Any mathematical structure that has a universal property has a unique representation, up to isomorphism.

1. Natural numbers

   initial algebra with one constant and one unary operation

2. Real numbers

   the complete Archimedean field

3. Inductive sets (lists, trees, . . .)

4. Cartesian product $A \times B$

5. Function space $A \to B$

Example: real numbers $\mathbb{R}$

A real number $x \in \mathbb{R}$ is realized by a pair (d, e):

$$(d, e) \Vdash_{\mathbb{R}} x \ ,$$

where:

- mantissa $d = d_0 d_1 d_2 \ldots$, with $d_i \in \{-1, 0, 1\}$
- exponent $e \in \mathbb{Z}$
- *signed* binary digit representation:

$$x = 2^e \cdot \sum_{k=0}^{\infty} \frac{d_i}{2^{k+1}}$$

Overview

✓ Building a Realizability World

☞ Life in a Realizability World

3. Practical Considerations

The Particle Physics of Realizability

"Our world is made of tiny invisible realizers."

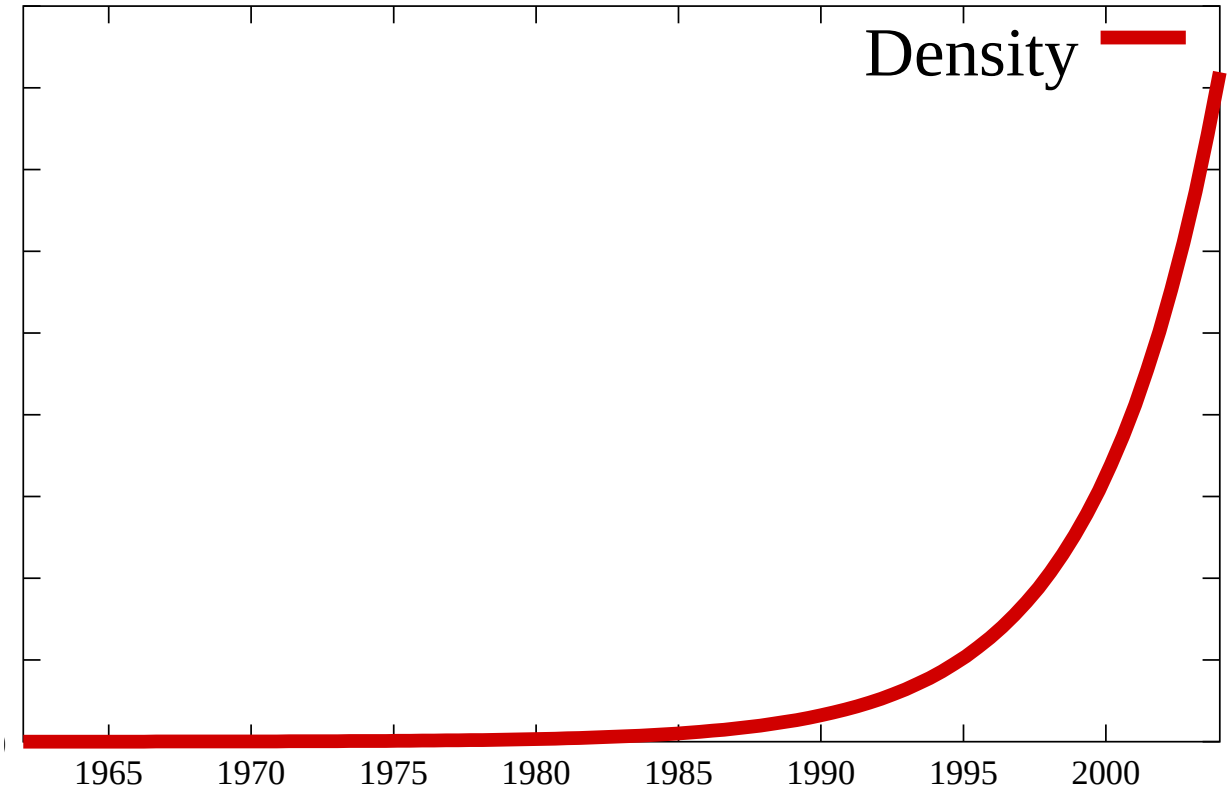
"The two basic realizers are S and K."

$$Kxy = x \qquad Sxyz = (xz)(yz)$$

The Cosmology of Realizability

"Our universe is becoming denser at an exponential rate."

(Gordon Moore, 1965)



The Language of Realizability

Constructive logic:

"The truth of a statement is witnessed by a program."

Example: a witness for

$$\forall x \in \mathbb{R}.(x < 0 \vee x \geq 0)$$

is a program p such that, for $(d, e) \Vdash_{\mathbb{R}} x$,

$$p(d, e) = 0 \qquad \text{if } x < 0$$

$$p(d, e) = 1 \qquad \text{if } x \geq 0$$

Intuitionistic Logic

The logic of realizability is *intuitionistic*.

The Law of Excluded Middle is not generally valid:

$$\varphi \vee \neg\varphi$$

Proof by contradiction in not generally valid:

$$\neg\neg\varphi \Longrightarrow \varphi$$

This is a good thing!

Markov's Principle

If elements of A can be enumerated and $\varphi(x)$ is a semi-decidable predicate then

$$(\neg\forall x \in A.\, \neg\varphi(x)) \quad \Longrightarrow \quad \exists x \in A.\, \varphi(x)$$

Witnessed by a program which iterates through all elements $x_1, x_2, \ldots$ of A and tests $\varphi(x_i)$ until one is found.

All functions are continuous

"All $f : \mathbb{R} \to \mathbb{R}$ are continuous."

"Only finite precision of input is required
for given finite precision of output."

Witnessed by a program which finds a $k \in \mathbb{N}$ such that f reads only k digits of input to produce n digits of output.

But only if the programming language has a `throw`–`catch` construct, or a similar control mechanism.

The "not not" translation

Classical logic can be translated into intuitionistic logic.

Classical	Intuitionistic
$\varphi \Longrightarrow \psi$	$\neg\neg(\varphi^* \Longrightarrow \psi^*)$
$\varphi \wedge \psi$	$\neg\neg(\varphi^* \wedge \psi^*)$
$\varphi \vee \psi$	$\neg\neg(\varphi^* \vee \psi^*)$
$\exists x.\, \phi(x)$	$\neg\neg\exists x.\, \phi(x)^*$
$\forall x.\, \phi(x)$	$\neg\neg\forall x.\, \phi(x)^*$

Not not a Classic Masterpiece

Not not, not to not be, or not to be:
not that is not the question:
Not not, whether not 'tis not nobler in the mind to suffer
The slings and arrows of outrageous fortune,
Or not to not take arms against a sea of troubles,
And not by not opposing not not end them?

(not not Hamlet)

What is still the same?

The "not not" stable statements do not change:

1. All equations and inequations:

   p witnesses "$a_1 = a_2$" if, and only if, a_1 is a_2

2. Any statement built from $=$, $<$, $\wedge$, $\Longrightarrow$ and $\forall$.

Finite combinatorics is pretty much the same.

Overview

- ✓ Building a Realizability World
- ✓ Life in a Realizability World
- ☞ Practical Considerations

Tailored for Computer Science

From the proof of a statement
we obtain an associated program witnessing it.

From the construction of a set or a function
we obtain an associated implementation.

We prove correctness of an implementation
by showing it realizes the desired specification.

A Question for Hardware Designers

Real numbers are represented with *signed* binary digits.

Would negative digits be useful in hardware implementation of floating point arithmetic?

A Challenge for Computational Geometers

Testing for collinearity is not just numerically unstable, it is *non-constructive*.

Prove theorems without using the dichotomy

$$\forall x \in \mathbb{R}. (x < 0 \vee x \geq 0)$$

Use instead

$$\forall \epsilon > 0. \forall x \in \mathbb{R}. (x < \epsilon \vee x > -\epsilon)$$

A Task for Programming Language Designers

Ordinary `if–then–else` control mechanism is inappropriate for exact arithmetic.

Design practical data structures for real numbers *and* invent new control mechanisms for programming with them.

In Conclusion

The world of realizability is *your* world.