Coherence Numbers of Domains

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Motivating the Audience

A classification of PER models on domains.

An unexpected theorem about reflexive domains.

Terminology

- (Scott) domain: ω-algebraic bounded complete dcpo
- Continuous retract: $D \lhd E$ ("E contains D")

$$D \underset{r}{\overset{i}{\rightleftarrows}} E \qquad\qquad r \circ i = \mathrm{id}_D$$

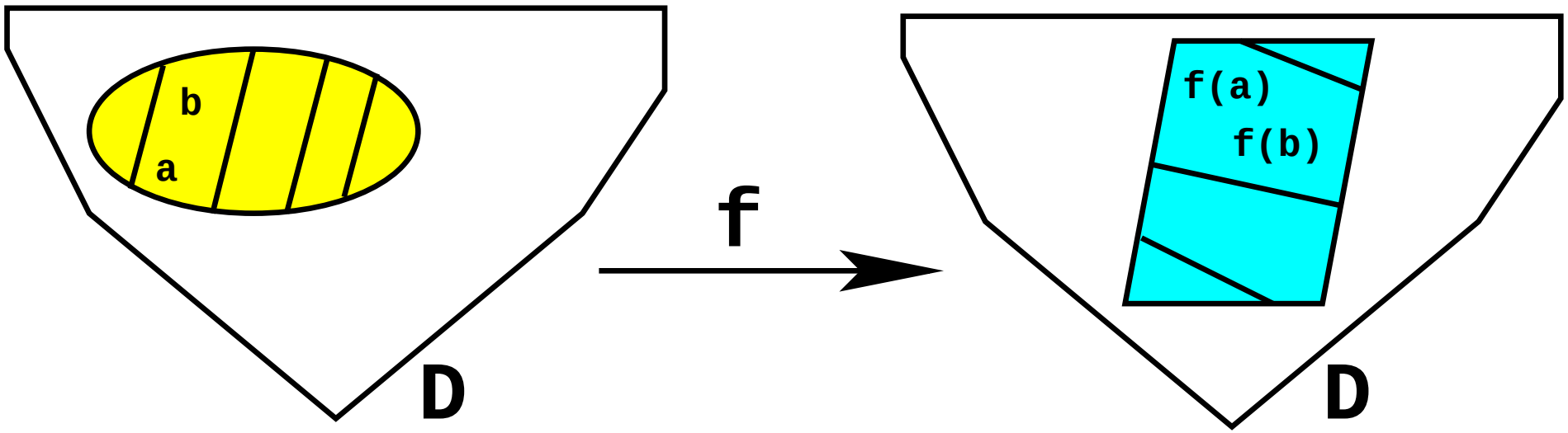
- Universal domain U for a class $\mathcal{C}$ of domains:
  $U \in \mathcal{C}$ and $E \lhd U$ for all $E \in \mathcal{C}$.
- Reflexive domain: $[D \to D] \lhd D$ and $D \neq \{\bot\}$

PER Models

A reflexive domain D is a model of untyped λ-calculus, therefore a *combinatory algebra*:

$$K = \lambda x.\, \lambda y.\, x \qquad\qquad S = \lambda x.\, \lambda y.\, \lambda z.\, (xz)(yz)$$

PER(D), category of *partial equivalence relations* on D:



How many PER models are there?

Given reflexive domains D and E, when is

$$\mathsf{PER}(D) \simeq \mathsf{PER}(E) \quad ?$$

Some well-known PER models:

- $\mathsf{PER}(\mathcal{P}\omega)$ where $\mathcal{P}\omega$ is the graph model
- $\mathsf{PER}(T^{\omega})$ where T^{ω} is Plotkin's universal coherent domain
- $\mathsf{PER}(U)$ where U is universal for Scott domains

These are all different.

The Answer

Theorem:

For reflexive D and E,

$$\mathsf{PER}(\mathsf{D}) \simeq \mathsf{PER}(\mathsf{E}) \quad \Longleftrightarrow \quad \mathsf{coh}(\mathsf{D}) = \mathsf{coh}(\mathsf{E})$$

where $\mathsf{coh}(\mathsf{D})$ is the *coherence number* of D,

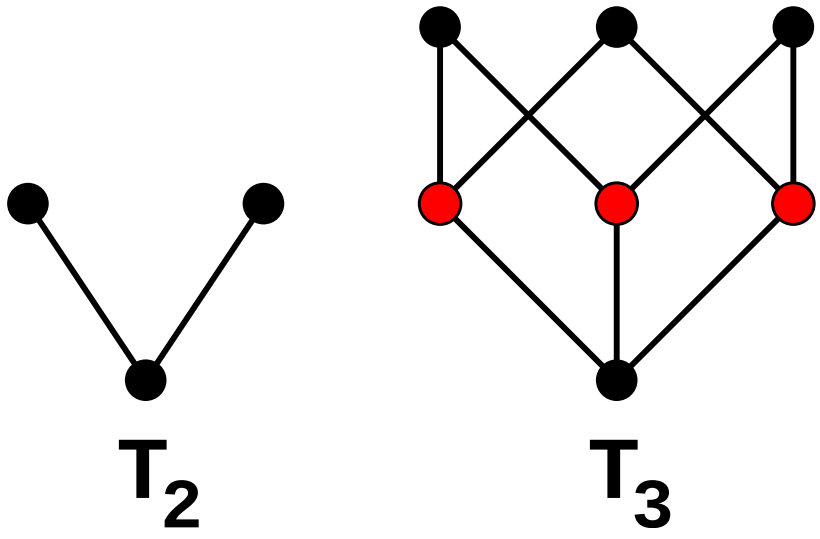
$1 \leq \mathsf{coh}(\mathsf{D}) \leq \omega$.

Coherent Domains

[Plotkin'78]

D is *coherent* when, for all $S \subseteq D$,

$$(\forall x_1, x_2 \in S.\{x_1, x_2\} \text{ bounded}) \Longrightarrow S \text{ bounded}$$



Coherence Numbers

Definition:

D is *n-coherent*, when for all $S \subseteq D$,

$$(\forall x_1, \ldots, x_n \in S. \{x_1, \ldots, x_n\} \text{ bounded}) \Longrightarrow S \text{ bounded}$$

Additionally, we say that every D is *ω-coherent*.

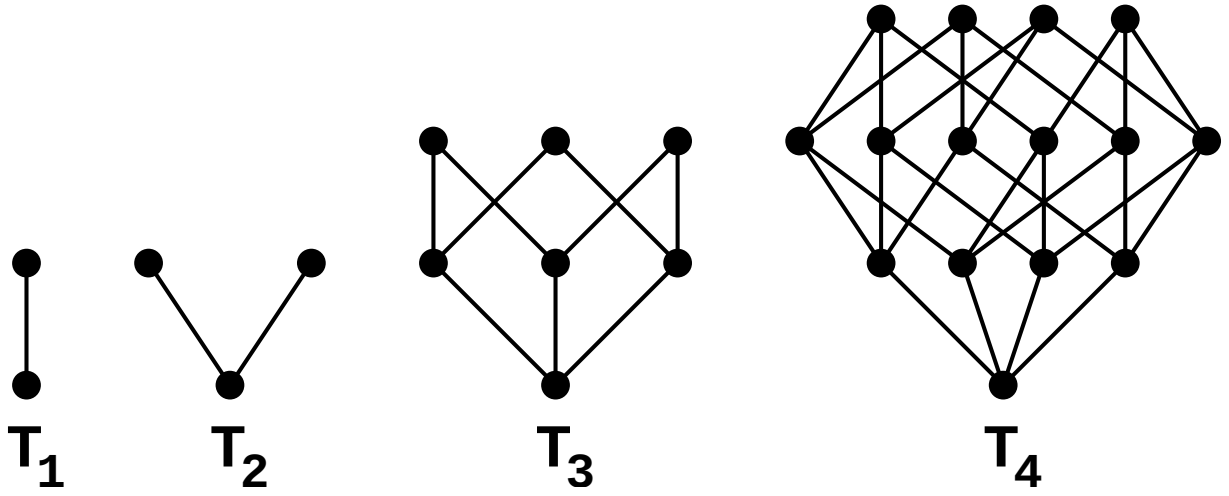
Coherence number:

$$\mathrm{coh}(D) = \min \left\{ 1 \leq n \leq \omega \;\middle|\; D \text{ is } n\text{-coherent} \right\}$$

Examples

1-coherent = lattice

2-coherent = Plotkin coherent



$$T_1 = \Sigma = \{\bot, \top\}$$

$$T_n = \mathcal{P}(\{1, \ldots, n\}) \setminus \{\{1, \ldots, n\}\} \qquad (2 \leq n < \omega)$$

Observations about Coherence Numbers

Proposition: $\mathsf{coh}(T_n) = n$

Proposition:

Let $2 \leq n < \omega$. Then $T_n \triangleleft D \iff \mathsf{coh}(D) \geq n$.

Proposition:

- $\mathsf{coh}(D \times E) = \max(\mathsf{coh}(D), \mathsf{coh}(E))$
- $\mathsf{coh}(D^\omega) = \mathsf{coh}(D)$
- $\mathsf{coh}([D \to E]) = \mathsf{coh}(E)$

n-Coherent Domains

Let Coh_n be the category of n-coherent domains.

$$\mathsf{Lat} = \mathsf{Coh}_1 \subseteq \mathsf{Coh}_2 \subseteq \mathsf{Coh}_3 \subseteq \cdots \subseteq \mathsf{Coh}_\omega = \mathsf{Dom}$$

Each Coh_n is a cartesian-closed subcategory of Dom.

Does Coh_n have a universal domain?

T_1^ω is universal for Coh_1.

T_2^ω is universal for Coh_2.

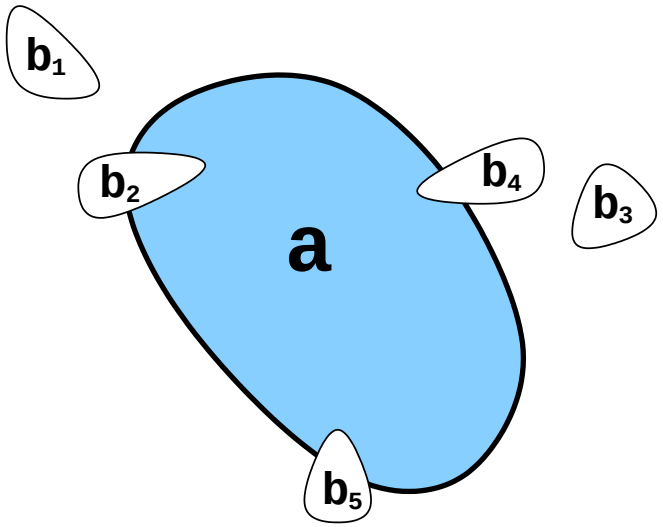
Therefore by induction...

Universal domain for Coh_n

Let $\mathbb{F}_n = \{x \subseteq \mathbb{N} \mid |x| < n\}$, and define:

$$\mathbb{C}_n = \{\langle a, B\rangle \in \mathcal{P}\mathbb{N} \times \mathcal{P}\mathbb{F}_n \mid \forall b \in B.\, b \not\subseteq a\}$$

A typical element $\langle a, \{b_1, b_2, \ldots\}\rangle$ of $\mathbb{C}_n$:



$$\mathbb{C}_1 \cong \mathcal{P}\mathbb{N} \cong \mathsf{T}_1^\omega \qquad \mathbb{C}_2 \cong \mathsf{T}_2^\omega$$

Universal domain for Coh_n

Theorem: $\mathbb{C}_n$ is universal for Coh_n.

Proof: a generalization of [Plotkin'78].

Theorem:

- $\mathbb{C}_n \triangleleft \mathsf{T}_n^\omega \quad$ for $n < \omega$,
- $\mathbb{C}_\omega \triangleleft \prod_{n<\omega} \mathsf{T}_n$.

Hence T_n^ω is universal for Coh_n,

and $\prod_{n<\omega} \mathsf{T}_n$ is universal for $\mathsf{Coh}_\omega = \mathsf{Dom}$.

A Theorem about Reflexive Domains

Theorem:

Let D be a domain with $n = \mathsf{coh}(D)$.

The following are equivalent:

1. D is reflexive
2. $D \times D \lhd D$
3. $E \times D \lhd D$ for some E with $\mathsf{coh}(E) = \mathsf{coh}(D)$
4. $T_k \times D \lhd D$ for all $k < 1 + n$
5. D is universal for Coh_n

Reflexive domains are easy to come by

Corollary:

D^ω is reflexive for any non-trivial domain D.

Proof: $\mathsf{D}^\omega \times \mathsf{D}^\omega \cong \mathsf{D}^\omega$

Corollary:

$[\mathsf{E} \to \mathsf{D}]$ is reflexive if $\max(\mathsf{E})$ is infinite and D non-trivial.

Proof: $\mathbb{N}_\perp \lhd \mathsf{E}$, therefore $\mathsf{D}^\omega \lhd [\mathbb{N}_\perp \to \mathsf{D}] \lhd [\mathsf{E} \to \mathsf{D}]$

Classification of PER models

Theorem:

For reflexive D and E,

$$\mathsf{coh}(\mathsf{D}) = \mathsf{coh}(\mathsf{E}) \quad \Longrightarrow \quad \mathsf{PER}(\mathsf{D}) \simeq \mathsf{PER}(\mathsf{E})\ .$$

Proof:

Sufficient to prove $\mathsf{D} \lhd \mathsf{E}$ and $\mathsf{E} \lhd \mathsf{D}$.

Let $\mathfrak{n} = \mathsf{coh}(\mathsf{D}) = \mathsf{coh}(\mathsf{E})$.

D reflexive $\Longrightarrow$ D universal for $\mathsf{Coh}_{\mathfrak{n}}$ $\Longrightarrow$ $\mathsf{E} \lhd \mathsf{D}$.

Similarly for $\mathsf{D} \lhd \mathsf{E}$. QED.

Classification of PER models

Every $\mathsf{PER}(\mathsf{D})$ is equivalent to precisely one of

$$\mathsf{PER}(\mathsf{T}_1^\omega)$$

$$\mathsf{PER}(\mathsf{T}_2^\omega)$$

$$\mathsf{PER}(\mathsf{T}_3^\omega)$$

$$\vdots$$

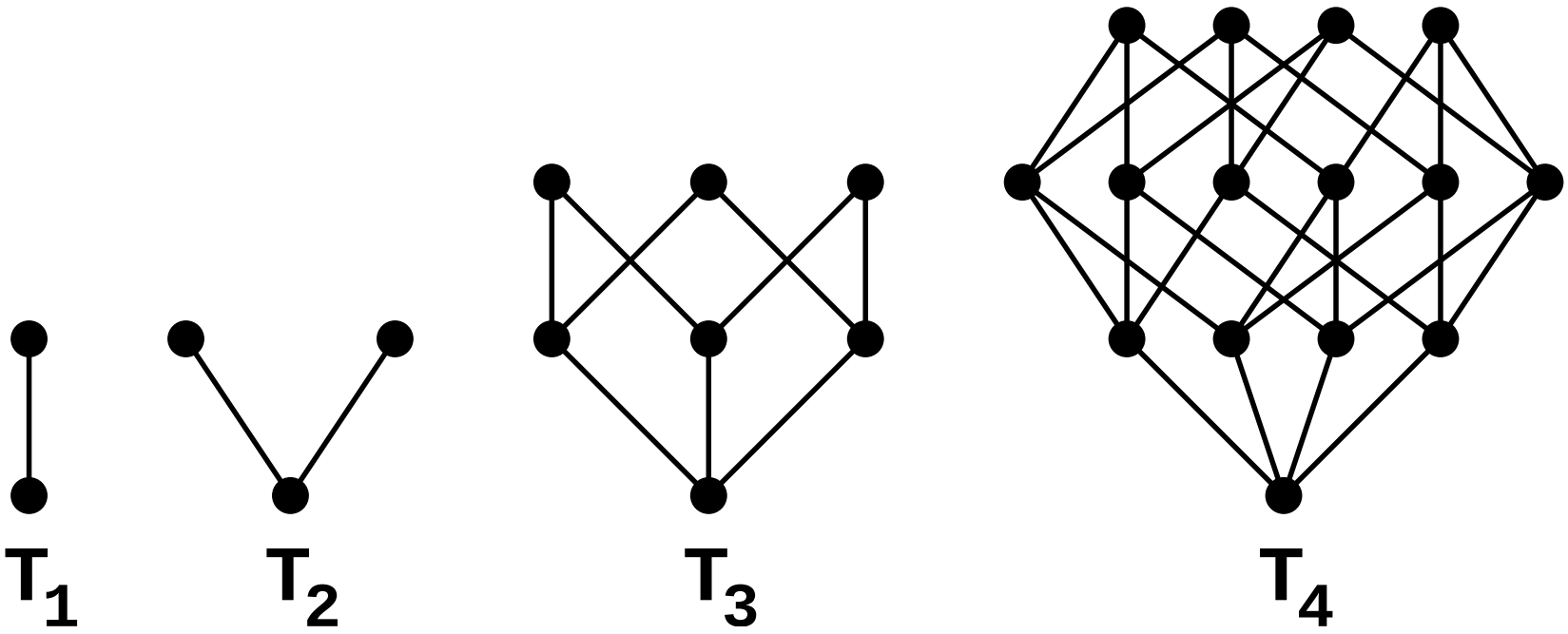
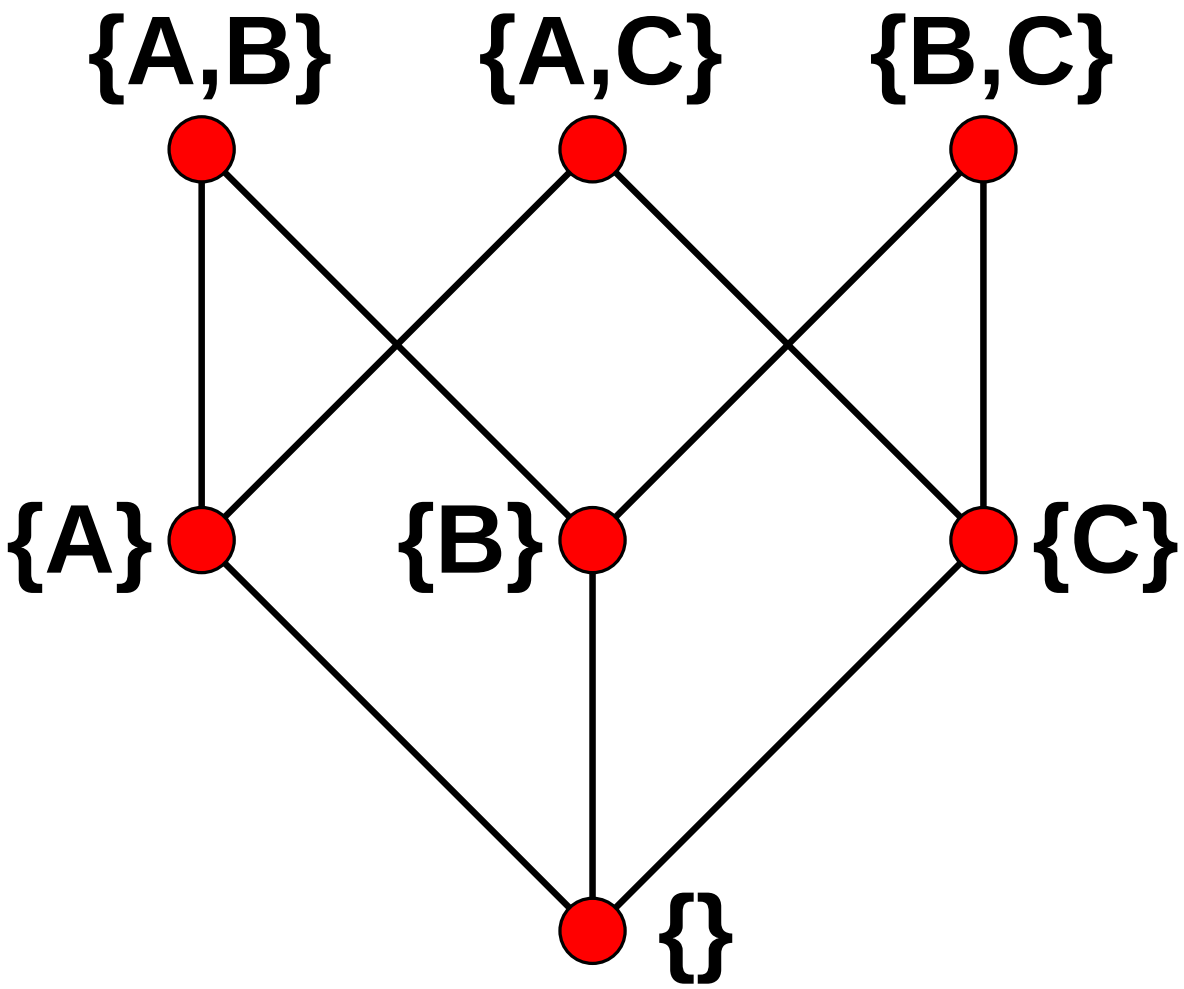
$$\mathsf{PER}(\textstyle\prod_{n<\omega} \mathsf{T}_n)$$

An analogous statement holds for realizability toposes.

Questions

What is this good for?

What is the computational understanding of T_k?



The ‘whodoneit’ interpretation