Coherence Numbers of Domains

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Motivating the Audience

A classification of PER models on domains.

An unexpected theorem about reflexive domains.

Terminology

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$$\begin{tikzcd} D \arrow[r, shift left, "i"] & E \arrow[l, shift left, "r"] \end{tikzcd} \qquad\qquad r \circ i = \mathrm{id}_D$$

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- Universal domain U for a class $\mathcal{C}$ of domains:

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- Reflexive domain: $[D \to D] \triangleleft D$ and $D \neq \{\bot\}$

PER Models

A reflexive domain D is a model of untyped λ-calculus, therefore a *combinatory algebra*:

$$K = \lambda x.\, \lambda y.\, x \qquad\qquad S = \lambda x.\, \lambda y.\, \lambda z.\, (xz)(yz)$$

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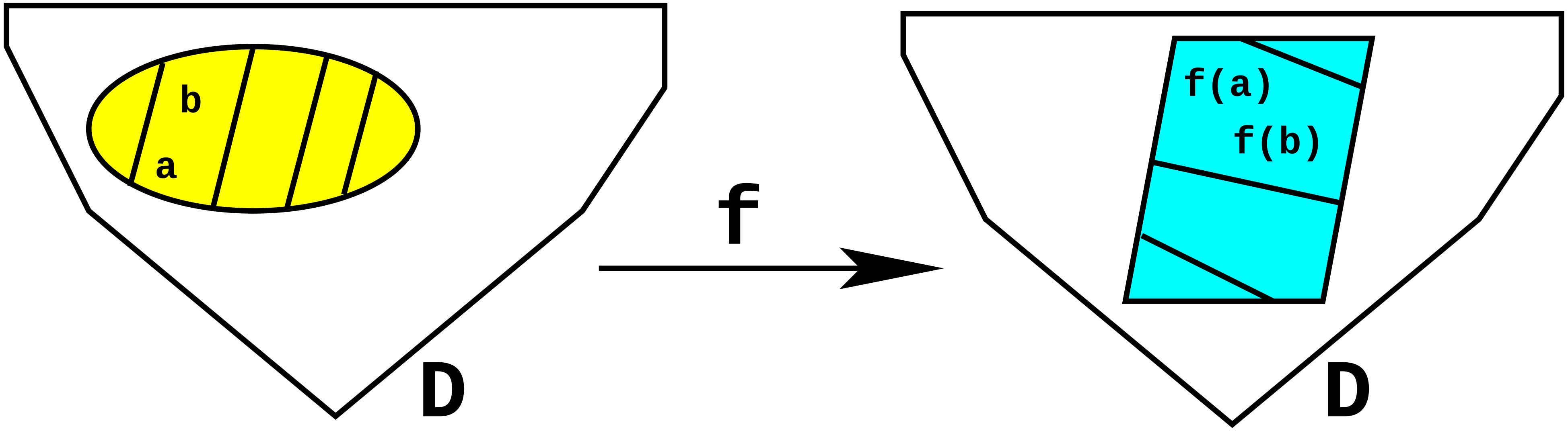
PER(D), category of *partial equivalence relations* on D:



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Given reflexive domains D and E, when is

$$\mathsf{PER}(\mathsf{D}) \simeq \mathsf{PER}(\mathsf{E}) \quad ?$$

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Some well-known PER models:

- $\mathsf{PER}(\mathcal{P}\omega)$ where $\mathcal{P}\omega$ is the graph model
- $\mathsf{PER}(\mathsf{T}^{\omega})$ where T^{ω} is Plotkin's universal coherent domain
- $\mathsf{PER}(\mathsf{U})$ where U is universal for Scott domains

These are all different.

The Answer

Theorem:

For reflexive D and E,

$$\mathsf{PER}(D) \simeq \mathsf{PER}(E) \quad \Longleftrightarrow \quad \mathsf{coh}(D) = \mathsf{coh}(E)$$

where $\mathsf{coh}(D)$ is the *coherence number* of D,

$1 \leq \mathsf{coh}(D) \leq \omega$.

Coherent Domains

[Plotkin'78]

D is *coherent* when, for all $S \subseteq D$,

$$(\forall x_1, x_2 \in S. \{x_1, x_2\} \text{ bounded}) \implies S \text{ bounded}$$

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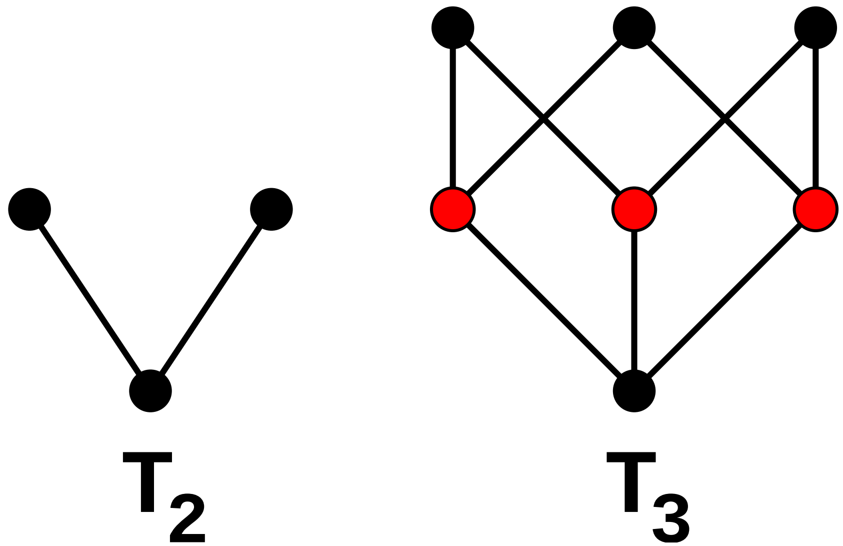

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Definition:

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Coherence number:

$$\mathrm{coh}(D) = \min \left\{ 1 \leq n \leq \omega \;\middle|\; D \text{ is } n\text{-coherent} \right\}$$

Examples

1-coherent = lattice

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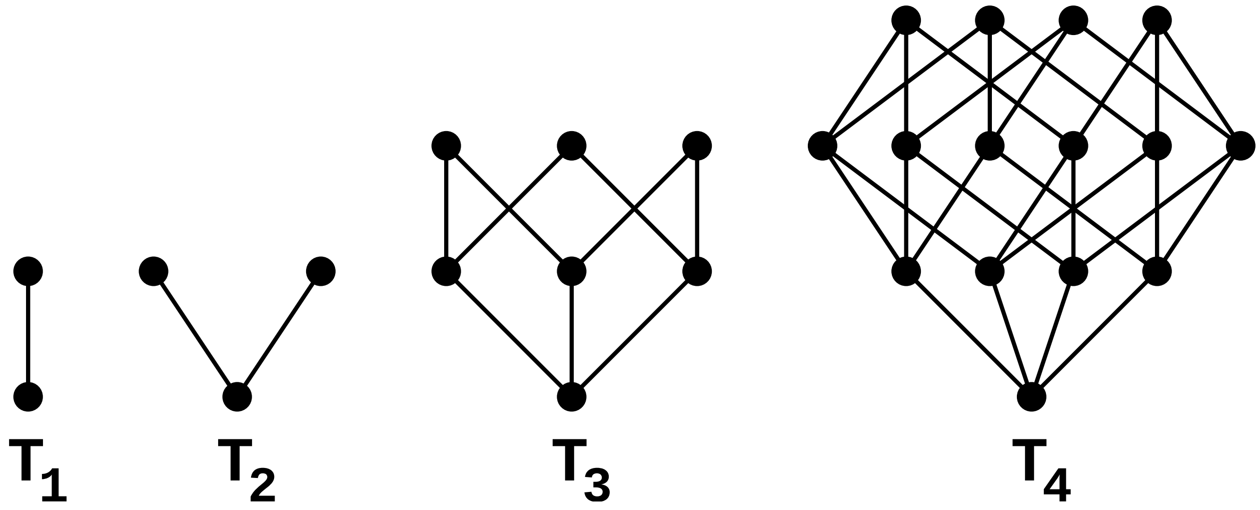

$$T_1 = \Sigma = \{\bot, \top\}$$

$$T_n = \mathcal{P}(\{1, \ldots, n\}) \setminus \{\{1, \ldots, n\}\} \qquad (2 \leq n < \omega)$$

Observations about Coherence Numbers

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Proposition:

- $\mathsf{coh}(D \times E) = \max(\mathsf{coh}(D), \mathsf{coh}(E))$
- $\mathsf{coh}(D^\omega) = \mathsf{coh}(D)$
- $\mathsf{coh}([D \to E]) = \mathsf{coh}(E)$

n-Coherent Domains

Let Coh_n be the category of n-coherent domains.

$$\mathsf{Lat} = \mathsf{Coh}_1 \subseteq \mathsf{Coh}_2 \subseteq \mathsf{Coh}_3 \subseteq \cdots \subseteq \mathsf{Coh}_\omega = \mathsf{Dom}$$

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Does Coh_n have a universal domain?

T_1^ω is universal for Coh_1.

T_2^ω is universal for Coh_2.

Therefore by induction...

Universal domain for Coh_n

Let $\mathbb{F}_n = \{x \subseteq \mathbb{N} \mid |x| < n\}$, and define:

$$\mathbb{C}_n = \{\langle a, B\rangle \in \mathcal{P}\mathbb{N} \times \mathcal{P}\mathbb{F}_n \mid \forall b \in B.\, b \not\subseteq a\}$$

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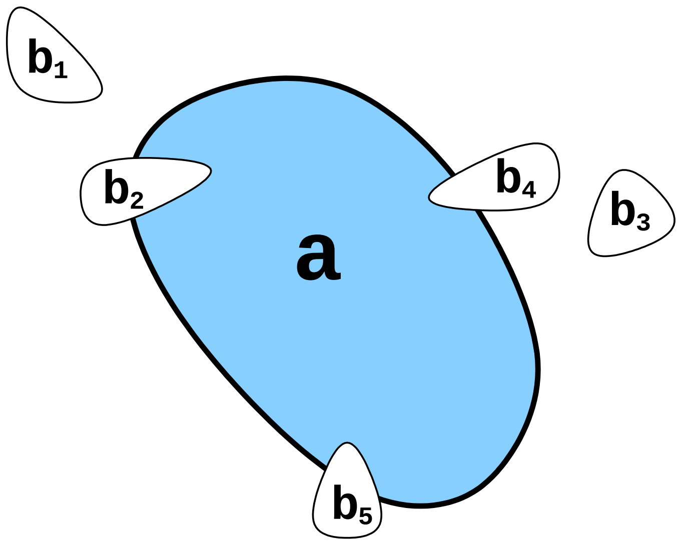
A typical element $\langle a, \{b_1, b_2, \ldots\}\rangle$ of $\mathbb{C}_n$:



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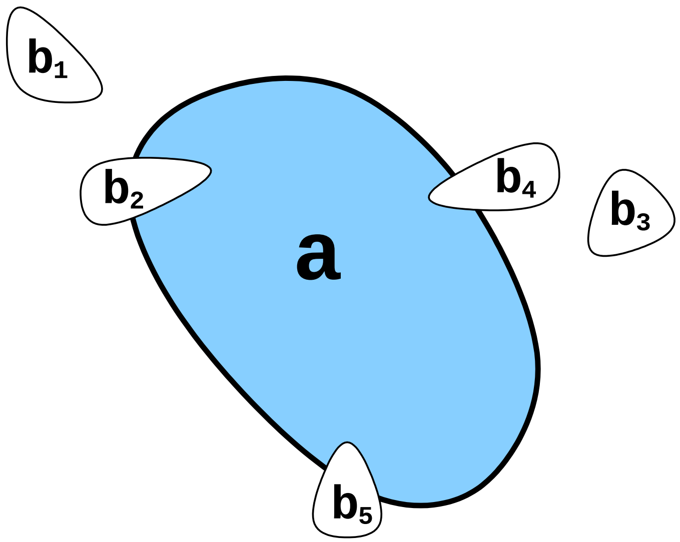

$$\mathbb{C}_1 \cong \mathcal{P}\mathbb{N} \cong \mathsf{T}_1^\omega \qquad \mathbb{C}_2 \cong \mathsf{T}_2^\omega$$

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Theorem:

- $\mathbb{C}_n \lhd T_n^\omega$ for $n < \omega$,
- $\mathbb{C}_\omega \lhd \prod_{n<\omega} T_n$.

Hence T_n^ω is universal for Coh_n,

and $\prod_{n<\omega} T_n$ is universal for $\mathrm{Coh}_\omega = \mathrm{Dom}$.

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Let D be a domain with $n = \mathsf{coh}(D)$.

The following are equivalent:

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4. $T_k \times D \lhd D$ for all $k < 1 + n$
5. D is universal for Coh_n

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$[E \to D]$ is reflexive if $\max(E)$ is infinite and D non-trivial.

Proof: $\mathbb{N}_\perp \lhd E$, therefore $D^\omega \lhd [\mathbb{N}_\perp \to D] \lhd [E \to D]$

Classification of PER models

Theorem:

For reflexive $\mathbb{D}$ and $\mathbb{E}$,

$$\mathsf{coh}(\mathbb{D}) = \mathsf{coh}(\mathbb{E}) \quad \Longrightarrow \quad \mathsf{PER}(\mathbb{D}) \simeq \mathsf{PER}(\mathbb{E}) .$$

Proof:

Sufficient to prove $\mathbb{D} \lhd \mathbb{E}$ and $\mathbb{E} \lhd \mathbb{D}$.

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Proof:

Sufficient to prove $D \lhd E$ and $E \lhd D$.

Let $n = \mathsf{coh}(D) = \mathsf{coh}(E)$.

D reflexive $\Longrightarrow$ D universal for Coh_n $\Longrightarrow$ $E \lhd D$.

Similarly for $D \lhd E$. QED.

Classification of PER models

Every $\mathsf{PER}(D)$ is equivalent to precisely one of

$$\mathsf{PER}(\mathsf{T}_1^\omega)$$

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An analogous statement holds for realizability toposes.

Questions

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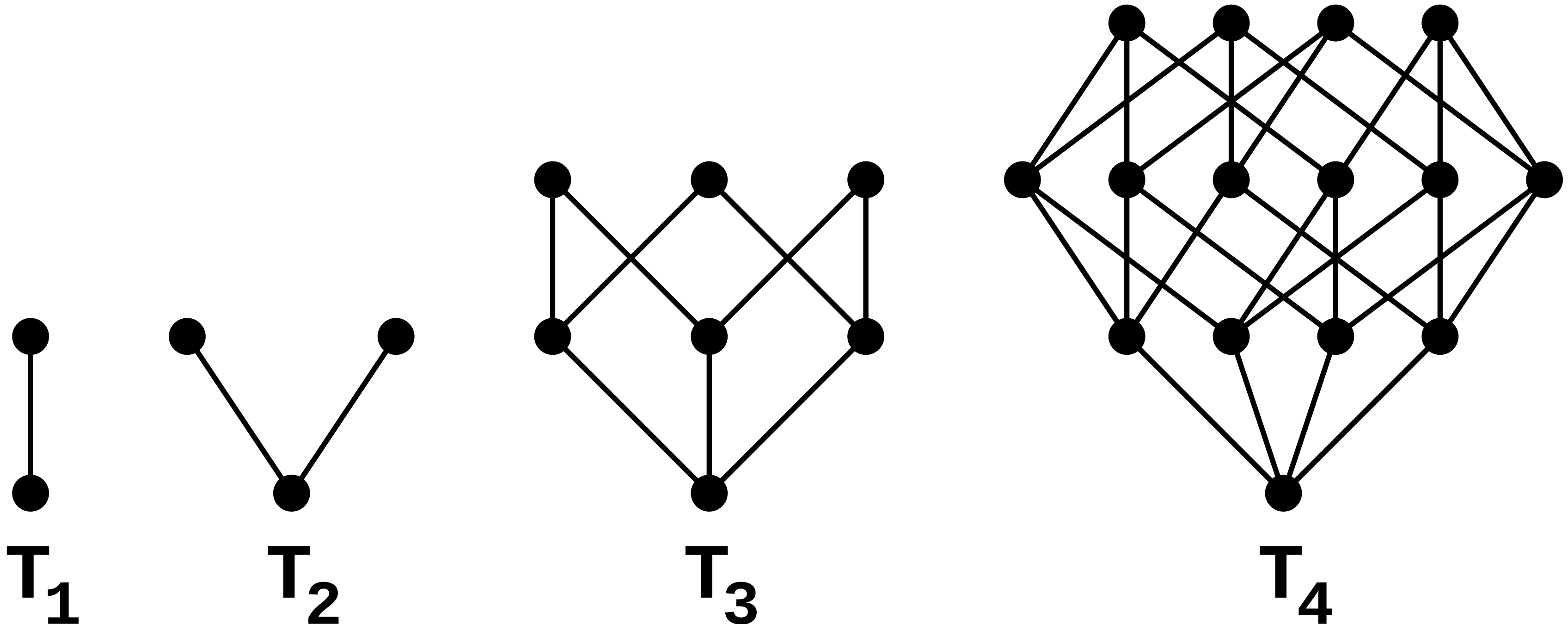
What is the computational understanding of T_k?



The ‘whodoneit’ interpretation