The Realizability Approach to Computable Analysis and Topology

Andrej Bauer

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School of Computer Science
Carnegie Mellon University
Pittsburgh, PA 15213

Thesis Committee

Dana S. Scott, chair
Steven M. Awodey
Lenore Blum
Abbas Edalat
Frank Pfenning

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To disertacijo posvečam mojemu očetu,
ki je moj prvi in najvplivnejši učitelj matematike.

I dedicate this dissertation to my father,
who is my first and most influential teacher of mathematics.

Abstract

In this dissertation, I explore aspects of computable analysis and topology in the framework of relative realizability. The computational models are partial combinatory algebras with subalgebras of computable elements, out of which categories of modest sets are constructed. The internal logic of these categories is suitable for developing a theory of computable analysis and topology, because it is equipped with a computability predicate and it supports many constructions needed in topology and analysis. In addition, a number of previously studied approaches to computable topology and analysis are special cases of the general theory of modest sets.

In the first part of the dissertation, I present categories of modest sets and axiomatize their internal logic, including the computability predicate. The logic is a predicative intuitionistic first-order logic with dependent types, subsets, quotients, inductive and coinductive types.

The second part of the dissertation investigates examples of categories of modest sets. I focus on equilogical spaces, and their relationship with domain theory and Type Two Effectivity (TTE). I show that domains with totality embed in equilogical spaces, and that the embedding preserves both simple and dependent types. I relate equilogical spaces and TTE in three ways: there is an applicative retraction between them, they share a common cartesian closed subcategory that contains all countably based T_0-spaces, and they are related by a logical transfer principle. These connections explain why domain theory and TTE agree so well.

In the last part of the dissertation, I demonstrate how to develop computable analysis and topology in the logic of modest sets. The theorems and constructions performed in this logic apply to all categories of modest sets. Furthermore, by working in the internal logic, rather than directly with specific examples of modest sets, we argue abstractly and conceptually about computability in analysis and topology, avoiding the unpleasant details of the underlying computational models, such as Gödel encodings and representations by sequences.

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Contents

Introduction

The idea of realizability originated in 1945 with Kleene's number realizability [Kle45]. Since then realizability has become a subject in itself, with numerous applications in logic, mathematics, and computer science. In this dissertation I use the tools of realizability to study computability in topology and analysis. Let me explain first, how realizability arises quite naturally in computer science.

One of the most common tasks in computer science is the design and implementation of data structures and algorithms. Let $\mathbb{C}$ be a chosen model of computation, such as the set of all Turing machines, the valid programs of a programming language, or well formed flow-chart diagrams that describe algorithms. We think of the elements of $\mathbb{C}$ as programs even though some of them might represent data—the distinction between programs and data is not important right now.

Suppose we wish to implement an abstract mathematical structure S in $\mathbb{C}$. For example, S could be the set of finite binary trees, or the triangulations of 3-dimensional polyhedra. What does it mean to implementation S in $\mathbb{C}$? It means that we represent, or *realize*, each element $s \in S$ by a program $p \in \mathbb{C}$. It is usually the case that an element $s \in S$ has many representations. For example, a common implementation of binary trees does not uniquely determine where in computer memory the representative of a given tree is located, which means that for every possible location we get a different representative. Thus, an implementation of S is a relation between $\mathbb{C}$ and S, called the *realizability relation* and written as $\Vdash_S$. The realizability relation $\Vdash_S$ relates implementations to elements; the reading of $p \Vdash_S s$ is "the program p realizes (represents, implements) the element $s \in S$". In order for a realizability relation to make sense it must satisfy two conditions: every element $s \in S$ has at least one realizer, and every program $p \in \mathbb{C}$ realizes at most one element of S.[1] A set with a realizability relation $(S, \Vdash_S)$ is called a *modest set*. It is "modest" because its cardinality cannot exceed the cardinality of $\mathbb{C}$, as follows from the second condition.

If $(S, \Vdash_S)$ and $(T, \Vdash_T)$ are modest sets, then a function $f \colon S \to T$ is said to be *realized* (represented, implemented) by a program $p \in \mathbb{C}$ when $a \Vdash_S x$ implies $p(a) \Vdash_S f(x)$. In words, p maps the realizers of x to the realizers of $f(x)$. We also say that p *tracks* f and write $p \Vdash_{S \to T} f$. A function between modest sets which has a realizer is called a *realized function*.

We refine the notion of a computational model by distinguishing between possible data and computable data. This is easiest to understand by example. Suppose $\mathbb{C}$ is the model of computation in which a Turing machine reads from an *infinite* input tape and writes onto an infinite output tape. For simplicity, we assume that the only two symbols that can be read from or written onto a tape are `0` and `1`. Such a tape may contain any infinite binary sequence, whether it is computable or not, but among all sequences we can hope to actually construct only the *computable* ones.[2] The

[1] The second condition is not strictly necessary, and leads to the category of *assemblies*.

[2] A sequence is computable when there exists a Turing machine that outputs that sequence, no matter what input

model $\mathbb{C}$ is then just the space of all binary sequences $\mathbb{C} = \{0,1\}^{\mathbb{N}}$, which correspond to infinite tapes, and it contains a *submodel* $\mathbb{C}_\sharp \subseteq \mathbb{C}$ consisting of all computable binary sequences.[3] This example suggests that in general a refined model of computation should be a pair of computational models $(\mathbb{C}, \mathbb{C}_\sharp)$ where $\mathbb{C}_\sharp \subseteq \mathbb{C}$ is a submodel of $\mathbb{C}$. We refer to the submodel $\mathbb{C}_\sharp$ as the *computable part* of $\mathbb{C}$, even though it might be unrelated to the usual notion of Turing computability.

The definition of modest sets and realized functions is refined accordingly. Modest sets represent data, whereas realized functions represent computations on data. Therefore, elements of modest sets are allowed to have any realizers, but we require that realized functions have only computable realizers. We refer to this kind of realizability as *relative realizability* over $(\mathbb{C}, \mathbb{C}_\sharp)$. In relative realizability we can talk about computable functions that take potentially non-computable data as input. Every modest set S has a *computable part* $\#S \subseteq S$, which contains those elements of S that have computable realizers, and has the realizability relation restricted to $\mathbb{C}_\sharp$. With the "sharp" operator $\#$ various computability notions can be expressed, e.g., $\#\mathbb{R}$ is the space of computable reals, $\#(\mathbb{R}^{\mathbb{R}})$ is the space of computable real functions, $(\#\mathbb{R})^{\#\mathbb{R}}$ is the space of all functions that map computable reals to computable reals, and $\#((\#\mathbb{R})^{\#\mathbb{R}})$ is the space of computable functions on computable reals.

We have been imprecise about what we mean by models and submodels of computation. In this dissertation a model of computation is a *partial combinatory algebra* (PCA). A PCA $\mathbb{A}$ is a set with a partial application operation $\cdot \colon \mathbb{A} \times \mathbb{A} \rightharpoonup \mathbb{A}$. The reading of $x \cdot y$ is "apply program x to data y", where the result may be undefined. A PCA must also have two distinguished *basic combinators* K and S that satisfy certain equations, cf. Definition 1.1.1. A submodel of a PCA $\mathbb{A}$ is a *subPCA* $\mathbb{A}_\sharp \subseteq \mathbb{A}$, which is a subset of $\mathbb{A}$ that is closed under application and contains the two basic combinators. The principal example of a PCA with a subPCA is the *graph model* $\mathcal{P}\mathbb{N}$ with the *recursively enumerable* submodel $\mathsf{RE} \subseteq \mathcal{P}\mathbb{N}$ consisting of recursively enumerable sets. PCAs are untyped models of computation since every element can be viewed as a program or as data. We could also develop realizability theory starting with *typed* models of computation, such as continuous domains or syntactic models of typed programming languages, but we do not do that because the examples we are most interested in are all (equivalent to) untyped models.

Just like ordinary set theory serves as a foundation for classical mathematics, the theory of modest sets serves as a foundation for *computation-aware mathematics*. If we systematically replace sets and functions with modest sets and realized functions then all mathematical structures and maps between them automatically carry realizability relations which tell us how to implement them in the chosen model of computation. However, it quickly turns out that it is rather cumbersome to work explicitly in terms of realizability relations because we have to deal with the peculiarities of the computational model $(\mathbb{C}, \mathbb{C}_\sharp)$. Ideally we would like to think of modest sets abstractly as just ordinary sets. Category theory and categorical logic tell us how this can be done. Modest sets and realized functions form a category $\mathsf{Mod}(\mathbb{C}, \mathbb{C}_\sharp)$ that is equipped with an *internal logic*. In this logic modest sets appear simply as *spaces of points* and realized functions as *maps between spaces*—there is no mention of realizability relations and realizers anywhere. We can always recover the realizability relations by computing the interpretation of the logic in the category of modest

it is given.

[3]It may seem puzzling that the model $\mathbb{C}$ consists of tapes only, and not also of Turing machines. This is so because the description of a Turing machine can be written onto a tape. Thus Turing machines are just tapes, of course properly interpreted. In modern computers this is what actually happens—both data and instructions are just sequences of 0's and 1's!

sets.

The plan then is to develop mathematical analysis and topology in the internal logic of modest sets. We adhere to the logical rules and reasoning principles that are valid in the logic of modest sets. In principle it could happen that these rules and principles were too weak to allow us to carry out the sort of constructions and arguments that are required in analysis and topology. Luckily, this is not the case at all. The logic of modest sets provides all the usual constructions of spaces: cartesian products, disjoint sums, function spaces, subspaces, quotient spaces, dependent products and dependent sums, inductive and coinductive types.[4] The main difference between classical set theory and the logic of modest sets is that the latter is an *intuitionistic* logic. This means that we cannot unrestrictedly use the Law of Excluded Middle, proof by contradiction, and the Axiom of Choice. In certain important cases these laws are still valid: the Law of Excluded Middle is valid for *decidable spaces*, and the Axiom of Choice is valid for the *projective spaces*. The natural numbers are both decidable and projective. In addition, we can prove equality of two points by contradiction, that is $\neg(x \neq y)$ implies $x = y$. Another reasoning principle which is valid in the logic of modest sets is *Markov's principle*: if a map $f \colon \mathbb{N} \to \mathbb{N}$ is not constantly zero then there exists $n \in \mathbb{N}$ such that $f(n) \neq 0$. Markov's principle significantly simplifies the theory of real numbers because it implies that the *apartness relation* and inequality on the reals coincide, cf. Proposition 5.5.19.

A number of approaches to computable analysis and topology are closely related to categories of modest sets. Domain theoretic models, such as effectively presented domains [Eda97] and domains with totality [Nor98a, Ber93, Ber97a], form subcategories of $\mathsf{Mod}(\mathcal{P}\mathbb{N}, \mathsf{RE})$. Domain representations [Bla97a, Bla97b] are equivalent to $\mathsf{Mod}(\mathbb{U}, \mathbb{U}_\sharp)$, where $\mathbb{U}$ is the universal Scott domain, and a PER model on a reflexive domain D is equivalent to $\mathsf{Mod}(D)$. The ambient category of Type Two Effectivity (TTE) [Wei00, Wei95, Wei85, Wei87, BW99, KW85] is $\mathsf{Mod}(\mathbb{B}, \mathbb{B}_\sharp)$, where $\mathbb{B} = \mathbb{N}^\mathbb{N}$ is the second Kleene algebra and $\mathbb{B}_\sharp$ is its effective version. In Recursive Mathematics [EGNR99a, EGNR99b] and in the theory of Spreen's effective T_0-spaces [Spr98] numbered sets play a central role. Numbered sets are just modest sets over the first Kleene algebra.[5] Realizability also covers the Blum-Shub-Smale model of real computation [BCSS97] and partial topological while^* algebras by Tucker and Zucker [TZ99], but perhaps less naturally so. Thus the realizability approach presented in this dissertation provides a unifying framework for a number of well-studied models of computable analysis and topology. It also relates constructive analysis[6] and computable analysis,[7] by showing how to interpret the former in the latter.

For the purposes of computable topology and analysis the most interesting examples of realizability models are those in which the underlying computational model is itself a topological space. The two principal examples are the graph model $\mathcal{P}\mathbb{N}$ and the second Kleene algebra $\mathbb{B}$. The corresponding categories of modest sets are Scott's equilogical spaces and representations (in the sense of TTE). These two categories have a very topological flavor and contain the category of countably based T_0-spaces. I compare these categories in Chapter 4.

There are some aspects of computation that I have not considered in this dissertation, most notably, questions of computational resources and computational complexity. One way to incor-

[4] You may have noticed that *powersets* are missing from this list. This is so because categories of modest sets fail to be toposes. A more advanced realizability construction yields a *realizability topos* in which a category of modest sets is included.

[5] This was already observed by Hyland [Hyl82].

[6] By "constructive analysis" we mean analysis developed in a constructive logic.

[7] By "computable analysis" we mean analysis developed in a model of computation, such as recursion theory or TTE.

porate a notion of computational resources would be to use *linear* PCAs, originally defined by Abramsky. Abramsky and Lenisa [AL00] studied PER models on linear combinatory algebras and established their basic properties. It would be interesting to see what kind of analysis and topology can be developed in such linear realizability models. In numerical analysis we are often interested in fairly abstract measures of complexity. For example, in iterative methods we count the number of iterations and take as the basic units the algebraic operations on real numbers, disregarding the computational complexity of the operations themselves. This sort of complexity analysis can be performed in the logic of modest sets just as well as in classical numerical analysis, but more work needs to be done in the area of intuitionistic numerical analysis.

Overview of the Chapters

The dissertation is divided into five chapters. The first three comprise the basic theory of modest sets. The fourth chapter deals with the two principal models of modest sets—equilogical spaces and TTE. In the last chapter I use the logic of modest sets to develop a selection of topics in computable topology and analysis. A detailed description of each chapter follows.

Chapter 1: Categories of Modest Sets.

We begin with the definition of partial combinatory algebra (PCA) and subalgebra, and state some basic properties of PCAs. Then we consider examples of PCAs: the first Kleene algebra $\mathbb{N}$, the graph model $\mathbb{P} = \mathcal{P}\mathbb{N}$ and its computable subPCA $\mathbb{P}_\sharp = \mathsf{RE}$, the universal Scott domain $\mathbb{U}$ and its computable part $\mathbb{U}_\sharp$, the second Kleene algebra $\mathbb{B} = \mathbb{N}^\mathbb{N}$ and its computable subPCA $\mathbb{B}_\sharp$ of total recursive functions, and lastly a PCA over a first-order structure. We prove the Embedding and Extension Theorems for $\mathbb{P}$ and $\mathbb{B}$, which we use in Chapter 4 to prove that $\mathsf{Mod}(\mathbb{P})$ is equivalent to the category of equilogical spaces, and $\mathsf{Mod}(\mathbb{B})$ is equivalent to the category of 0-equilogical spaces.

In the second section we define categories of modest sets and show that modest sets can also be viewed as partial equivalence relations and as representations. We recall the basic constructions in categories of modest sets, and prove that a category of modest sets has inductive and coinductive types.

The last section of the chapter reviews the definition and basic properties of applicative morphisms between PCAs. Longley's original definition is extended to the case of relative realizability. An applicative morphism between PCAs induces a functor between the corresponding categories of modest sets, and an adjunction of applicative morphisms induces a functorial adjunction.

Chapter 2: A Logic for Modest Sets.

The internal logic of modest sets is presented by an informal and rigorous axiomatic method. The logic of modest sets is an intuitionistic first-order logic with $\neg\neg$-stable equality. The following simple types are axiomatized: function spaces, products, disjoint sums, the empty and the unit spaces, subspaces, and quotient spaces. We examine the basic properties of maps and prove that every map factors uniquely (up to isomorphism) into a quotient map, bijection, and an embedding.

Next, we present dependent sums and products, and axiomatize inductive and coinductive types as initial algebras and final coalgebras, respectively. We prove that the Axiom of Inductive Types implies an induction principle, and that the Axiom of Coinductive Types implies a coinduction principle. Natural numbers and finite lists are presented as examples of inductive spaces, and infinite streams and spreads are presented as examples of coinductive spaces.

In the last section we introduce the computability predicate $\#$, the Axiom of Computability, and deduce basic facts about computability of maps and natural numbers. We relate decidable predicates on a space with the space of maps into 2, define decidable spaces, and show that the natural numbers are a decidable space. Finally, we postulate Markov's principle, define projective spaces and relate them to the axiom of choice, and state the Axioms of Projective Spaces and Number Choice.

Chapter 3: The Realizability Interpretation of the Logic of Modest Sets.

We explain how the logic of modest sets is interpreted in a category of modest sets $\mathsf{Mod}(\mathbb{A}, \mathbb{A}_\sharp)$. We adopt the realizability interpretation of first-order logic. A space A is interpreted as a modest set $[\![A]\!]$ and a dependent type is interpreted as a uniform family of modest sets. The simple and complex types are interpreted by the corresponding category-theoretic constructions: function spaces are interpreted as exponentials, products and disjoint sums as categorical products and coproducts, the empty space as the initial object, the unit space as the terminal object, subspaces as subobjects, quotient spaces as coequalizers. Dependent types, inductive and coinductive spaces are interpreted as the corresponding categorical versions. The computability predicate $\#_A$ on space A is interpreted as the subobject $\#[\![A]\!] \hookrightarrow [\![A]\!]$, where $\#[\![A]\!]$ is the computable part of $[\![A]\!]$.

We show that Markov's Principle, the Axiom of Projective Spaces, and Number Choice are valid in the realizability interpretation. Lastly, we internalize the realizability interpretation by introducing the realizability operator into the logic of modest sets.

Chapter 4: Equilogical Spaces and Related Categories.

Our principal realizability model is the category of equilogical spaces Equ.[8] The following categories are equivalent formulations of equilogical spaces: equivalence relations on countably based T_0-spaces, partial equivalence relations on countably based algebraic lattices, partial equivalence relations on the graph model $\mathbb{P}$, modest sets over the graph model $\mathbb{P}$, and dense partial equivalence relations on Scott domains. We state and prove the basic properties of Equ.

We define the category of effective T_0-spaces $\mathsf{Top}_{\mathsf{eff}}$ and show that the category of effective equilogical spaces $\mathsf{Equ}_{\mathsf{eff}}$, formed by equivalence relations on effective T_0-spaces, is equivalent to $\mathsf{Mod}(\mathbb{P}, \mathbb{P}_\sharp)$. The embedding of $\mathsf{Top}_{\mathsf{eff}}$ into $\mathsf{Equ}_{\mathsf{eff}}$ preserves limits, coproducts and all exponentials that exist in $\mathsf{Top}_{\mathsf{eff}}$. The category of effectively presented continuous domains is a full subcategory of $\mathsf{Top}_{\mathsf{eff}}$, and the embedding preserves products and exponentials.

We prove that dense and codense totalities on Scott domains embed into Equ and that the embedding preserves the cartesian closed structure. We extend this result to dependent types and show that dependent sums and products on dense, codense, consistent and natural dependent totalities agree with those in equilogical spaces. It follows that Kleene-Kreisel countable functionals of finite and dependent types are formed by repeated exponentiation and dependent product formation, starting with the natural numbers object.

The category of partial equivalence relations on effective Scott domains $\mathsf{PER}(\mathsf{Dom}_{\mathsf{eff}})$, also known as the category of domain representations, is equivalent to $\mathsf{Mod}(\mathbb{U}, \mathbb{U}_\sharp)$, where $\mathbb{U}$ is the

[8] We only consider *countably based* equilogical spaces.

universal Scott domain and $\mathbb{U}_\sharp$ is its computable part. The category $\mathsf{Equ}_{\mathsf{eff}}$ is a full subcategory of $\mathsf{PER}(\mathsf{Dom}_{\mathsf{eff}})$ because it is equivalent to the category of dense partial equivalence relations on effective Scott domains. The inclusion functor is induced by an applicative inclusion between $(\mathbb{P}, \mathbb{P}_\sharp)$ and $(\mathbb{U}, \mathbb{U}_\sharp)$, therefore it has a left adjoint, which means that $\mathsf{Equ}_{\mathsf{eff}}$ is a reflective subcategory of $\mathsf{PER}(\mathsf{Dom}_{\mathsf{eff}})$. This is also the case for the relationship between the non-effective versions of these categories, Equ and $\mathsf{PER}(\omega\mathsf{Dom})$. Moreover, the inclusion of Equ into $\mathsf{PER}(\omega\mathsf{Dom})$ has both a left and a right adjoint. The category Equ is equivalent to the category of partial equivalence relations on Scott domains and *partial* maps between them. We also show that Equ and $\mathsf{PER}(\omega\mathsf{Dom})$ are not equivalent.

In the second section we compare equilogical spaces and Type Two Effectivity, which is the study of computable analysis and topology in $\mathsf{Mod}(\mathbb{B}, \mathbb{B}_\sharp)$. The category $\mathsf{Mod}(\mathbb{B}, \mathbb{B}_\sharp)$ embeds fully and faithfully into $\mathsf{Equ}_{\mathsf{eff}}$. The inclusion has a right adjoint, and the adjointness is induced by an applicative retraction between $(\mathbb{P}, \mathbb{P}_\sharp)$ and $(\mathbb{B}, \mathbb{B}_\sharp)$. The category $\mathsf{Mod}(\mathbb{B}, \mathbb{B}_\sharp)$ is equivalent to the category of 0-equilogical spaces $\mathsf{0Equ}$. A 0-equilogical space is a 0-dimensional countably based T_0-spaces with an equivalence relation. Let $\omega\mathsf{Top}_0$ be the category of countably based T_0-spaces and continuous maps, and let Seq be the category of sequential spaces. Menni and Simpson [MS00] defined a category PQ_0, which is the largest common cartesian closed subcategory of Equ and Seq. Similarly, Schröder [Sch00] defined the category AdmSeq which is a common cartesian closed subcategory of $\mathsf{Mod}(\mathbb{B})$ and Seq. Both categories contain $\omega\mathsf{Top}_0$ as a full subcategory. We prove that PQ_0 and AdmSeq coincide. This result tells us that Equ and $\mathsf{Mod}(\mathbb{B})$ are equivalent as far as the cartesian closed structure over $\omega\mathsf{Top}_0$ is concerned.

In the last section we use topos-theoretic tools to further compare modest sets over different PCAs. The motivating example is the comparison of Equ and $\mathsf{Mod}(\mathbb{B})$. We build a category of sheaves on a PCA and apply the theory of localic local maps between toposes [ABS99] to obtain a logical transfer principle between $\mathsf{Mod}(\mathbb{P})$ and $\mathsf{Mod}(\mathbb{B})$. This principle says that the so called local sentences are valid in one category if, and only if, they are valid in the other one.

Chapter 5: Computable Topology and Analysis.

In this chapter we developed a selection of topics in computable topology and analysis in the logic of modest sets. We compute the interpretations of a number of examples in order to show that topology and analysis developed in the logic of modest sets correspond to the usual computable topology and analysis in various categories of modest sets.

First we consider the intuitionistic theory of countable sets, prove a Minimization Principle for decidable predicates on $\mathbb{N}$, and show that countable sets in $\mathsf{Mod}(\mathbb{N})$ are, up to effective isomorphism, those numbered sets who numbering is a total function.

In the second section we look at the generic convergent sequence $\mathbb{N}^+$. Classically this is the one-point compactification of the natural numbers. In the logic of modest sets $\mathbb{N}^+$ is defined as a coinductive type.

The third section is devoted to semidecidable predicates and dominances. We study the "standard" dominance Σ, obtained as a suitable quotient of the Cantor space. The standard dominance satisfies Phoa's principle if, and only if, a weak continuity principle is satisfied in the logic of modest sets. It is also related to non-existence of discontinuous functions on real numbers. From a dominance Σ we obtain a notion of Σ-partial maps and a lifting functor. The standard dominance also serves for classifying the intrinsically open subspaces of a space—for any space A the space

Σ^A acts as an intrinsic topology on A.

The sections on countable sets, convergent sequences, and semidecidable predicates are brought together in the fourth section, in which a theory of countably based spaces in the logic of modest sets is introduced. There are two formulations of countably based spaces and continuous maps—a pointwise one and a point-free one. The point-free version has better properties. It turns out that the interpretation of pointwise countably based spaces in $\mathsf{Mod}(\mathbb{N})$ gives exactly Spreen's T_0-spaces [Spr98], whereas the interpretation of the point-free version in $\mathsf{Mod}(\mathbb{N})$ corresponds exactly to the theory of RE-T_0-spaces, which are the regular projectives in $\mathsf{Mod}(\mathsf{RE})$. We also show that the point-wise and the point-free version are not equivalent by giving an example of a Spreen T_0-space which is not an RE-T_0-space.

The fifth section discusses the real numbers. We construct the reals using the usual Cauchy completion of the rational numbers. We show that this construction is isomorphic to the signed binary digit representation of the reals that is often used in exact real arithmetic. We focus on the algebraic structure of the space of real numbers. We review the intuitionistic theory of ordered fields, and prove that up to isomorphism there is only one Cauchy complete Archimedean field. As expected, the Cauchy reals form such a field. It follows from Markov's principle that the apartness relation and inequality coincide in an Archimedean field. In the last part of the section we prove that the non-existence of discontinuous real functions is equivalent to Phoa's principle for the standard dominance.

Intuitionistic theory of metric spaces is presented in the sixth section. As is well known, uniform continuity plays the role of continuity in an intuitionistic setting. We compute a representation for the space of uniformly continuous maps between metric spaces. Then we prove Banach's Fixed Point Theorem for contracting maps on a complete metric space, and conclude the section with the definition and examples of complete totally bounded metric spaces, which are the intuitionistic analogue of compact metric spaces.

In the last section we investigate computability of subspaces. In the logic of modest sets the full powersets are not available, and only spaces of restricted kinds of subspaces exist. Such spaces are called *hyperspaces*. We compute representations of the following hyperspaces: the hyperspace of open subspaces of a countably based space, the hyperspace of formal balls of a metric space, the hyperspace of complete located subspaces of an inhabited metric space,[9] the upper space of a complete metric space, and the hyperspace of solids.

Contributions and Related Work

The idea of realizability originated in 1945 with Kleene's number realizability [Kle45]. Since then realizability has become a subject in itself, with numerous applications in logic, mathematics, and computer science. See [BvORS99] for a comprehensive bibliography on realizability. In this dissertation I use the tools of realizability to study computability in topology and analysis. The idea of applying realizability in this way is not original—already in 1959 Kreisel [Kre59] formulated an interpretation of analysis by means of functionals of finite type. What is perhaps original is the realization that many well known and widely studied approaches to computable topology and analysis can be treated uniformly by relative realizability in categories of modest sets. In late 1996 Scott [Sco96, BBS98] defined equilogical spaces and proved that they form a cartesian closed category. It was soon realized that equilogical spaces were closely related to realizability. Birkedal

[9] The complete located subspaces are the intuitionistic version of closed subspaces.

investigated a general notion of relative realizability in his dissertation [Bir99], which served as a foundation for much of my work. Parts of this dissertation are not original, some are joint work, and some are original results. More specifically, the contributions are as follows.

Chapter 1: The overview of partial combinatory algebras is based on Longley [Lon94]; the PCA structure of continuous reflexive posets is not original; Subsection 1.1.3 on the graph model is based on Dana Scott's 1996 lectures in domain theory; Subsection 1.1.4 on the universal domain is based on Gunter and Scott [GS90]; Subsection 1.1.5 about the partial universal domain is new, as far as I know, and was discovered jointly with Dana Scott; Subsection 1.1.6 about the second Kleene algebra does not contain any new material; in Subsection 1.1.7 I define a PCA over a first-order structure, and to the best of my knowledge this is a new construction. Section 1.2 and the construction of modest sets is folklore by now. Section 1.3 is an exercise in category theory, but finding the proof that modest sets have inductive and coinductive types was not all that easy, and should count as original work, done jointly with Lars Birkedal. Dana Scott first had the idea for the computability operator #, and the notion was then developed by the Logic of Types and Computation group at Carnegie Mellon University. Section 1.4 is about applicative morphisms and is based on Longley [Lon94] who also first defined and proved the basic properties of applicative morphisms; I have adapted the definition to relative realizability; I first learned about the applicative adjunction between $\mathbb{P}$ and $\mathbb{B}$ from Peter Lietz, but I am quite certain it had been known to other people before that; the applicative inclusion between $\mathbb{P}$ and $\mathbb{U}$, the applicative equivalence of $\mathbb{P}$ and $\mathbb{V}$, and the applicative equivalence of reflexive continuous lattices are original work.

Chapter 2: My presentation of the logic of modest sets is based on Birkedal [Bir99, Appendix A], where the details of the interpretation of the logic of equilogical spaces have been worked out. The axioms pretty much just mirror the categorical structure of modest sets. The proof of Theorem 2.1.28 on the canonical factorization of maps, the proof of Theorem 2.2.2 about the induction principle for inductive types, and the proof of Theorem 2.2.6 about the coinduction principle are worth noticing and marking as my original work. The formulation of the Axiom of Computability is original, and therefore so are the derivations of the basic facts about computability in the logic of modest sets.

Chapter 3: The realizability interpretation of the logic of modest sets is based on Birkedal [Bir99, Appendix A]. The original part of this chapter is the presentation of the realizability operator in the form given in Section 3.6.

Chapter 4: Equilogical spaces were defined by Dana Scott in 1996. The Logic of Types and Computation group at Carnegie Mellon University jointly explored equilogical spaces. While others studied more general aspects of realizability and equilogical spaces, I focused on the relationship between equilogical spaces and other frameworks for computable topology. I formulated the notion of effective equilogical spaces as presented in Subsection 4.1.2. In Subsections 4.1.3 and 4.1.4 I present my original work on the relation between equilogical spaces and effectively presented domains, and on equilogical spaces and domains with totality—except for the part about totalities with dependent types which is joint work with Lars Birkedal. I thank Ulrich Berger, Dag Normann, and Alex Simpson for discussing totality on domains with me. The comparison of domain representations and equilogical spaces is original; I gratefully acknowledge stimulating discussions about domain representations with Jens Blanck . In Section 4.2 I examine the relationship between equilogical spaces and TTE. I originally learned about an applicative retraction between $\mathsf{Mod}(\mathbb{P})$ and $\mathsf{Mod}(\mathbb{B})$ from Peter Lietz, with whom I also worked out the idea that $\mathsf{Mod}(\mathbb{B})$ is the category of 0-equilogical spaces. The results from Subsection 4.2.4 are mine, unless a reference for the original source is given. I believe that Theorem 4.2.25 deserves special notice. I thank Matthias Schröder and Alex Simpson for discussing it with me. Section 4.3 is joint work with Steve Awodey.

Chapter 5: Much of this chapter is based on standard presentations of constructive mathematics, in particular Troelstra and van Dalen [TvD88a, TvD88b] and Bishop and Bridges [BB85]. Section 5.1 is based on McCarty's work [McC84] on set theory in the effective topos. Section 5.2 is original. Section 5.3 is based on [Ros86, vOS98, Hyl92, Lon94]; I selected and reformulated the axioms for the dominance to make them better suited for the logic of modest sets. As far as I know, Theorem 5.3.20 about the standard dominance is original, and so is the equivalence of Phoa's principle to various other statements, such as the weak continuity principle, the decidability of $2^{\mathbb{N}}$, and the non-existence of discontinuous maps. Section 5.4 on countably based spaces is original and was inspired by Dieter Spreen's T_0-spaces [Spr98]. His work guided me in finding the correct formulation of countably based spaces. I thank Douglas Cenzer and Dieter Spreen for helpfully discussing the proof of Theorem 5.4.22 with me. Sections 5.5 and 5.6 are based on standard presentations of real numbers and metric spaces [TvD88a, TvD88b, BB85]. The original part of these sections are the relationship between metric and (intrinsic) topology, and the statements about computability. The idea of hyperspaces is not new, and the hyperspaces considered in Section 5.7 are all standard ones, but they are rarely treated within an intuitionistic logic, like in this section.

Having listed quite specifically all the bits and pieces that are and are not original in my dissertation, I would like to point out that a considerable amount of creative work was put into organizing and collecting the material, whether it be original or not.

Apart from the new technical results presented in this dissertation, I believe and hope that I have demonstrated two main points. First, that relative realizability and categories of modest sets are an outstandingly natural, general, and useful framework for computable topology and analysis that unifies many existing approaches to this subject. Second, that it is advantageous to use constructive logic—in particular, the logic of modest sets—because it helps us choose the correct definitions, makes the reasoning more abstract, closer to the usual mathematical practice, and independent of the details of the underlying computational model.

Chapter 1

Categories of Modest Sets

1.1 Partial Combinatory Algebras

The study of computability starts with a notion of computation. We take *partial combinatory algebras* (PCA) as our models of computation.[1] A *partial* function $f\colon A \rightharpoonup B$ is a function that is defined on a subset $\mathsf{dom}(f) \subseteq A$, called the *domain* of f. Sometimes there is confusion between the domain $\mathsf{dom}(f)$ and the set A, which is also called the domain. In such cases we call $\mathsf{dom}(f)$ the *support* of f. If $f\colon A \rightharpoonup B$ is a partial function and $x \in A$, we write $fx\!\downarrow$ to indicate that fx is defined. For an expression e, we also write $e\!\downarrow$ to indicate that e and all of its subexpressions are defined. The symbol $\downarrow$ is sometimes inserted into larger expressions, for example, $fx\!\downarrow = y$ means that fx is defined and is equal to y. If e_1 and e_2 are two expressions whose values are possibly undefined, we write $e_1 \simeq e_2$ to indicate that either e_1 and e_2 are both undefined, or they are both defined and equal.

Definition 1.1.1 A *partial combinatory algebra (PCA)* $(\mathbb{A}, \cdot, \mathsf{K}, \mathsf{S})$ is a set $\mathbb{A}$ with a *partial* binary operation $\square \cdot \square\colon \mathbb{A} \times \mathbb{A} \rightharpoonup \mathbb{A}$ and two distinguished elements $\mathsf{K}, \mathsf{S} \in \mathbb{A}$. We usually write xy instead of $x \cdot y$, and assume that application associates to the left. A PCA is required to satisfy, for all $x, y, z \in \mathbb{A}$,

$$\mathsf{K}xy \simeq x\ , \qquad \mathsf{S}xyz \simeq (xz)(yz)\ , \qquad \mathsf{S}xy\!\downarrow .$$

When application is total $\mathbb{A}$ is a *(total) combinatory algebra (CA)*. A *subPCA* $\mathbb{A}'$ of a PCA $(\mathbb{A}, \cdot, \mathsf{K}, \mathsf{S})$ is a subset $\mathbb{A}' \subseteq \mathbb{A}$ that contains K and S, and is closed under application.

It may seem that PCAs are not much of a model of computation, since we only require two distinguished elements, the *combinators* S and K. However, we can build up the identity function, pairs, conditionals, natural numbers, and recursion just by combining the two basic combinators. In order to do this, we follow Longley [Lon94] and introduce the notation $\lambda^* x.\, e$ where x is a variable and e is an expression involving variables, elements of $\mathbb{A}$, and application. The meaning of $\lambda^* x.\, e$ is defined inductively: $\lambda^* x.\, x = \mathsf{I} = \mathsf{SKK}$; $\lambda^* x.\, y = \mathsf{K}y$ if y is a constant or a variable other than x; $\lambda^* x.\, e_1 e_2 = \mathsf{S}(\lambda^* x.\, e_1)(\lambda^* x.\, e_2)$. We abbreviate $\lambda^* x.\, \lambda^* y.\, e$ as $\lambda^* xy.\, e$, and similarly for more than

[1] PCAs are untyped models of computation. Longley [Lon99] generalized the definition of PCAs to that of *typed PCAs*, which he called "typed partial combinatory systems". We are not concerned with the typed models because all of our examples are untyped.

two variables. The notation $\lambda^* x.\, e$ is meta-notation for an expression involving K, S, variables, and elements of $\mathbb{A}$. It suggests a relation to the untyped λ-calculus, but we must be careful as β-reduction is only valid in restricted cases.[2] The notation $\lambda^* x.\, e$ saves a lot of space and makes expressions much more comprehensible, as even the translation of a simple term like $\lambda^* xyz.\,(zxy)$ is quite unwieldy,[3]

$$\begin{aligned}\lambda^* xyz.\,(zxy) = \ &\mathsf{S(S(KS)(S(S(KS)(S(KK)(KS)))(S(S(KS)(S(S(KS)(S(KK)(KS)))}\\ &\mathsf{(S(S(KS)(S(S(KS)(S(KK)(KS)))(S(KK)(KK))))(S(KK)(KK)))))}\\ &\mathsf{(S(S(KS)(S(KK)(KK)))(S(KK)(SKK))))))(S(S(KS)(S(KK)(KK)))}\\ &\mathsf{(S(S(KS)(KK))(KK)))}.\end{aligned}$$

We review from [Lon94, Chapter 1] how to encode some basic programming constructs in a PCA. A *pairing* for $\mathbb{A}$ is a triple of elements $\mathsf{pair}, \mathsf{fst}, \mathsf{snd} \in \mathbb{A}$ such that, for all $x, y \in \mathbb{A}$,

$$\mathsf{pair}\, xy\downarrow\,, \qquad \mathsf{fst}\,(\mathsf{pair}\, xy) = x\ , \qquad \mathsf{snd}\,(\mathsf{pair}\, xy) = y\ .$$

Every PCA has a pairing $\mathsf{pair} = \lambda^* xyz.\, zxy$, $\mathsf{fst} = \lambda^* z.\, z(\lambda^* xy.\, x)$, $\mathsf{snd} = \lambda^* z.\, z(\lambda^* xy.\, y)$. We write $\langle x, y\rangle$ instead of $\mathsf{pair}\, xy$.

Similarly, every PCA has *Booleans* $\mathsf{if}, \mathsf{true}, \mathsf{false} \in \mathbb{A}$ that satisfy, for all $x, y, z \in \mathbb{A}$,

$$\mathsf{if}\ x\ y\downarrow\,, \qquad \mathsf{if}\ \mathsf{true}\ y\ z = y\ , \qquad \mathsf{if}\ \mathsf{false}\ y\ z = z\ .$$

For example, we can take $\mathsf{true} = \lambda^* yz.\, y$, $\mathsf{false} = \lambda^* yz.\, z$, and $\mathsf{if} = \lambda^* xyz.\, xyz$.

The *Curry numerals* are defined for each $n \in \mathbb{N}$ by $\overline{0} = \mathsf{I} = \mathsf{SKK}$ and $\overline{n+1} = \langle \mathsf{false}, \overline{n}\rangle$. There exist elements $\mathsf{succ}, \mathsf{pred}, \mathsf{iszero} \in \mathbb{A}$ such that, for all $n \in \mathbb{N}$,

$$\begin{aligned}\mathsf{succ}\ \overline{n} &= \overline{n+1}\\ \mathsf{pred}\ \overline{n} &= \begin{cases}\overline{0} & \text{if } n = 0\\ \overline{n-1} & \text{if } n > 0\end{cases}\\ \mathsf{iszero}\ \overline{n} &= \begin{cases}\mathsf{true} & \text{if } n = 0\\ \mathsf{false} & \text{if } n > 0\end{cases}\end{aligned}$$

To see this, take $\mathsf{succ} = \lambda^* x.\, \langle \mathsf{false}, x\rangle$, $\mathsf{iszero} = \mathsf{fst}$, and $\mathsf{pred} = \lambda^* x.\, \mathsf{if}\ (\mathsf{iszero}\, x)\overline{0}(\mathsf{snd}\ x)$.

In a PCA we can define functions by recursion by using the *fixed point combinators* Y *and* Z, defined by

$$\begin{aligned} W &= \lambda^* xy.\, y(xxy)\ , & \mathsf{Y} &= WW\ ,\\ X &= \lambda^* xyz.\, y(xxy)z\ , & \mathsf{Z} &= XX\ .\end{aligned}$$

---

[2] See [Lon94, Chapter 1] for further details about the notation $\lambda^* x.\, e$.

[3] The following *Mathematica* program was used to translate terms:

```
lam[x_Symbol, x_Symbol] := s[k][k]
lam[x_Symbol, y_Symbol] := k[y]
lam[x_Symbol, f_[g_]] := s[lam[x,f]][lam[x,g]]
lam[vars_List, f_] := Fold[lam[#2,#1]&, f, Reverse[vars]]
```

These combinators satisfy, for all $f \in \mathbb{A}$,

$$\mathsf{Y}f \simeq f(\mathsf{Y}f)\ , \qquad \mathsf{Z}f\!\downarrow\ , \qquad (\mathsf{Z}f)z \simeq f(\mathsf{Z}f)z\ .$$

Finally, let us see how to define functions by primitive recursion. The element

$$\mathsf{rec} = \lambda^* xfm.\, ((\mathsf{Z}R)xfm\mathsf{I})\ ,$$

where $R = \lambda^* rxfm.\, \mathsf{if}(\mathsf{iszero}\, m)(\mathsf{K}x)(\lambda^* y.\, f(\mathsf{pred}\, m)(rxf(\mathsf{pred}\, m)\mathsf{I}))$, satisfies

$$\mathsf{rec}\ x\ f\ \overline{0} = x\ , \qquad \mathsf{rec}\ x\ f\ \overline{n+1} \simeq f\ \overline{n}\ (\mathsf{rec}\ x\ f\ \overline{n}).$$

It turns out that every partial recursive function can be encoded in a PCA, and so PCAs are Turing complete [Bee85, VI.2.8]. Let us now consider some examples of PCAs.

1.1.1 The First Kleene Algebra $\mathbb{N}$

Let $\mathcal{P}^{(1)}$ be the set of partial recursive functions, and let $\varphi \colon \mathbb{N} \to \mathcal{P}^{(1)}$ be a standard enumeration of partial recursive functions.[4] Define the *Kleene application* $\{\Box\}\,\Box \colon \mathbb{N} \times \mathbb{N} \rightharpoonup \mathbb{N}$ by

$$\{m\}\, n = \varphi_m n.$$

The existence of the combinators K and S is a consequence of the s-m-n theorem. We call this PCA the *first Kleene algebra*. It should always be clear from the context whether $\mathbb{N}$ denotes the first Kleene algebra or the set of natural numbers.

1.1.2 Reflexive Continuous Posets

A *poset* $(P, \leq)$, or a *partially ordered set*, is a set with a reflexive, transitive, antisymmetric relation. A *directed set* of a poset is a non-empty subset $S \subseteq P$ such that for all $x, y \in S$ there exists $z \in S$ so that $x \leq z$ and $y \leq z$. A *complete poset* (CPO) is a poset in which suprema of directed subsets exist. A *continuous function* between CPOs is a function that preserves directed suprema. Given CPOs D and E, the set of continuous functions E^D with the pointwise ordering is again a CPO.

A CPO D is *reflexive* when D^D is a retract of D, which means that there are a section $s \colon D^D \to D$ and a retraction $r \colon D \to D^D$ such that $r \circ s = 1_{D^D}$. Apart from the trivial one-point reflexive CPO there also exist non-trivial reflexive CPOs, as was shown by Scott [Sco72]. A non-trivial reflexive CPO D is a model for the untyped λ-calculus.[5] Suppose M is a term in the untyped λ-calculus whose freely occurring variables are among $x_1, \ldots, x_n$. An *environment* for M is a map $\eta \colon \{x_1, \ldots, x_n\} \to D$. If x_{n+1} is a variable, and $t \in D$, then $\eta' = \langle \eta, x_{n+1} := t\rangle$ is the environment that extends η by mapping x_{n+1} to t. The empty environment $\eta \colon \{\} \to D$ is denoted by $\langle\,\rangle$. For every term M we define inductively $[\![M]\!]$ to be a map from environments for M to D as follows:

$$\begin{aligned}
[\![x]\!]\eta &= \eta x\ ,\\
[\![MN]\!]\eta &= (r([\![M]\!]\eta))([\![N]\!]\eta)\ ,\\
[\![\lambda x.\, M]\!]\eta &= s(\lambda t \in D\,.\,([\![M]\!]\langle \eta, x := t\rangle))\ .
\end{aligned}$$

Every model of the untyped λ-calculus is a *total* combinatory algebra where application in the algebra is the same as application in the λ-calculus model, and the combinators are

$$\mathsf{K} = \lambda xy.\, x\ , \qquad\qquad \mathsf{S} = \lambda xyz.\, ((xz)(yz))\ .$$

[4] For background on recursion theory see [Soa87]

[5] For background on λ-calculus see [Bar85] and [AC98]

1.1.3 The Graph Model $\mathbb{P}$

The *graph model* $\mathbb{P} = \mathcal{P}\mathbb{N}$ is an example of a reflexive CPO. We study it closely, as we are going to build our principal example of modest sets out of it. The PCA is called the *graph* model because the section $\Gamma\colon (\mathbb{P} \to \mathbb{P}) \to \mathbb{P}$ encodes a continuous function by its graph. For further material on the graph model see [Sco76].

Let $\mathbb{P} = \mathcal{P}\mathbb{N}$ be the powerset of natural numbers ordered by inclusion. It is an algebraic lattice whose compact elements are the finite subsets of $\mathbb{N}$.[6] We denote with $\mathbb{P}_0$ the set of all compact elements of $\mathbb{P}$. The lattice $\mathbb{P}$ is also a topological space for the Scott topology, whose basic open sets are the upper sets of finite sets

$$\uparrow\{n_0, \dots, n_k\} = \left\{x \in \mathbb{P} \;\middle|\; \{n_0, \dots, n_k\} \subseteq x\right\} \;.$$

We write $\uparrow n$ instead of $\uparrow\{n\}$. It is clear that $\mathbb{P}$ is a countably based T_0-space.

An *enumeration operator* is a map $f\colon \mathbb{P} \to \mathbb{P}$ with the property

$$fx = \bigcup\left\{fy \mid y \ll x\right\} \;,$$

for all $x \in \mathbb{P}$. Here $y \ll x$ means that y is a finite subset of x. The enumeration operators are exactly the continuous maps $\mathbb{P} \to \mathbb{P}$ for the Scott topology on $\mathbb{P}$. They form an algebraic lattice $\mathbb{P}^{\mathbb{P}}$ under the pointwise ordering, i.e., $f \leq g$ exactly when $fx \subseteq gx$ for all $x \in \mathbb{P}$. We show that $\mathbb{P}^{\mathbb{P}}$ is a retract of $\mathbb{P}$. To do this, we first need to look at coding of pairs and sequences with natural numbers. Define the coding function $\langle \Box, \Box \rangle\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ as

$$\langle m, n \rangle = 2^m(2n + 1) \;.$$

Every natural number except zero is the code of a unique pair.

There is an effective bijection $\mathsf{finset}\colon \mathbb{N} \to \mathbb{P}_0$ between natural numbers and finite sets of natural numbers. By 'effective' we mean that there is a recursive function which takes an index $n \in \mathbb{N}$ and computes the code of a list $[m_1, \dots, m_k]$ such that $\mathsf{finset}\, n = \{m_1, \dots, m_k\}$. It can be further assumed that $m_1 < \cdots < m_k$. For example, we could use the following coding function finset, defined by its inverse:

$$\mathsf{finset}^{-1}(x) = \sum_{k \in x} 2^k \;.$$

Next we consider coding functions in $\mathbb{P}$. Define a pairing function $\langle \Box, \Box \rangle\colon \mathbb{P} \times \mathbb{P} \to \mathbb{P}$ by

$$\langle x, y \rangle = \left\{2n \;\middle|\; n \in x\right\} \cup \left\{2m + 1 \;\middle|\; m \in y\right\} \;.$$

The map $\langle \Box, \Box \rangle$ is an isomorphism of lattices $\mathbb{P} \times \mathbb{P}$ and $\mathbb{P}$. Let $\pi_0, \pi_1\colon \mathbb{P} \to \mathbb{P}$ be the compositions of the inverse isomorphism $\langle \Box, \Box \rangle^{-1}$ with the two canonical projections $\mathbb{P} \times \mathbb{P} \to \mathbb{P}$.

The lattice of continuous maps $\mathbb{P}^{\mathbb{P}}$ is a retract of $\mathbb{P}$. It is embedded into $\mathbb{P}$ by a map $\Gamma\colon \mathbb{P}^{\mathbb{P}} \to \mathbb{P}$ defined by

$$\Gamma f = \left\{\langle m, n \rangle \;\middle|\; n \in \mathbb{N} \text{ and } m \in f(\mathsf{finset}\, n)\right\} .$$

The set Γf is called the *graph* of f. Note that Γf is defined whenever the values of f are defined on $\mathbb{P}_0$. For a continuous map $f\colon \mathbb{P} \to \mathbb{P}$ the graph Γf uniquely determines f because the value of

[6] See Subsection 1.1.4 on page 27 for the definitions of domain-theoretic terms used here.

f at any element $x \in \mathbb{P}$ is the union of values of f at finite subsets of x. The retraction $\Lambda\colon \mathbb{P} \to \mathbb{P}^{\mathbb{P}}$ is defined by

$$\Lambda x\colon y \mapsto \left\{m \in \mathbb{N} \;\middle|\; \exists\, n \in \mathbb{N}\,.\,(\langle m, n\rangle \in x \text{ and } \mathsf{finset}\, n \subseteq y)\right\}.$$

We can use the pairing function $\langle \Box, \Box \rangle$ and the section-retraction pair Γ, Λ to define a model of the untyped λ-calculus with surjective pairing. As described in Subsection 1.1.2, the interpretation of a term in an environment η, which is a mapping from variables to elements of $\mathbb{P}$, is defined as follows:

$$\begin{aligned}
[\![v]\!]\eta &= \eta(v)\\
[\![\lambda v.\, M]\!]\eta &= \Gamma(\lambda x \in \mathbb{P}\,.\,([\![M]\!][\eta, v \mapsto x]))\\
[\![MN]\!]\eta &= (\Lambda([\![M]\!]\eta))([\![N]\!]\eta)\\
[\![\langle M, N\rangle]\!]\eta &= \langle [\![M]\!]\eta, [\![N]\!]\eta\rangle\\
[\![\mathsf{fst}\, M]\!]\eta &= \pi_0([\![M]\!]\eta)\\
[\![\mathsf{snd}\, M]\!]\eta &= \pi_1([\![M]\!]\eta)
\end{aligned}$$

Hence $\mathbb{P}$ is a combinatory algebra. Recall that $\mathsf{K} = \lambda xy.\, x$ and $\mathsf{S} = \lambda xyz.\,(xz)(yz)$.

We now turn to the topological properties of $\mathbb{P}$. We have already mentioned that $\mathbb{P}$ is a countably based T_0-space. In fact, it is the universal space of this kind—every countably based T_0-space can be embedded in $\mathbb{P}$.

Theorem 1.1.2 (Embedding Theorem) *Every countably based T_0-space X can be embedded in $\mathbb{P}$. There is a bijective correspondence between embeddings $e\colon X \hookrightarrow \mathbb{P}$ and enumerations of countable subbases $S_\Box\colon \mathbb{N} \to \mathcal{O}(X)$. The subbase corresponding to an embedding $e\colon X \hookrightarrow \mathbb{P}$ is*

$$S_n = e^*(\uparrow n)\,, \tag{1.1}$$

and the embedding determined by an enumeration $S\colon \mathbb{N} \to \mathcal{O}(X)$ of a subbase is

$$et = \left\{n \in \mathbb{N} \;\middle|\; t \in S_n\right\}\,. \tag{1.2}$$

Proof. It is obvious that a topological space which is homeomorphic to a subspace of $\mathbb{P}$ is countably based and T_0. Conversely, suppose X is a T_0-space with countable subbase $S\colon \mathbb{N} \to \mathcal{O}(X)$. The map $e\colon X \hookrightarrow \mathbb{P}$ defined by (1.2) is injective because X is a T_0-space. It is continuous because the inverse image $e^*(\uparrow n)$ of a subbasic open set $\uparrow n$ is open:

$$e^*(\uparrow n) = \left\{t \in X \;\middle|\; et \in \uparrow n\right\} = \left\{t \in X \;\middle|\; n \in et\right\} = \left\{t \in X \;\middle|\; t \in S_n\right\} = S_n\,.$$

The map e is an embedding because it maps a basic open set $U = S_{n_1} \cap \cdots \cap S_{n_k}$ to

$$e_*(U) = e_*(X) \cap \uparrow\{n_1, \ldots, n_k\}\,.$$

Indeed, if $x = e(t) \in e_*(U)$ then $t \in S_{n_1} \cap \cdots \cap S_{n_k}$, hence

$$x \in \uparrow n_1 \cap \cdots \cap \uparrow n_k = \uparrow\{n_1, \ldots, n_k\}\,.$$

On the other hand, if $x = et \in e_*(X)$ and $x \in \uparrow\{n_1, \ldots, n_k\}$, then $n_i \in et$ for every $i = 1, \ldots, k$, which means that $t \in S_{n_1} \cap \cdots \cap S_{n_k} = U$.

Now suppose that $e\colon X \hookrightarrow \mathbb{P}$ is an embedding. Let $S\colon \mathbb{N} \to \mathcal{O}(X)$ be the map defined by (1.1). The family $S_0, S_1, \ldots$ is a subbase for X because e is an embedding. All that remains to be shown is that $et = \{n \in \mathbb{N} \mid t \in e^*(\uparrow n)\}$ for all $t \in X$:

$$et = \{n \in \mathbb{N} \mid n \in et\} = \{n \in \mathbb{N} \mid et \in \uparrow n\} = \{n \in \mathbb{N} \mid t \in e^*(\uparrow n)\} \ .$$

■

We conclude this subsection with a theorem that complements the Embedding Theorem because it says that a continuous map from a subspace of $\mathbb{P}$ can be extended to $\mathbb{P}$.

Theorem 1.1.3 (Extension Theorem) *Every continuous map $f\colon X \to Y$ between subspaces X and Y of $\mathbb{P}$ has a continuous extension $F\colon \mathbb{P} \to \mathbb{P}$, which means that $Fx = fx$ for all $x \in X$.*

Proof. Suppose X and Y are subspaces of $\mathbb{P}$ and $f\colon X \to Y$ is a continuous map between them. A continuous extension $F\colon \mathbb{P} \to \mathbb{P}$ is explicitly defined as

$$Fx = \bigcup \Big\{ \bigcap \{fz \mid z \in X \cap \uparrow y\} \ \Big| \ y \ll x \Big\} .$$

Clearly, F is continuous because the inner intersection defines a monotone map, so that Fx is defined as a directed supremum over the compact elements below x of monotonically increasing values. We show that $Fx = fx$ for all $x \in X$. If $x \in X$ then for every $y \ll x$ it is the case that $x \in X \cap \uparrow y$, hence

$$\bigcap \{fz \mid z \in X \cap \uparrow y\} \subseteq fx \ .$$

This shows that $Fx \subseteq fx$ when $x \in X$. On the other hand, suppose $n \in fx$. Then $fx \in \uparrow n$ and $x \in f^*(\uparrow n)$, and because f is continuous there exists $y \ll x$ such that $x \in \uparrow y \subseteq f^*(\uparrow n)$. Now for every $z \in X \cap \uparrow y$ it is the case that

$$z \in X \cap \uparrow y \subseteq X \cap f^*(\uparrow n) \subseteq f^*(\uparrow n) \ ,$$

so $fz \in \uparrow n$ and $n \in fz$. This shows that $n \in \bigcap \{fz \mid z \in X \cap \uparrow y\}$, from which it follows that $n \in Fx$, and finally $fx \subseteq Fx$. ■

The R.E. Graph Model $\mathbb{P}_\sharp$

The recursively enumerable graph model $\mathbb{P}_\sharp$ is like $\mathbb{P}$ except that we take only the recursively enumerable subsets of $\mathbb{N}$. We denote the PCA $(\mathsf{RE}, \cdot, \mathsf{K}, \mathsf{S})$ by $\mathbb{P}_\sharp$ to indicate that it is a *computable subPCA* of $\mathbb{P}$, and to distinguish the PCA from its underlying set RE.

Recall that a set $x \in \mathbb{P}$ is *recursively enumerable* (r.e.) if there exists a recursive function $e\colon \mathbb{N} \to \mathbb{N}$ such that

$$x = \{n \in \mathbb{N} \mid \exists\, m \in \mathbb{N} \,.\, e(m) = n + 1\} \ .$$

The family of all recursively enumerable sets is denoted by RE. It is easily checked that the pairing function $\langle \Box, \Box \rangle$ defined on $\mathbb{P}$ restricts to a bijection from $\mathsf{RE} \times \mathsf{RE}$ to RE. An enumeration operator $f\colon \mathsf{RE} \to \mathsf{RE}$ is *computable* when its graph

$$\Gamma f = \{\langle m, n\rangle \mid n \in \mathbb{N} \text{ and } m \in f(\mathsf{finset}\, n)\}$$

is an r.e. set. We denote the set of all computable enumeration operators by $\#(\mathbb{P}^{\mathbb{P}})$. It is not hard to see that Γ and Λ restrict to maps $\Gamma\colon \#(\mathbb{P}^{\mathbb{P}}) \to \mathsf{RE}$ and $\Lambda\colon \mathsf{RE} \to \#(\mathbb{P}^{\mathbb{P}})$. Similarly, the pairing $\langle \Box, \Box \rangle$ and the projections π_0, π_1 restrict to a bijection $\langle \Box, \Box \rangle\colon \mathsf{RE} \times \mathsf{RE} \to \mathsf{RE}$ and projections $\pi_0, \pi_1\colon \mathsf{RE} \times \mathsf{RE} \to \mathsf{RE}$, respectively. This means that RE is a model for the untyped λ-calculus because the interpretation for $\mathbb{P}$ restricts to one for RE.

1.1.4 The Universal Domain $\mathbb{U}$

First we review the basic definitions, terminology, and notation from domain theory. For further material on domain theory see [AC98]. The present subsection is based on [GS90], where proofs and missing details can be found.

Let $(P, \leq)$ be a partially ordered set. A *directed set* $S \subseteq P$ is a non-empty subset such that for all $x, y \in S$ there exists $z \in S$ such that $x, y \leq z$. A partially ordered set is *directed complete* if every directed subset has a supremum. The supremum of a directed set S is denoted by $\bigvee S$.

A subset $S \subseteq P$ is *bounded*, or *consistent*, when it has an upper bound, which means that there exists $x \in P$ such that $y \leq x$ for all $y \in S$. We write $x \uparrow y$ when $\{x, y\}$ is bounded. A partially ordered set is *bounded complete* when every bounded subset has a least upper bound.

Suppose P is directed complete. An element $a \in P$ is *compact*, or *finite*, when for every directed subsets $S \subseteq P$, if $a \leq \bigvee S$ then there exists $x \in P$ such that $a \leq x$. The set of compact elements of P is denoted by $\mathcal{K}(P)$. The notation $a \ll x$ means that $a \leq x$ and $a \in \mathcal{K}(P)$. A partially ordered set is *algebraic* when every element is the supremum of the compact elements below it, i.e., for all $x \in P$, $x = \bigvee \{a \in \mathcal{K}(P) \mid a \ll x\}$. We say that an algebraic poset is *countably based* when there exists a countable set $B \subseteq \mathcal{K}(P)$ such that every element is the supremum of elements of B that are below it.

A *Scott domain*, or just a *domain*, is a bounded complete, directed complete, countably based, algebraic, partially ordered non-empty set $(D, \leq)$. The *Scott topology* on D is defined by the topological basis consisting of the basic open sets $\uparrow a = \{x \in D \mid a \leq x\}$, where $a \in \mathcal{K}(D)$. The least element of D exists, as it is the least upper bound of the empty set, and is denoted by $\bot_D$.

An *effective domain* is a domain D with an enumeration $b\colon \mathbb{N} \to \mathcal{K}(D)$ of its compact elements such that the relation $b_m \uparrow b_n$ is decidable in $\langle m, n \rangle \in \mathbb{N} \times \mathbb{N}$, the relation $b_m \leq b_n$ is r.e. in $\langle m, n \rangle \in \mathbb{N} \times \mathbb{N}$, and the join operation $b_m \vee b_n$ is a recursive function $\mathbb{N}^2 \to \mathbb{N}$. A continuous map $f\colon (C, c) \to (D, d)$ between effective domains is *computable* when the relation $d_m \leq f c_n$ is r.e. in $\langle m, n \rangle \in \mathbb{N} \times \mathbb{N}$. The category of effective domains and computable maps is denoted by $\mathsf{Dom}_{\mathsf{eff}}$. An *embedding-projection pair* is a pair (i, p) of continuous maps between domains, $i\colon D \to E$, $p\colon E \to D$, such that $p \circ i = 1_D$ and $i \circ p \leq 1_E$, where i is the embedding and p is the projection.

Definition 1.1.4 A *universal domain* is a domain $\mathbb{U}$ such that for every domain D there exists an embedding-projection pair

$$D \underset{p_D}{\overset{i_D}{\rightleftarrows}} \mathbb{U} \ .$$

There exists a universal domain, see [GS90, SHLG94]. In particular, the lattice of open subsets of the Cantor space $2^{\mathbb{N}}$ with the top element removed,

$$\mathbb{U} = \mathcal{O}(2^{\mathbb{N}}) \setminus \{2^{\mathbb{N}}\} \ ,$$

is a universal domain.[7] The lattice of clopen subsets of $2^{\mathbb{N}}$ is a countable Boolean algebra, and is exactly the set of compact elements of $\mathcal{O}(2^{\mathbb{N}})$. Hence, $\mathcal{K}(\mathbb{U})$ is the set of clopen subsets of $2^{\mathbb{N}}$ with $2^{\mathbb{N}}$ itself excluded. The elements of $\mathcal{K}(\mathbb{U})$ can be effectively enumerated in such a way that the Boolean operations on them are computable, and equality and inclusion are decidable relations. Let $\langle B_n \rangle_{n \in \mathbb{N}}$ be such an enumeration. It follows that $\mathbb{U}$ is an effective algebraic domain. In fact, it can be shown that $\mathbb{U}$ is a universal effective domain.[8] For every effective domain D there exists a computable embedding-projection pair (i, p) from D to $\mathbb{U}$.

A point $x \in \mathbb{U}$ is said to be *computable* when the set of clopens that are contained in x is r.e. The *computable part* $\mathbb{U}_\sharp$ of $\mathbb{U}$ is the set of all computable points of $\mathbb{U}$. The domain $\mathbb{U} \to \mathbb{U}$ of continuous endomaps is an effective domain, therefore there exists a computable embedding-projection pair

$$(\mathbb{U} \to \mathbb{U}) \underset{p_{\mathbb{U}\to\mathbb{U}}}{\overset{i_{\mathbb{U}\to\mathbb{U}}}{\rightleftarrows}} \mathbb{U} \ .$$

This means that $\mathbb{U}$ is a reflexive CPO and thus a model of the untyped λ-calculus, as explained in Section 1.1.2. Thus, $\mathbb{U}$ is a combinatory algebra with the application operation defined by, for $x, y \in \mathbb{U}$,

$$x \cdot y = (p_{\mathbb{U}\to\mathbb{U}} x) y \ .$$

The combinators are defined like in any model of the untyped λ-calculus,

$$\mathsf{K} = \lambda xy.\, x \ , \qquad \mathsf{S} = \lambda xyz.\, (xy)(yz) \ .$$

Application, K, and S are computable, therefore $\mathbb{U}_\sharp$ is a subPCA of $\mathbb{U}$.

1.1.5 The Partial Universal Domain $\mathbb{V}$

The universal domain $\mathbb{U}$ can be equipped with another PCA structure, which is different from the one described in the previous section. In order to avoid confusion, we denote this PCA by $\mathbb{V}$. So let $\mathbb{V} = \mathbb{U}$ and $\mathbb{V}_\sharp = \mathbb{U}_\sharp$.

The lattice $\mathcal{O}(2^{\mathbb{N}})$ of open subsets of the Cantor space $2^{\mathbb{N}}$ has a compact top element because the Cantor space is compact. Hence, if we add the compact top element to $\mathbb{V} = \mathcal{O}(2^{\mathbb{N}}) \setminus \{2^{\mathbb{N}}\}$, we recover $\mathcal{O}(2^{\mathbb{N}}) = \mathbb{V}^\top$. The space $\mathbb{V} \to \mathbb{V}^\top$ of continuous maps from $\mathbb{V}$ to $\mathbb{V}^\top$ is an effective domain. Therefore, there exists a computable embedding-projection pair

$$(\mathbb{V} \to \mathbb{V}^\top) \underset{p_{\mathbb{V}\to\mathbb{V}^\top}}{\overset{i_{\mathbb{V}\to\mathbb{V}^\top}}{\rightleftarrows}} \mathbb{V} \ .$$

Define a binary operation $\square \star \square \colon \mathbb{V} \times \mathbb{V} \to \mathbb{V}^\top$ for $x, y \in \mathbb{V}$ by

$$x \star \top = \top \star y = \top \star \top = \top \ , \qquad\qquad x \star y = (p_{\mathbb{V}\to\mathbb{V}^\top} x) y \ .$$

Suppose M is a term in the untyped λ-calculus whose freely occurring variables are among $x_1, \ldots, x_n$. An *environment* for M is a map $\eta \colon \{x_1, \ldots, x_n\} \to \mathbb{V}$. If x_{n+1} is a variable, and $t \in \mathbb{V}$, then $\eta' = \langle \eta, x_{n+1} := t \rangle$ is the environment defined by

$$\eta' x_i = \begin{cases} t & \text{if } i = n+1 \ , \\ \eta x_i & \text{if } 1 \le i \le n \ . \end{cases}$$

[7] Alternatively we can view $\mathbb{U}$ as the *upper space* of $2^{\mathbb{N}}$.

[8] See [SHLG94, Chapter 7].

The empty environment $\eta\colon \{\} \to \mathbb{V}$ is denoted by $\langle\,\rangle$. For every term M we define inductively $[\![M]\!]$ to be a map from environments for M to $\mathbb{V}^\top$ as follows:

$$\begin{aligned}
[\![x]\!]\eta &= \eta x \ , \\
[\![MN]\!]\eta &= ([\![M]\!]\eta) \star ([\![N]\!]\eta) \ , \\
[\![\lambda x.\, M]\!]\eta &= i_{\mathbb{V}\to\mathbb{V}^\top}(\lambda t \in \mathbb{V}\,.\,([\![M]\!]\langle \eta, x := t\rangle)) \ .
\end{aligned}$$

Note that $[\![\lambda x.\, M]\!]\eta$ is never equal to $\top$. Finally, we define the partial combinatory structure on $\mathbb{V}$. Partial application $\Box \cdot \Box\colon \mathbb{V} \times \mathbb{V} \rightharpoonup \mathbb{V}$ is defined for $x, y \in \mathbb{V}$ by

$$x \cdot y = \begin{cases} x \star y & \text{if } x \star y \neq \top \ , \\ \text{undefined} & \text{otherwise} \ . \end{cases}$$

The combinators K and S are

$$\mathsf{K} = [\![\lambda xy.\, x]\!]\,\langle\,\rangle \ , \qquad \mathsf{S} = [\![\lambda xyz.\,(xz)(yz)]\!]\,\langle\,\rangle \ .$$

For all $u, v \in \mathbb{V}$,

$$(\mathsf{K} \star u) \star v = ([\![\lambda y.\, x]\!]\langle x := u\rangle) \star v = [\![x]\!]\langle x := u, y := v\rangle = u \ .$$

Since $\mathsf{K} \star u = [\![\lambda y.\, x]\!]\langle x := u\rangle$, $\mathsf{K} \star u \neq \top$. Therefore, $\mathsf{K} \cdot u \cdot v = u$. For all $u, v \in \mathbb{V}$,

$$(\mathsf{S} \star u) \star v = [\![\lambda z.\,(xz)(yz)]\!]\langle x := u, y := v\rangle \ ,$$

hence $\mathsf{S} \star u \neq \top$ and $(\mathsf{S} \star u) \star v \neq \top$. Therefore $\mathsf{S} \cdot u \cdot v$ is always defined and is equal to $(\mathsf{S} \star u) \star v$. Suppose that for $w \in \mathbb{V}$, $(u \cdot w) \cdot (v \cdot w)$ is defined. Then both $u \cdot w$ and $v \cdot w$ are defined, hence

$$\begin{aligned}
&(u \cdot w) \cdot (v \cdot w) = (u \star w) \star (v \star w) = \\
&\qquad [\![(xz)(yz)]\!]\langle x := u, y := v, z := w\rangle = ([\![\lambda z.\,(xz)(yz)]\!]\langle x := u, y := v\rangle) \star w = \\
&\qquad (([\![\lambda yz.\,(xz)(yz)]\!]\langle x := u\rangle) \star v) \star w = ((([\![\lambda xyz.\,(xz)(yz)]\!]\langle\,\rangle) \star u) \star v) \star w = \\
&\qquad\qquad (((\mathsf{S} \star u) \star v) \star w) = (\mathsf{S} \cdot u \cdot v) \cdot w \ .
\end{aligned}$$

Conversely, if $(\mathsf{S}\cdot u\cdot v)\cdot w$ is defined, then read the above derivation backwards to see that $(u\cdot w)\cdot(v\cdot w)$ is defined and equal to $(\mathsf{S} \cdot u \cdot v) \cdot w$.

The computable part $\mathbb{V}_\sharp$ is a subPCA of $\mathbb{V}$ because application, K and S are all defined in terms of computable maps $i_{\mathbb{V}\to\mathbb{V}^\top}$ and $p_{\mathbb{V}\to\mathbb{V}^\top}$, therefore they are computable.

The realized partial maps $\mathbb{V} \rightharpoonup \mathbb{V}$ are exactly those partial continuous maps that are defined on a closed subset of $\mathbb{V}$. Indeed, let $u \in \mathbb{V}$ and $fv = u \cdot v$. Let $g\colon \mathbb{V} \to \mathbb{V}^\top$ be defined by $gv = u \star v$. Then fv is defined and equals gv if, and only if, $gv \neq \top$. Because $\top$ is a compact element of $\mathbb{V}^\top$, $g^*\{\top\}$ is an open subset of $\mathbb{V}$. Therefore, f is defined on a closed subset of $\mathbb{V}$. Conversely, suppose $D \subseteq \mathbb{V}$ is a closed subset and $f\colon D \to \mathbb{V}$ a continuous map. By Lemma 4.1.25 we can extend f to a continuous map $g\colon \mathbb{V} \to \mathbb{V}^\top$ by

$$gv = \begin{cases} fv & \text{if } fv \text{ is defined} \ , \\ \top & \text{otherwise} \ . \end{cases}$$

The partial map f is realized by $i_{\mathbb{V}\to\mathbb{V}^\top} g$.

1.1.6 The Second Kleene Algebra $\mathbb{B}$

The second Kleene algebra $\mathbb{B}$ has as its underlying set the Baire space. Before defining the PCA structure of $\mathbb{B}$, we consider some basic topological properties of the Baire space.

The Baire space $\mathbb{B} = \mathbb{N}^{\mathbb{N}}$ is the set of all infinite sequences of natural numbers, equipped with the product topology. Let $\mathbb{N}^*$ be the set of all finite sequences of natural numbers. If $a, b \in \mathbb{N}^*$ we write $a \sqsubseteq b$ when a is a prefix of b. The length of a finite sequence a is denoted by $|a|$. Similarly, we write $a \sqsubseteq \alpha$ when a is a prefix of an infinite sequence $\alpha \in \mathbb{B}$.

A countable topological base for $\mathbb{B}$ consists of the basic open sets

$$[a_0, \ldots, a_k]{::}\mathbb{B} = \left\{[a_0, \ldots, a_{k-1}]{::}\beta \;\middle|\; \beta \in \mathbb{B}\right\} = \left\{\alpha \in \mathbb{B} \;\middle|\; [a_0, \ldots, a_{k-1}] \sqsubseteq \alpha\right\} .$$

The expression $a{::}\beta$ denotes the concatenation of the finite sequence $a \in \mathbb{N}^*$ with the infinite sequence $\beta \in \mathbb{B}$. Sometimes we abuse notation and write $n{::}\beta$ instead of $[n]{::}\beta$ for $n \in \mathbb{N}$ and $\beta \in \mathbb{B}$. The base $\left\{a{::}\mathbb{B} \;\middle|\; a \in \mathbb{N}^*\right\}$ is a clopen countable base for the topology of $\mathbb{B}$, which means that $\mathbb{B}$ is a countably based 0-dimensional T_0-space. Recall that a space is *0-dimensional* when its clopen subsets form a base for its topology.

Theorem 1.1.5 (Embedding Theorem for $\mathbb{B}$) *A topological space is a 0-dimensional countably based T_0-space if, and only if, it embeds into $\mathbb{B}$.*

Proof. Clearly, every subspace of $\mathbb{B}$ is a countably based 0-dimensional T_0-space. Suppose X is a countably based 0-dimensional T_0-space. Let $\left\{U_k \;\middle|\; k \in \mathbb{N}\right\}$ be a countable base for X consisting of clopen sets. We define an embedding $e\colon X \to \mathbb{B}$ by

$$ex = \lambda n \in \mathbb{N} .\, (\texttt{if } x \in U_n \texttt{ then } 1 \texttt{ else } 0) .$$

The map e is injective because X is a T_0-space. It is continuous because

$$e^*([a_0, \ldots, a_n]{::}\mathbb{B}) = \bigcap \left\{U_k \;\middle|\; 1 \leq k \leq n \text{ and } a_k = 1\right\} .$$

It is an open map because $e_*(U_n) = \left\{\alpha \in \mathbb{N}^{\mathbb{N}} \;\middle|\; \alpha n = 1\right\}$ is an open set. ∎

Given a finite sequence of numbers $a = [a_0, \ldots, a_{k-1}]$, let $\mathsf{seq}\, a$ be the encoding of a as a natural number, for example

$$\mathsf{seq}\,[a_0, \ldots, a_{k-1}] = \prod_{i=0}^{k-1} {p_i}^{1+a_i} ,$$

where p_i is the i-th prime number. For $\alpha \in \mathbb{B}$ let $\overline{\alpha}n = \mathsf{seq}\,[\alpha 0, \ldots, \alpha(n-1)]$. For $\alpha, \beta \in \mathbb{B}$ let $\alpha \star \beta$ be defined by

$$\alpha \star \beta = n \iff \exists\, m \in \mathbb{N} .\, \left(\alpha(\overline{\beta}m) = n + 1 \wedge \forall\, k < m \,.\, \alpha(\overline{\beta}k) = 0\right) .$$

If there is no $m \in \mathbb{N}$ that satisfies the above condition, then $\alpha \star \beta$ is undefined. Thus, $\star$ is a partial function $\mathbb{B} \times \mathbb{B} \rightharpoonup \mathbb{N}$. It is continuous because the value of $\alpha \star \beta$ depends only on finite prefixes of α and β. The *continuous function application* $\square \mid \square\colon \mathbb{B} \times \mathbb{B} \to \mathbb{N} \rightharpoonup \mathbb{N}$ is defined by

$$(\alpha \mid \beta)n = \alpha \star (n{::}\beta);.$$

The Baire space $\mathbb{B}$ together with $|$ is a PCA, where $\alpha \mid \beta$ is considered to be undefined when $\alpha \mid \beta$ is not a total function. Instead of specifically defining the combinators K and S we characterize those partial functions $\mathbb{B} \rightharpoonup \mathbb{B}$ that are represented by elements of $\mathbb{B}$.[9] Every $\alpha \in \mathbb{B}$ represents a partial function $\boldsymbol{\eta}_\alpha \colon \mathbb{B} \rightharpoonup \mathbb{B}$ defined by

$$\boldsymbol{\eta}_\alpha \beta = \alpha \mid \beta \ ,$$

We say that a partial function $f \colon \mathbb{B} \rightharpoonup \mathbb{B}$ is *realized* when there exists $\alpha \in \mathbb{B}$ such that $f = \boldsymbol{\eta}_\alpha$. Such an α is called a *realizer* for f. A partial function $f \colon X \rightharpoonup Y$ is said to be *continuous* when it is continuous as a total map $f \colon \mathsf{dom}(f) \to Y$. Note that there is no restriction on the domain $\mathsf{dom}(f)$.

Theorem 1.1.6 (Extension Theorem for $\mathbb{B}$) *Every partial continuous map $f \colon \mathbb{B} \rightharpoonup \mathbb{B}$ can be extended to a realized one.*

Proof. Suppose $f \colon \mathbb{B} \rightharpoonup \mathbb{B}$ is a partial continuous map. Consider the set $A \subseteq \mathbb{N}^* \times \mathbb{N}^2$ defined by

$$A = \left\{ \langle a, i, j \rangle \in \mathbb{N}^* \times \mathbb{N}^* \;\middle|\; a{::}\mathbb{B} \cap \mathsf{dom}(f) \neq \emptyset \text{ and } \forall\, \alpha \in (a{::}\mathbb{B} \cap \mathsf{dom}(f))\,.\,((f\alpha)i = j) \right\}$$

If $\langle a, i, j \rangle \in A$, $\langle a', i, j' \rangle \in A$ and $a \sqsubseteq a'$ then $j = j'$ because there exists $\alpha \in a'{::}\mathbb{B} \cap \mathsf{dom}(f) \subseteq a{::}\mathbb{B} \cap \mathsf{dom}(f)$ such that $j = (f\alpha)i = j'$. We define a sequence $\phi \in \mathbb{B}$ as follows. For every $\langle a, i, j \rangle \in A$ let $\phi(\mathsf{seq}\,(i{::}a)) = j + 1$, and for all other arguments let $\phi n = 0$. Suppose that $\phi(\mathsf{seq}\,(i{::}a)) = j + 1$ for some $i, j \in \mathbb{N}$ and $a \in \mathbb{N}^*$. Then for every prefix $a' \sqsubseteq a$, $\phi(\mathsf{seq}\,(i{::}a')) = 0$ or $\phi(\mathsf{seq}\,(i{::}a')) = j + 1$. Thus, if $\langle a, i, j \rangle \in A$ and $a \sqsubseteq \alpha$ then $\phi \star (i{::}\alpha) = j$. We show that $(\boldsymbol{\eta}_\phi \alpha)i = (f\alpha)i$ for all $\alpha \in \mathsf{dom}(f)$ and all $i \in \mathbb{N}$. Because f is continuous, for all $\alpha \in \mathsf{dom}(f)$ and $i \in \mathbb{N}$ there exists $\langle a, i, j \rangle \in A$ such that $a \sqsubseteq \alpha$ and $(f\alpha)i = j$. Now we get $(\boldsymbol{\eta}_\phi \alpha)i = (\phi \mid \alpha)i = \phi \star (i{::}\alpha) = j = (f\alpha)i$. ■

Recall that a G_δ-set is a countable intersection of open sets.

Proposition 1.1.7 *If $U \subseteq \mathbb{B}$ is a G_δ-set then the function $u \colon \mathbb{B} \rightharpoonup \mathbb{B}$ defined by*

$$u\alpha = \begin{cases} \lambda n.\, 1 & \alpha \in U \ , \\ \textit{undefined} & \textit{otherwise} \end{cases}$$

is realized.

Proof. The set U is a countable intersection of countable unions of basic open sets

$$U = \bigcap_{i \in \mathbb{N}} \bigcup_{j \in \mathbb{N}} a_{i,j}{::}\mathbb{B} \ .$$

Define a sequence $v \in \mathbb{B}$ for all $i, j \in \mathbb{N}$ by $v(\mathsf{seq}\,(i{::}a_{i,j})) = 2$, and set $vn = 0$ for all other arguments n. Clearly, if $\boldsymbol{\eta}_v \alpha$ is total then its value is $\lambda n.\, 1$, so we only need to verify that $\mathsf{dom}(\boldsymbol{\eta}_v) = U$. If $\alpha \in \mathsf{dom}(\boldsymbol{\eta}_v)$ then $v \star (i{::}\alpha)$ is defined for every $i \in \mathbb{N}$, therefore there exists $ci \in \mathbb{N}$ such that $v(\mathsf{seq}\,(i{::}[\alpha 0, \ldots, \alpha(ci)])) = 2$, which implies that $\alpha \in a_{i,ci}$. Hence

$$\alpha \in \bigcap_{i \in \mathbb{N}} a_{i,ci}{::}\mathbb{B} \subseteq U.$$

Conversely, if $\alpha \in U$ then for every $i \in \mathbb{N}$ there exists some $ci \in \mathbb{N}$ such that $\alpha \in a_{i,ci}$. For every $i \in \mathbb{N}$, $v(\mathsf{seq}\,(i{::}[\alpha 0, \ldots, \alpha(ci)])) = 2$, therefore $(\boldsymbol{\eta}_v \alpha)i = v \star (i{::}\alpha) = 1$. Hence $\alpha \in \mathsf{dom}(\boldsymbol{\eta}_v)$. ■

[9] See [KV65] for details about the partial combinatory structure of $\mathbb{B}$.

Corollary 1.1.8 *Suppose $\alpha \in \mathbb{B}$ and $U \subseteq \mathbb{B}$ is a G_δ-set. Then there exists $\beta \in \mathbb{B}$ such that $\boldsymbol{\eta}_\alpha \gamma = \boldsymbol{\eta}_\beta \gamma$ for all $\gamma \in \mathsf{dom}(\boldsymbol{\eta}_\alpha) \cap U$ and $\mathsf{dom}(\boldsymbol{\eta}_\beta) = U \cap \mathsf{dom}(\boldsymbol{\eta}_\alpha)$.*

Proof. By Proposition 1.1.7 there exists $\upsilon \in \mathbb{B}$ such that for all $\beta \in \mathbb{B}$

$$\boldsymbol{\eta}_\upsilon \beta = \begin{cases} \lambda n.\, 1 & \beta \in U\ , \\ \text{undefined} & \text{otherwise}\ . \end{cases}$$

It suffices to show that the function $f \colon \mathbb{B} \rightharpoonup \mathbb{B}$ defined by

$$(f\beta)n = ((\eta_\upsilon \beta)n) \cdot ((\eta_\alpha \beta)n)$$

is realized. This is so because coordinate-wise multiplication of sequences is realized, and so are pairing and composition. ∎

Theorem 1.1.9 *A partial function $f \colon \mathbb{B} \rightharpoonup \mathbb{B}$ is realized if, and only if, f is continuous and its domain is a G_δ-set.*

Proof. First we show that $\boldsymbol{\eta}_\alpha$ is a continuous map whose domain is a G_δ-set. It is continuous because the value of $(\boldsymbol{\eta}_\alpha \beta)n$ depends only on n and finite prefixes of α and β. The domain of $\boldsymbol{\eta}_\alpha$ is the G_δ-set

$$\begin{aligned} \mathsf{dom}(\boldsymbol{\eta}_\alpha) &= \left\{\beta \in \mathbb{B} \;\middle|\; \forall\, n \in \mathbb{N} \,.\, ((\alpha \mid \beta)n \text{ defined})\right\} \\ &= \bigcap_{n \in \mathbb{N}} \left\{\beta \in \mathbb{B} \;\middle|\; (\alpha \mid \beta)n \text{ defined}\right\} = \bigcap_{n \in \mathbb{N}} \bigcup_{m \in \mathbb{N}} \left\{\beta \in \mathbb{B} \;\middle|\; \alpha \star (n{::}\beta) = m\right\}\ . \end{aligned}$$

Each of the sets $\left\{\beta \in \mathbb{B} \;\middle|\; \alpha \star (n{::}\beta) = m\right\}$ is open because $\star$ and $::$ are continuous operations.

Now let $f \colon \mathbb{B} \rightharpoonup \mathbb{B}$ be a partial continuous function whose domain is a G_δ-set. By Theorem 1.1.6 there exists $\phi \in \mathbb{B}$ such that $f\alpha = \boldsymbol{\eta}_\phi \alpha$ for all $\alpha \in \mathsf{dom}(f)$. By Corollary 1.1.8 there exists $\psi \in \mathbb{B}$ such that $\mathsf{dom}(\boldsymbol{\eta}_\psi) = \mathsf{dom}(f)$ and $\boldsymbol{\eta}_\psi \alpha = \boldsymbol{\eta}_\phi \alpha$ for every $\alpha \in \mathsf{dom}(f)$. ∎

The Effective Second Kleene Algebra $\mathbb{B}_\sharp$

As we have seen the second Kleene algebra $\mathbb{B}$ enjoys its own versions of the Embedding and Extension Theorems. In this respect it is similar to the graph model $\mathbb{P}$, and just like the graph model, it also has a natural notion of computable elements, namely the total recursive functions. In fact, the set of recursive functions by $\mathbb{B}_\sharp$ is a subPCA of $\mathbb{B}$. We omit the details of the proof, suffice it to say that the continuous function application $\mid$ was given by an effective procedure, and that the combinators K and S are also constructed explicitly.

1.1.7 PCA over a First-order Structure

A *many-sorted first-order structure* is a structure

$$\mathcal{S} = (B_1, \ldots, B_k, f_1, \ldots, f_m, R_1, \ldots, R_n)$$

where

(1) B_i, $i = 1, \ldots, k$, is a non-empty set. The sets B_i are called the *basic sorts*. We require that the basic sorts are pairwise disjoint. A *sort* is finite cartesian product of the basic sorts. We use letters S and T to denote sorts. The empty cartesian product is a sort; it is the singleton set $1 = \{\star\}$.

(2) f_j, $j = 1, \ldots, m$, is a *partial* function $f_j \colon S_j \rightharpoonup B_{i_j}$ from a sort S_j to a basic sort B_{i_j}. A function that maps from the singleton set $1 \to T$ is just a *constant* element of T. The functions f_j are called the *basic operations* and *basic constants*.

(3) R_k, $k = 1, \ldots, n$, is a *partial* relation on a sort S_k, i.e., a partial function $R_k \colon S_k \rightharpoonup \{\mathsf{false}, \mathsf{true}\}$. The relations R_k are called the *basic relations*.

We do *not* assume that equality is always available as a basic relation.

Example 1.1.10 The structure of *the ordered field of real numbers*

$$\mathcal{R} = (\mathbb{R}, 0, 1, +, -, \times, /, =, <)$$

consists of the set of real numbers $\mathbb{R}$, the basic constants 0 and 1, the basic arithmetic operations of addition $+$, subtraction $-$, multiplication $\times$, division $/$, and the basic relations of equality $=$ and comparison $<$.

Example 1.1.11 In the previous example the basic relations $=$ and $<$ are discontinuous when viewed as boolean functions from $\mathbb{R}^2$ to the discrete space $B = \{\mathsf{false}, \mathsf{true}\}$. We can change this by omitting the equality relation and replacing $<$ with the *partial* comparison

$$(x < y) = \begin{cases} \mathsf{true} & \text{if } x \text{ is strictly smaller than } y \ , \\ \mathsf{false} & \text{if } y \text{ is strictly smaller than } x \ , \\ \bot & \text{if } x = y \ . \end{cases}$$

Here $\bot$ stands for "undefined", and the topology on $B_\bot = \{\text{undefined}, \mathsf{false}, \mathsf{true}\}$ is determined by the basic open sets $B_\bot$, $\{\mathsf{false}\}$, $\{\mathsf{true}\}$. This way we get the *topological algebra of the real numbers*

$$\mathcal{R}_{\mathrm{t}} = (\mathbb{R}, 0, 1, +, -, \times, /, <) \ .$$

We now describe a *Turing machine over a first-order structure* $\mathcal{S}$, or shortly an $\mathcal{S}$-TM, which is an ordinary TM enhanced with the basic sorts, operations, and relations from $\mathcal{S}$. It has one input tape, a fixed finite number of working tapes, and one output tape. The tapes are infinite and it is not important whether we take them to be infinite only in one direction or both. Each tape has a read/write head associated with it. When we say "read from tape T" or "write onto tape T" we always mean to read from the current position of the head on that tape. Tape cells may contain the following values:

(1) ϵ, which indicates the empty tape cell,

(2) the symbols `0` and `1`,

(3) for each basic sort B_i, and for each element $x \in B_i$, the value x can be written onto a cell.

Instead of the symbols 0 and 1, we could have chosen any other finite alphabet with at least two elements. Apart from the usual instructions, the program for an $\mathcal{S}$-TM may contain the following special instructions:

(1) For every $x \in B_i$, and every tape τ, an instruction that writes x onto τ. Every such element x that appears in a program is called a *machine constant*. Since the program for a $\mathcal{S}$-TM machine is finite, it contains finitely many machine constants.

(2) For every two tapes τ, τ', an instruction that copies the value from tape τ onto tape τ'.

(3) For every two tapes τ, τ' and every basic operation f_j, an instruction that computes the value of f_j, taking as the arguments the values written immediately to the right of head τ. If the value is defined, it is written onto tape τ', otherwise the machine diverges.

(4) For every tape τ and every basic relation R_k, an instruction that computes the value of R_k, taking as the arguments the values written immediately to the right of head τ. If the value is true the machine goes to state P, if the value is false the machine goes to state Q, and if the value is undefined, the machine diverges.

An ordinary Turing machine can be encoded as a single natural number. When this is done in a reasonable way, we obtain the first Kleene algebra $\mathbb{N}$, by interpreting natural numbers as codes for Turing machines. We would like to do the same for Turing machines over $\mathcal{S}$. However, since an $\mathcal{S}$-TM may contain a finite number of machine constants, it cannot be encoded by a single natural number. Instead, it can be encoded as a pair $\langle p, x\rangle$ where $p \in \mathbb{N}$ is a natural number that describes the program, except for the machine constants, and $x = (x_1, \dots, x_m)$ is the list of machine constants that appear in the program.

We define a PCA $\mathsf{TM}(\mathcal{S})$, called *the first Kleene algebra over* $\mathcal{S}$. We omit a formal definition of all the coding tricks required to define the PCA. The underlying set of $\mathsf{TM}(\mathcal{S})$ is

$$|\mathsf{TM}(\mathcal{S})| = \mathbb{N} \times (B_1 + \dots + B_n)^* \;,$$

where $(B_1 + \dots + B_n)^*$ is the set of all finite sequences of elements of the disjoint union $B_1 + \dots + B_n$. If $\langle p, x\rangle, \langle q, y\rangle \in |\mathsf{TM}(\mathcal{S})|$ then the application

$$\{\langle p, x\rangle\}\, \langle q, y\rangle \;, \tag{1.3}$$

is defined as follows. Interpret the pair $\langle p, x\rangle$ as the code of an $\mathcal{S}$-TM T. Write the number q and the list $y = (y_1, \dots, y_k)$ onto the input tape of T, using a suitable encoding to write down q and to indicate where the input starts and ends. Run the machine T. If T terminates successfully, and immediately to the left of the output head a pair $\langle r, z\rangle \in |\mathsf{TM}(\mathcal{S})|$ is written, of course suitably encoded, then the value of (1.3) is defined to be $\langle r, z\rangle$. Otherwise, the value is undefined.

Example 1.1.12 A Turing machine over the structure $\mathcal{R}$ of the ordered field of reals corresponds to a *BSS Real RAM machine* by Blum, Shub, and Smale.[10] The original definition is in terms of flow-chart diagrams and random-access memory. The two definitions are equivalent up to polynomial time speed-up. In this model of computation a real number can be written in constant space, a single tape cell even. Furthermore, the basic arithmetic operations require a single step to compute, there are discontinuous computable functions, and equality on $\mathbb{R}$ is decidable.

[10] References on BSS [BCSS97].

Example 1.1.13 The PCA $\mathsf{TM}(\mathcal{R}_t)$ corresponds to the while^* programs [TZ99]. Briefly, the while^* programs over a partial topological algebra are simple imperative programs with simple control structures, such as while and $\mathsf{if\text{-}then\text{-}else}$, and unlimited amount of random access memory. The programs also use the basic constants, functions and relations from the partial topological algebra as primitive operations. In fact, any partial topological algebra, as defined by Tucker and Zucker, can be viewed as a first-order structure. Computability by while^* programs is then equivalent to computability by Turing machines over the first-order structure.

1.2 Modest Sets

In order to study computability of classical mathematical structures, such as the real numbers and spaces of smooth functions, we first need to link them up with models of computability. Classical mathematics is developed within the realm of set theory. We refine the usual notion of a set by keeping track of how the elements of a set are represented in a model of computation. Thus, every set S comes equipped with a *realizability relation* $\Vdash_S$ which relates the elements of a PCA $\mathbb{A}$ with the elements of S. A set together with its realizability relation is called a *modest set.* If $x \in S$ and $a \in \mathbb{A}$, the meaning of $a \Vdash_S x$ is "the element x is realized by a". We think of a as the implementation, or the *realizer*, of the element x. For example, the number $x = 42$ might be implemented by the binary string $a = \mathtt{101010}$, which we would express as $\mathtt{101010} \Vdash_{\mathbb{N}} 42$. We require that every element is realized by at least one program and that every program realizes at most one element of a given set. An element of a set may be realized by several programs. It is important to allow this because some sets may not have canonical realizers. For example, a partial recursive function $\mathbb{N} \rightharpoonup \mathbb{N}$ can be implemented by many different programs, and all of them should be allowed as valid realizers for the function.

We also need to keep track of how functions are implemented in the computational model. We say that a function $f : S \to T$ is *tracked* by $a \in \mathbb{A}$ in case that $b \Vdash_S x$ implies $a \cdot b \Vdash_T fx$. Of course, there may be functions that are not tracked by any element of $\mathbb{A}$. This is well known in recursion theory where only a subset of all the number-theoretic functions can be implemented by a Turing machine. When a function is tracked by a program we say that it is a *realized function.* We now give a formal definition.

Definition 1.2.1 Let $\mathbb{A}$ be a PCA and $\mathbb{A}_\sharp \subseteq \mathbb{A}$ a subPCA. A *modest set* $(S, \Vdash_S)$ over $\mathbb{A}$ is a set S with a *realizability relation* $\Vdash_S \subseteq \mathbb{A} \times S$ such that, for all $x, y \in S$,

$$x = y \iff \exists\, a \in \mathbb{A}\,.\,(a \Vdash_S x \land a \Vdash_S y) \ . \tag{1.4}$$

An $\mathbb{A}_\sharp$*-realizable function* $f : (S, \Vdash_S) \to (T, \Vdash_T)$ between modest sets is a function $f : S \to T$ between the underlying sets that is *tracked* by some $a \in \mathbb{A}_\sharp$, which means that, for all $x \in S$, $b \in \mathbb{A}$,

$$b \Vdash_S x \Longrightarrow (a \cdot b){\downarrow} \land a \cdot b \Vdash_T fx \ .$$

The category of modest sets over $\mathbb{A}$ and $\mathbb{A}_\sharp$-realizable functions is denoted by $\mathsf{Mod}(\mathbb{A}, \mathbb{A}_\sharp)$. We abbreviate $\mathsf{Mod}(\mathbb{A}, \mathbb{A})$ by $\mathsf{Mod}(\mathbb{A})$.

The *existence predicate* for S is the function $\mathsf{E}_S : S \to \mathcal{P}\mathbb{A}$ defined by $\mathsf{E}_S x = \{a \in \mathbb{A} \mid a \Vdash_S x\}$. Condition (1.4) is equivalent to the requirement that, for every $x \in S$, $\mathsf{E}_S x \neq \emptyset$ and that, for all $x, y \in S$,

$$x \neq y \Longrightarrow \mathsf{E}_S x \cap \mathsf{E}_S y = \emptyset \ .$$

A modest set is determined by its existence predicate. When no confusion can arise we drop the subscripts in $\Vdash_S$ and E_S, and refer to a modest set $(S, \Vdash_S)$ simply as S. Sometimes we denote the underlying set of a modest set $(S, \Vdash_S)$ by $|S|$ instead of S. To see that $\mathsf{Mod}(\mathbb{A}, \mathbb{A}_\sharp)$ is a category, observe that the identity function $1_S \colon S \to S$ is realized by the combinator I, and that the composition of $f \colon S \to T$ and $g \colon T \to U$ is realized by $\lambda^* x.\, a(bx) = \mathsf{S}(\mathsf{K}a)(\mathsf{S}(\mathsf{K}b)(\mathsf{SKK}))$, where a and b are realizers for f and g, respectively.

Let us explain the motivation for taking modest sets over $\mathbb{A}$ but restricting the realizers for functions to a subPCA $\mathbb{A}_\sharp$. We can think of modest sets as *data structures*. There is no need to restrict the model so that only the computable data are representable. Indeed, a realistic model of computation presumably should not assume that all data from the real world, such as the stock market index, or the results of a quantum mechanics experiment, are always computable. On the other hand, morphisms between modest sets can be viewed as *programs*, and clearly they should be restricted to those functions that can actually be realized in a model of computation. These ideas can be summarized by the slogan

"Topological objects, computable morphisms!"

The slogan calls for "topological" objects to remind us that in realistic examples information flow is finite, i.e., it is only possible to communicate a finite amount of information about the data in finite time. This restriction induces a natural topology on the data, if we take as subbasic open sets those properties of data that can be communicated in finite time. With this topology all computable maps turn out to be continuous, which happens because a finite result of a computable process only depends on a finite amount of input. Since we view objects as data structures, they are naturally equipped with a topology.[11] In the abstract case, an arbitrary PCA $\mathbb{A}$ might of course violate the assumption of finite information flow, but most models of computation that are considered to be realistic, such as domain theory and Type Two Effectivity, reflect the idea of finite information flow.

The slogan is demonstrated by the difference between the internal and external function spaces. For example, in $\mathsf{Mod}(\mathbb{P}, \mathbb{P}_\sharp)$ the exponential object $\mathbb{N}^\mathbb{N}$, where $\mathbb{N}$ is the natural numbers object, turns out to be the Baire space which consists of all infinite sequences of natural numbers, but the homset $\mathsf{Hom}(\mathbb{N}, \mathbb{N})$ is the set $\mathcal{R}^{(1)}$ of total recursive functions.

Modest sets can also be described as partial equivalence relations, and as representations. We shall use these alternative descriptions when convenient.

1.2.1 Modest Sets as Partial Equivalence Relations

A *partial equivalence relation* (per) is a symmetric and transitive relation. We use capital letters A, B, $C, \ldots$ to denote partial equivalence relations, but we write $x =_A y$ instead of $x\ A\ y$. The category of partial equivalence relations $\mathsf{PER}(\mathbb{A}, \mathbb{A}_\sharp)$ has as objects partial equivalence relations on $\mathbb{A}$, and as morphisms equivalence classes of equivalence preserving elements of $\mathbb{A}_\sharp$. More precisely, suppose A and B are partial equivalence relations on $\mathbb{A}$. We say that $f \in \mathbb{A}_\sharp$ is equivalence preserving when, for all $a, b \in \mathbb{A}$,

$$a =_A b \Longrightarrow (f \cdot a)\!\downarrow\, =_B (f \cdot b)\!\downarrow .$$

[11] In Section 5.3 such an *intrinsic* topology is formalized.

Two equivalence preserving elements $f, g \in \mathbb{A}_\sharp$ are considered to be equivalent when, for all $a, b \in \mathbb{A}$,

$$a =_A b \Longrightarrow (f \cdot a)\!\downarrow =_B (g \cdot b)\!\downarrow .$$

A morphism $A \to B$ is an equivalence class of equivalence preserving elements of $\mathbb{A}_\sharp$. Composition of morphisms $[f]\colon A \to B$ and $[g]\colon B \to C$ is the morphism $[\lambda^* x.\, g(fx)] : A \to C$.

Proposition 1.2.2 *The categories* $\mathsf{Mod}(\mathbb{A}, \mathbb{A}_\sharp)$ *and* $\mathsf{PER}(\mathbb{A}, \mathbb{A}_\sharp)$ *are equivalent.*

Proof. The existence predicate E of a modest set S over a PCA $\mathbb{A}$ satisfies, for all $x, y \in S$,

$$\mathsf{E}x \neq \emptyset , \qquad\qquad x \neq y \Longrightarrow \mathsf{E}x \cap \mathsf{E}y = \emptyset .$$

Hence, the family $\{\mathsf{E}x \mid x \in S\}$ consists of non-empty pairwise disjoint subsets of $\mathbb{A}$. When these sets are viewed as equivalence classes, they determine a partial equivalence relation S on $\mathbb{A}$, defined by

$$a =_S b \iff \exists\, x \in S\,.\,(a \Vdash x \wedge b \Vdash x) \;.$$

A partial equivalence relation A on $\mathbb{A}$ determines a modest set as follows. Let $[a]_A$ denote the equivalence class of $a \in \mathbb{A}$, assuming $a =_A a$. The modest set determined by A is the set of equivalence classes

$$|A| = \{[a]_A \mid a \in \mathbb{A} \wedge a \sim a\}$$

with the existence predicate being simply the identity, $\mathsf{E}_A[a]_A = [a]_A$.

A morphism $f\colon S \to T$ between modest sets corresponds to a morphism $[a]\colon S \to T$ of pers, where $a \in \mathbb{A}_\sharp$ is a realizer for f. Clearly, all realizers for f determine the same morphism between pers. In the other direction, a morphism $[a]\colon A \to B$ between pers corresponds to the morphism $|A| \to |B|$ defined by $[b]_A \mapsto [a \cdot b]_B$. It is evident that these correspondences constitute an equivalence of categories. ∎

1.2.2 Modest Sets as Representations

Let $\mathbb{A}$ be a PCA and $\mathbb{A}_\sharp \subseteq \mathbb{A}$ a subPCA. A *representation* of a set S over a PCA $\mathbb{A}$ is a partial surjection $\delta_S\colon \mathbb{A} \rightharpoonup S$. A morphism $f\colon (S, \delta_S) \to (T, \delta_T)$ between representations is a function $f\colon S \to T$ that is *tracked* by some $a \in \mathbb{A}_\sharp$, which means that, for all $b \in \mathsf{dom}(\delta_S)$,

$$f(\delta_S b) = \delta_T(a \cdot b\!\downarrow) \;.$$

The category of representations and morphisms over $\mathbb{A}_\sharp \subseteq \mathbb{A}$ is clearly equivalent to the category $\mathsf{Mod}(\mathbb{A}, \mathbb{A}_\sharp)$, since a modest set $(S, \Vdash)$ corresponds to the representation (S, δ), defined by

$$\delta a = x \iff a \Vdash x \;, \tag{1.5}$$

and vice versa, a representation (S, δ) corresponds to the modest set $(S, \Vdash)$, again defined by (1.5).

1.3 Properties of Modest Sets

This section contains a brief review of the basic categorical constructions on modest sets. Many proofs are omitted since they are easy and readily available elsewhere [Lon94, Bir99, Pho92].

1.3.1 Finite Limits and Colimits

Monos and Epis

A realizable map $f\colon S \to T$ is a mono if, and only if, it is injective. It is an epi if, and only if, it is surjective. Indeed, this follows easily from the fact that morphisms are just functions between sets, and that equality and composition of functions are the usual set-theoretic ones.

Equalizers and coequalizers

The equalizer of two maps $f, g\colon S \to T$ is the modest set

$$K = \{x \in S \mid fx = gx\}$$

whose existence predicate is inherited from S, $\mathsf{E}_K x = \mathsf{E}_S x$. The equalizer map $k\colon K \to S$ is the subset inclusion $K \subseteq S$, and is realized by I.

The coequalizer of maps $f, g\colon S \to T$ is constructed as follows. Let $\sim$ be the smallest equivalence relation on $|T|$ that satisfies $fx \sim gx$ for all $x \in |S|$. The coequalizer of f and g is the modest set Q whose underlying set is $|T|/{\sim}$, the set of $\sim$-equivalence classes, and the existence predicate is

$$\mathsf{E}_Q[x] = \bigcup_{y \in [x]} \mathsf{E}_T y \; .$$

The coequalizer map $q\colon T \to Q$ is the canonical quotient map $qx = [x]$, and is realized by I.

The Initial and the Terminal Object

A modest set is an initial object if, and only if, its underlying set is empty. We denote the initial object by $\mathsf{0}$.

A modest set is a terminal object if, and only if, its underlying set is a singleton, and the single element has a realizer from $\mathbb{A}_\sharp$. Specifically, we define the terminal object 1 to be the singleton set $\{\star\}$ with existence predicate $\mathsf{E}_1\star = \{\mathsf{K}\}$.

Binary Products and Coproducts

The product of modest sets S and T is the usual cartesian product $S \times T$ with the realizability relation defined by

$$\langle a, b\rangle \Vdash_{S\times T} \langle x, y\rangle \iff a \Vdash_S x \wedge b \Vdash_T y \; ,$$

where $a, b \in \mathbb{A}$, and $\langle a, b\rangle$ is the pairing of a and b, as described in Section 1.1. The projections $\mathsf{fst} : S \times T \to S$ and $\mathsf{snd} : S \times T \to T$ are the usual set-theoretic projections, which are realized by the combinators fst and snd, respectively.

The coproduct of modest sets S and T is the usual disjoint sum

$$S + T = \{\mathsf{inl}\, x \mid x \in S\} \cup \{\mathsf{inr}\, y \mid y \in T\} \; ,$$

where $\mathsf{inl}\colon S \to S + T$ and $\mathsf{inr}\colon T \to S + T$ are the canonical inclusions. The realizability relation on $S + T$ is defined by

$$\langle \mathsf{false}, a\rangle \Vdash_{S+T} \mathsf{inl}\, x \iff a \Vdash_S x \; , \qquad \langle \mathsf{true}, b\rangle \Vdash_{S+T} \mathsf{inr}\, y \iff b \Vdash_S y \; .$$

The inclusion maps inl and inr are realized by $\lambda^* u.\, \langle \mathsf{false}, u\rangle$ and $\lambda^* u.\, \langle \mathsf{true}, v\rangle$, respectively.

Pullbacks and Pushouts

A pullback

$$\begin{tikzcd} P \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & S \arrow[d, "f"] \\ T \arrow[r, "g"'] & U \end{tikzcd}$$

is constructed as follows. The modest set P is the set

$$P = \{\langle x, y\rangle \in S \times T \mid fx = gy\}$$

with the realizability relation inherited from $S \times T$,

$$\langle a, b\rangle \Vdash_P \langle x, y\rangle \iff a \Vdash_S x \wedge b \Vdash_T y \ .$$

The maps p_1 and p_2 are the first and the second projection, respectively. They are realized by fst and snd.

A pushout

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d, "g"'] & T \arrow[d, "q_1"] \\ U \arrow[r, "q_2"'] \arrow[ur, phantom, "\ulcorner", very near end] & Q \end{tikzcd}$$

is constructed as follows. Let $\sim$ be the smallest equivalence relation on $|T + U|$ that satisfies $\mathsf{inr}(fx) \sim \mathsf{inl}(gx)$ for all $x \in |S|$. The modest set Q is the set $|T+U|/\sim$ with the existence predicate

$$\mathsf{E}_Q[z] = \bigcup_{\mathsf{inl}\, x \sim z} \{\langle \mathsf{false}, a\rangle \mid a \in \mathsf{E}_T x\} \ \cup \bigcup_{\mathsf{inr}\, u \sim z} \{\langle \mathsf{true}, a\rangle \mid a \in \mathsf{E}_U y\} \ .$$

The maps q_1 and q_2 are the canonical injections $|S| \hookrightarrow |S+T|$ and $|T| \hookrightarrow |S+T|$, and they are realized by $\lambda^* u.\, \langle \mathsf{false}, u\rangle$ and $\lambda^* u.\, \langle \mathsf{true}, v\rangle$, respectively.

Regular Monos and Epis

Recall that a morphism is a regular mono if, and only if, it is an equalizer of a pair of maps. The dual notions is that of a regular epi, which is an arrow that is a coequalizer.

A *canonical inclusion* is a morphism $i \colon S \to T$ where $|S| \subseteq |T|$, i is the canonical set-theoretic inclusion map, and E_S is the restriction of E_T to $|S|$. The canonical inclusion map is realized by the combinator I.

A realized map $f \colon S \to T$ is a regular mono if, and only if, it is isomorphic to a canonical inclusion. This means that there is an embedding $i \colon S' \to T'$ and a pair of isomorphisms $s \colon S \to S'$,

$t\colon T \to T'$ such that the following diagram commutes:

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d, "s"'] & T \arrow[d, "t"] \\ S' \arrow[r, "i"'] & T' \end{tikzcd}$$

It is easy to check that a canonical inclusion is indeed a regular mono, and thus every morphisms that is isomorphic to an embedding is a regular mono as well. The converse follows from the earlier construction of equalizers, where an equalizer was seen to be a canonical inclusion.

A *quotient map* is a morphism of the form $q\colon S \to S/{\sim}$ that maps an element x to its equivalence class $qx = [x]$ under the equivalence relation $\sim$ on $|S|$. Here the existence predicate on $S/{\sim}$ is defined by

$$\mathsf{E}_{S/\sim}[x] = \bigcup_{y \in [x]} \mathsf{E}_S y \ .$$

A morphism $f\colon S \to T$ is a regular epi if, and only if, it is equivalent to a quotient map. Again, this equivalence is evident from the construction of coequalizers as quotient maps.

1.3.2 The Locally Cartesian Closed Structure

The exponential T^S of modest sets S and T is the modest set

$$T^S = \big\{ f\colon |S| \to |T| \;\big|\; f \text{ is } \mathbb{A}\text{-realized} \big\} \ .$$

The realizability relation on T^S is defined by

$$a \Vdash_{T^S} f \iff \forall\, x \in S \,.\, \forall\, b \in \mathbb{A} \,.\, (b \Vdash_S x \Longrightarrow (a \cdot b){\downarrow} \Vdash_T fx) \ .$$

Note that T^S contains those functions from $|S|$ to $|T|$ that are realized by elements of $\mathbb{A}$, and not just $\mathbb{A}_\sharp$. The evaluation map $\epsilon\colon T^S \times S \to T$ is the usual set-theoretic one, $\epsilon(f, x) = fx$, and is realized by $\lambda^* u.\, (\mathsf{fst}\, u)(\mathsf{snd}\, u)$. Let us recall what universal property T^S with ϵ satisfies. For every modest set U and every morphism $f\colon U \times S \to T$ there exists exactly one morphism $\widetilde{f}\colon U \to T^S$ such that the following diagram commutes:

$$\begin{tikzcd} T^S \times S \arrow[dr, "\epsilon"] & \\ U \times S \arrow[u, "\widetilde{f} \times 1_S"] \arrow[r, "f"'] & T \end{tikzcd}$$

The maps $\widetilde{f}$ and f are *transposes* of each other. In modest sets the transpose $\widetilde{f}$ is like in sets, $\widetilde{f}x = \lambda y \in S \,.\, f(x, y)$. It is realized by $\lambda^* uv.\, a\langle u, v\rangle$, where a is a realizer for f. Sometimes we denote the exponential T^S by $S \to T$.

A *dependent type* is a family of objects, indexed by an object.[12] More precisely, if I is a modest set and for each $i \in I$ we have a modest set $S(i)$, then $\{S(i) \mid i \in I\}$ is a dependent type.[13] We indicate the fact that S is a dependent type indexed by $i \in I$ by writing $S(i:I)$ or $\{S(i) \mid i \in I\}$.

Suppose $S(i:I)$ and $T(j:J)$ are dependent types. A morphism $f = (r_f, (f_i)_{i \in I}) \colon S \to T$ between dependent types consists of a *reindexing morphism* $r_f \colon I \to J$, which is just an ordinary morphism of modest sets, together with a family of functions

$$f_i \colon T(i) \to S(r_f i) ,$$

one for each $i \in I$, such that the family $(f_i)_{i \in I}$ is *uniformly realized*, which means: there exists $a \in \mathbb{A}_\sharp$ such that, for all $i \in I$ and all $b \Vdash_I i$, $a \cdot b$ realizes f_i. Morphisms between dependent types are composed in the obvious way, and together with dependent types form a category.

As an example of a dependent type, we consider the *inverse image* of a morphism $f \colon A \to B$. For every $y \in B$, we can define a modest set

$$f^* y = \{x \in A \mid fx = y\} ,$$

with the existence predicate $\mathsf{E}_{f^*y} x = \mathsf{E}_A x$. This makes f^* into a dependent type, indexed by $y \in B$.

Another, trivial example of a dependent type is the constant dependent type $T(i) = T$, $i \in I$. It is isomorphic to the dependent type fst^* for the first projection $\mathsf{fst} \colon I \times T \to I$.

Suppose $T(i:I)$ is a dependent type. The *dependent sum* $\sum_{i:I} T(i)$ is the modest set

$$\textstyle\sum_{i:I} T(i) = \{\langle i, x\rangle \mid i \in I \land x \in T(i)\}$$

with the realizability relation

$$\langle a, b\rangle \Vdash_{\sum_{i:I} T(i)} \langle i, x\rangle \iff a \Vdash_I i \text{ and } b \Vdash_{T(i)} x .$$

There are two *projection morphisms* $\mathsf{fst} : \sum_{i:I} T(i) \to I$ and $\mathsf{snd}_i \colon \sum_{i:I} T(i) \to T(i)$, defined by

$$\mathsf{fst}\,\langle i, x\rangle = i , \qquad \mathsf{snd}\,\langle i, x\rangle = x .$$

Whereas fst is an ordinary morphism of modest sets, snd is a morphism of dependent types, where the dependent sum $\sum_{i:I} T(i)$ is viewed as the dependent type

$$\mathsf{fst}^* i = \{pair j, x \in \textstyle\sum_{i:I} T(i) \mid i = j\} . \qquad (i \in I)$$

Dependent sums are at once a generalization of products and coproducts. If A and B are modest sets, then the binary product $A \times B$ is isomorphic to the dependent sum of the constant dependent type,

$$A \times B = \textstyle\sum_{x:A} B .$$

[12]The usual way to handle dependent types in category theory is with slice categories, i.e., a dependent type indexed by I corresponds to a morphism $T \to I$. The presentation of dependent types as *uniform families* given here is equivalent to the usual approach with slice categories, in a precise way—they are equivalent as fibrations, see [Bir99, BBS98] for details. We chose uniform families because they correspond more directly to the intuition that a dependent type is just a family of types.

[13]In mathematical practice dependent types are commonplace. For example, whenever we speak of an n-dimensional Euclidean space $\mathbb{R}^n$, that is a dependent type indexed by $n \in \mathbb{N}$. In computer science a common dependent type is $\mathsf{array}[n]$, the type of arrays of length n.

The binary coproduct $A+B$ is isomorphic to the coproduct of the dependent type $T_0 = A$, $T_1 = B$ over 2,

$$\textstyle\sum_{i \in 2} T_i = T_0 + T_1 = A + B \; .$$

The *dependent product* $\prod_{i\,:\,I} T(i)$ of a dependent type $T(i : I)$ is the modest set

$$\textstyle\prod_{i\,:\,I} T(i) = \left\{ f \colon I \to \coprod_{i \in I} T(i) \;\middle|\; f \text{ is realized and } \forall\, i \in I \,.\, fi \in T(i) \right\}$$

with the realizability relation

$$a \Vdash_{\prod_{i\,:\,I} T(i)} f \iff \forall\, i \in I \,.\, \forall\, b \in \mathsf{E}_I i \,.\, \big(a \cdot b \Vdash_{T(i)} fi\big) \; .$$

Here $\coprod_{i \in I} T(i)$ is a disjoint union of the sets $T(i)$. A function $f \colon I \to \coprod_{i \in I} T(i)$ such that $fi \in T(i)$ for all $i \in I$ is called a *choice function* because it chooses an element of $T(i)$ for each $i \in I$. With the dependent product is associated the *evaluation morphism*

$$\epsilon \colon \; (\textstyle\prod_{i\,:\,I} T(i)) \times I \to T(i) \; ,$$

defined by $\epsilon\langle f, i\rangle = fi$. It is realized by the combinator $\lambda^* ab.\, ab$.

Dependent products are a generalization of exponentials. Indeed, it is not hard to see that the exponential B^A can be written as a dependent product of the constant dependent types, $B^A = \prod_{x\,:\,A} B$.

1.3.3 The Regular Structure

A category is *regular* when every arrow has a kernel pair, every kernel pair has a coequalizer, and the pullback of a regular epimorphism along any morphism exists and is again a regular epimorphism [Bor94, Definition 2.1.1]. It is often required that a regular category have all finite limits. In the case of modest sets this is the case. In a regular category every morphism can be decomposed in a unique way into an epimorphism followed by a regular monomorphism.

We already know that $\mathsf{Mod}(\mathbb{A}, \mathbb{A}_\sharp)$ has all finite limits and colimits, so in order to see that it is a regular category it only remains to be shown that regular epimorphisms are stable under pullbacks. In fact, pullbacks preserve coequalizer diagrams. Suppose we pull back a coequalizer diagram as in the diagram below.

$$\begin{tikzcd}
q^*A \arrow[r] \arrow[d, shift right, "q^*k_1"'] \arrow[d, shift left, "q^*k_2"] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, shift right, "k_1"'] \arrow[d, shift left, "k_2"] \\
g^*B \arrow[r, "q"] \arrow[d, "g^*f"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "f"] \\
D \arrow[r, "g"'] & C
\end{tikzcd}$$

We want to show that the left-hand column forms a coequalizer diagram if the right-hand one does. This can be verified by writing down the explicit construction of the pullbacks and comparing it to the construction of coequalizers. For a proof see [Lon94, Subsection 1.2.2].

The regular structure together with the locally cartesian closed structure and disjoint, stable coproduct suffices for an interpretation of intuitionistic first-order logic in $\mathsf{Mod}(\mathbb{A}, \mathbb{A}_\sharp)$. For details see [BBS98, Bir99].

1.3.4 Projective Modest Sets

Definition 1.3.1 A modest set P is *projective* when for every regular epi $f\colon A \to B$ and every morphism $g\colon P \to B$, there exists a (not necessarily unique) morphism $\overline{g}\colon P \to A$ such that the following diagram commutes:

$$\begin{tikzcd} & P \arrow[dl, "\overline{g}"'] \arrow[d, "g"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

Note: what we call projective modest set is usually called regular projective modest set. But since regular projectives are the only kind we ever consider we call them simply projective.

Definition 1.3.2 A modest set S is *canonically separated* when every element of S has exactly one realizer. A modest set is *separated* when it is isomorphic to a canonically separated modest set.

We say that the modest set S is *covered* by the modest set T via q when $q\colon T \twoheadrightarrow S$ is a regular epi. We also say that T is a *cover* of S.

Proposition 1.3.3 *Every modest set is covered by a canonically separated modest set.*

Proof. Let S be a modest set. Define the modest set S_0 to be the set

$$S_0 = \bigcup_{x \in S} \mathsf{E}_S x$$

with the existence predicate $\mathsf{E}_{S_0} a = \{a\}$. For every $a \in S_0$ there exists a unique $x \in S$ such that $a \Vdash_S x$. Let $q_S\colon S_0 \to S$ be the function

$$q_S a = (\text{the } x \in S \text{ such that } a \Vdash_S x)\ .$$

It is realized by the identity combinator I. It is obvious that q_S is a regular epi. ∎

The cover S_0 in the previous proof is called the *canonical cover of* S_0.

Theorem 1.3.4 *The following are equivalent for a modest set P:*

(1) P is projective.

(2) P is separated.

(3) The canonical cover $q_P\colon P_0 \twoheadrightarrow P$ is a retraction.

(4) $\mathsf{Hom}(P, f)\colon \mathsf{Hom}(P, A) \to \mathsf{Hom}(P, B)$ is a surjective map, for every regular epi $f\colon A \twoheadrightarrow B$.

Proof. Recall that the definition of $\mathsf{Hom}(P, f)$: the function $\mathsf{Hom}(P, f)$ maps a morphism $h\colon P \to A$ to the morphism $f \circ h\colon P \to B$. The equivalence of (1) and (4) is obvious because (4) is just the definition of projective objects, phrased in terms of Hom-sets.

Let us prove that (1) implies (2). Suppose P is projective. Let $q_P\colon P_0 \twoheadrightarrow P$ be the canonical cover of P. Because P is projective and q_P is regular epi, there exists $s\colon P \to P_0$ such that $q_P \circ s = 1_P$. Let R be the image of s, i.e., the modest set $|R| = \{sx \mid x \in P\}$ with the existence predicate $\mathsf{E}_R a = \mathsf{E}_{P_0} a$. Clearly, R is canonically separated. The map $s\colon S \to R$ and the restriction $q_P{\restriction}_R\colon R \to P$ are inverses of each other, as is easily checked. Therefore P is separated because it is isomorphic to R.

It suffices to prove that (2) implies (1) just for canonically separated modest sets. Furthermore, every regular epi is isomorphic to one that is realized by the identity combinator I. It is enough to prove the claim for such regular epis. So suppose P is canonically separated, $f\colon A \twoheadrightarrow B$ is a regular epi that is realized by the identity combinator I, and $g\colon P \to B$ is an arbitrary morphism. Let b be a realizer for g. We claim that b tracks a morphism from P to A. Indeed, suppose $a \Vdash_P x$ and $a' \Vdash_P x$. Then $a = a'$ because P is canonically separated. Because b tracks g, $b \cdot a$ is defined, and because I realizes f, $b \cdot a$ realizes some element of A. Therefore, b tracks some function $\overline{g}\colon P \to A$. It is obvious that $f \circ \overline{g} = g$ because f is realized by I while both g and $\overline{g}$ are realized by the same realizer b.

Next we prove that (1) implies (3). Suppose P is projective. Since the cover $q_P\colon P_0 \to P$ is a regular epi there exists a map $s\colon P \to P_0$ such that $q_P \circ s = 1_P$, hence q_P is a retraction.

Lastly, we prove that (3) implies (1). Suppose the canonical cover $q_S\colon S_0 \to S$ is a retraction, and let $s\colon S \to S_0$ be a section, i.e., $q_S \circ s = 1_S$. Let $f\colon A \twoheadrightarrow B$ be a regular epi and $g\colon S \to B$ a morphism. Since (2) implies (1) the canonical cover S_0 is projective. Therefore there exists a morphism $\overline{g}\colon S_0 \to A$ such that the following square commutes:

$$
\begin{tikzcd}
S_0 \arrow[r, "q_S"', shift right] \arrow[d, "\overline{g}"'] & S \arrow[l, dotted, "s"', shift right] \arrow[d, "g"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}
$$

The morphism $\overline{g} \circ s$ satisfies $f \circ (\overline{g} \circ s) = g$. ■

Proposition 1.3.5 *A dependent sum and a product of projective spaces is projective. A regular subobject of a projective space is projective.*

Proof. It suffices to prove the proposition for canonically separated modest sets. If A and B are canonically separated modest sets then their product is canonically separated because the existence predicate on $A \times B$ is

$$\mathsf{E}_{A\times B}\langle x, y\rangle = (\mathsf{E}_A x) \times (\mathsf{E}_B x) \ ,$$

and a cartesian product of two singleton sets $\mathsf{E}_A x$ and $\mathsf{E}_B x$ is a singleton. A similar argument works for dependent sums. A regular subobject B of a canonically separated modest set A is canonically separated because its existence predicate is just the restriction of E_A to B. ■

1.3.5 Factorization of Morphisms

Let $f\colon S \to T$ be an arbitrary morphism. As is well known, the set-theoretic function $f\colon |S| \to |T|$ can be decomposed into a quotient, bijection, and a canonical inclusion as follows. Let $\sim$ be the equivalence relation on $|S|$ defined by

$$x \sim y \iff fx = fy ,$$

and let $q\colon |S| \to |S|/{\sim}$ be the canonical quotient map. Let $|T'|$ be the image of f,

$$|T'| = \{fx \mid x \in |S|\} \subseteq |T| ,$$

and let $i\colon |T'| \to |T|$ be the subset inclusion. The function f factors as

$$|S| \xrightarrow{\;q\;} |S|/{\sim} \xrightarrow{\;f'\;} |T'| \xrightarrow{\;i\;} |T|$$

where f' is defined by $f'[x] = fx$. This factorization can be realized in modest sets. Let $S/{\sim}$ be the modest set whose underlying set is $|S|/{\sim}$ and the existence predicate is

$$\mathsf{E}_{S/\sim}[x] = \bigcup_{y \sim x} \mathsf{E}_S(y) .$$

Let T' be the modest set whose underlying set is $|T'|$ and the existence predicate is the restriction on E_T to $|T'|$. The quotient map $q\colon S \to S/{\sim}$ and the inclusion $i\colon T' \to T$ are both realized by I. The function f' has the same realizers as f, hence it is a morphism $f'\colon S/{\sim} \to T'$. We obtain a factorization

$$S \xrightarrow{\;q\;} S/{\sim} \xrightarrow{\;f'\;} T' \xrightarrow{\;i\;} T .$$

where q is a quotient map, f' is mono and epi, and i is an embedding. Note that even though f' is mono and epi, it need not be an isomorphism. This factorization is unique up to isomorphism. The embedding $i\colon T' \to T$ is called the *image of* f. The modest set T' is denoted by $\mathsf{im}(f)$.

1.3.6 Inductive and Coinductive Types

In this section we study *inductive types*, also known as well-founded types or W-types, which are a generalization of common inductive data types, such as natural numbers, binary trees, and finite lists. We also study *coinductive types*, which are the dual notion.

Inductive Types

Let $f\colon B \to A$ be a morphism. The *polynomial functor* P_f associated with f, maps a modest set C to

$$P_f C = \textstyle\sum_{x \in A} C^{f^* x}$$

and a morphism $g\colon C \to D$ to $P_f g$ with

$$(P_f g)\langle x, u\rangle = \langle x, g \circ u\rangle .$$

A P_f*-algebra* is a morphism $c\colon P_f C \to C$. A morphism of P_f-algebras is a map $g\colon C \to D$ such that the following diagram commutes:

$$\begin{tikzcd} P_f C \arrow[r, "c"] \arrow[d, "P_f g"'] & C \arrow[d, "g"] \\ P_f D \arrow[r, "d"] & D \end{tikzcd}$$

Algebras for P_f and morphisms between them form a category. An *inductive type* with signature f is an initial algebra for P_f. If it exists, it is denoted by W_f and its *structure map* is denoted by $\mathsf{w}_f\colon P_f\mathsf{W}_f \to \mathsf{W}_f$.

The initial algebra W_f is the initial object in the category of P_f-algebras and so it has the following universal property: for every algebra $c\colon P_f C \to C$ there exists a unique morphism $r\colon \mathsf{W}_f \to C$ such that the following diagram commutes:

$$\begin{tikzcd} P_f\mathsf{W}_f \arrow[r, "\mathsf{w}_f"] \arrow[d, "P_f r"'] & \mathsf{W}_f \arrow[d, "r"] \\ P_f C \arrow[r, "c"] & C \end{tikzcd}$$

We say that r is defined *by recursion* on W_f. By Lambek's Theorem for initial algebras, the structure map $\mathsf{w}_f\colon P_f\mathsf{W}_f \to \mathsf{W}_f$ of an initial algebra is an isomorphism, hence W_f satisfies the recursive equation

$$\mathsf{W}_f \cong P_f\mathsf{W}_f = \textstyle\sum_{x\in A}{\mathsf{W}_f}^{f^*x} \ .$$

Example 1.3.6 Let A be a modest set and define $f = \mathsf{inl} : A \to A + 1$. Then

$$P_f C = \textstyle\sum_{x\in A+1} C^{f^*x} = C^{f^*\star} + \sum_{x\in A} C^{\mathsf{inl}^*x} = C^0 + \sum_{x\in A} C^1 = 1 + A \times C \ .$$

For $g\colon C \to D$, we have $P_f g = 1_1 + 1_A \times g$. We denote the inductive type W_f, if it exists, by $\mathsf{List}(A)$. The structure map $\mathsf{w}_f\colon 1 + A \times \mathsf{List}(A) \to \mathsf{List}(A)$ can be seen as two maps $\mathsf{nil}\colon 1 \to \mathsf{List}(A)$ and $\mathsf{cons}\colon A \times \mathsf{List}(A) \to \mathsf{List}(A)$. The inductive type $\mathsf{List}(A)$ satisfies the equation[14]

$$\mathsf{List}(A) \cong 1 + A \times \mathsf{List}(A) \ .$$

Suppose $[c_0, c_1]\colon 1 + A \times C \to C$ is an algebra for P_f. Then the recursive definition of $h\colon \mathsf{List}(A) \to C$ from $[c_0, c_1]$ can be written in the familiar form

$$h\,\mathsf{nil} = c_0 \ , \qquad\qquad h(\mathsf{cons}\langle x, l\rangle) = c_1\langle x, hl\rangle \ .$$

[14]The analogy with definition of the list data type in a programming language, say in SML, is obvious:

$$\texttt{datatype}\ \alpha\ \texttt{list} = \texttt{nil} \mid \texttt{cons of}\ \alpha * \alpha\ \texttt{list} \ .$$

Example 1.3.7 Let $2 = 1 + 1 = \{0, 1\}$ and consider the map $f\colon 2 \to 2$ defined by $fx = 1$. The corresponding polynomial functor is

$$P_f C = \textstyle\sum_{x \in \{0,1\}} C^{f^* x} = C^0 + C^2 = 1 + C^2 \;.$$

The inductive type W_f, if it exists, is denoted by Tree. It is the type of finite binary trees and it satisfies the recursive equation

$$\mathsf{Tree} = 1 + \mathsf{Tree} \times \mathsf{Tree} \;.$$

Theorem 1.3.8 *In modest sets all inductive types exists.*

Proof. For the purposes of this proof we view modest sets as partial equivalence relations on $\mathbb{A}$. Recall that a partial equivalence relation (per) on $\mathbb{A}$ is a subset of $\mathbb{A} \times \mathbb{A}$ that is symmetric and transitive. Let $\mathsf{PER}(\mathbb{A})$ be the family of all pers on $\mathbb{A}$. The set $\mathsf{PER}(\mathbb{A})$ ordered by set-theoretic inclusion $\subseteq$ is a complete lattice because an arbitrary intersections of pers is a per. We denote the elements of $\mathsf{PER}(\mathbb{A})$ with capital letters R, P, Q, ..., but write them as $=_R$, $=_P$, $=_Q$ when they play the role of a per in an expression.

Let $f\colon B \to A$ be a morphism in $\mathsf{Mod}(\mathbb{A}, \mathbb{A}_\sharp)$ and let $f_0 \in \mathbb{A}_\sharp$ be a realizer for it. The polynomial functor P_f can be viewed as an operator $P_f\colon \mathsf{PER}(\mathbb{A}) \to \mathsf{PER}(\mathbb{A})$, characterized by

$$\begin{aligned}\langle a, c\rangle =_{P_f R} \langle a', c'\rangle \iff\;& a =_A a' \wedge \\ & \forall\, b, b' \in \mathbb{A} \,.\, \big(b =_B b' \wedge f_0 b =_A a \Longrightarrow cb{\downarrow} =_R c' b'{\downarrow}\big) \;.\end{aligned}$$

Strictly speaking, the above definition of P_f depends on the choice of the realizer f_0. However, we do not have to worry about this. We are going to construct a per and prove that it is the initial algebra for P_f. The construction depends on the choice of f_0, but as long as we show that the resulting per has the required universal property, this does not matter. A different choice of the realizer for f would give us an isomorphic copy of W_f because there is only one initial algebra, up to isomorphism.

The operator $P_f\colon \mathsf{PER}(\mathbb{A}) \to \mathsf{PER}(\mathbb{A})$ is monotone, which is easily checked. By Tarski's Fixed Point Theorem every monotone operator on a complete lattice has a least fixed point. So let W be the least fixed point of P_f. The per W is computed explicitly as the supremum of the increasing chain

$$\emptyset \subseteq P_f\emptyset \subseteq P_f^2\emptyset \subseteq P_f^3\emptyset \subseteq \cdots \subseteq \textstyle\bigcup_{n=0}^\infty P_f^n\emptyset \subseteq P_f\left(\bigcup_{n=0}^\infty P_f^n\emptyset\right) \subseteq P_f^2\left(\bigcup_{n=0}^\infty P_f^n\emptyset\right) \subseteq \cdots$$

In general, the chain may have to be iterated into large transfinite ordinal numbers.

We defined P_f on pers but we can equally well apply it to any binary relation on $\mathbb{A}$. If R is a binary relation on $\mathbb{A}$, let σR be its symmetric closure, and let τR be its transitive closure. It is not hard to check that P_f satisfies

$$\sigma(P_f R) \subseteq P_f(\sigma R) \;, \qquad\qquad \tau(P_f R) \subseteq P_f(\tau R) \;.$$

and thus also $\tau(\sigma(P_f R)) \subseteq P_f(\tau(\sigma R))$. This observation is useful when we have a per $R = \tau(\sigma R_0)$ which is the transitive symmetric closure of a binary relation R_0, and we want to show that $R \subseteq P_f R$, as it suffices to check that $R_0 \subseteq P_f R_0$.

Since $P_f W = W$ it is easy to get a candidate for the structure morphism on W—we simply take the identity $w = 1_W = [\mathsf{I}]\colon P_F W \to W$. It remains to show that $w\colon P_f W \to W$ has the desired universal property.

First we show uniqueness of morphisms defined by recursion on W. Suppose $[v]\colon P_f V \to V$ is an algebra for P_f and that $[s], [t]\colon W \to V$ are morphism of algebras. Let Q be the per defined as follows: $\langle a, c\rangle =_Q \langle a', c'\rangle$ if, and only if, the following four conditions hold:

(1) $\langle a, s \circ c\rangle =_{P_f V} \langle a', s \circ c'\rangle$,

(2) $\langle a, t \circ c\rangle =_{P_f V} \langle a', t \circ c'\rangle$,

(3) $\langle a, s \circ c\rangle =_{P_f V} \langle a', t \circ c'\rangle$,

(4) $\langle a', s \circ c'\rangle =_{P_f V} \langle a, t \circ c\rangle$.

Here we used the abbreviation $g \circ h = \lambda^* x.\, g(hx)$. If $\langle a, c\rangle =_Q \langle a', c'\rangle$ then by the first and the second condition

$$s\langle a, c\rangle =_V v\langle a, s \circ c\rangle =_V v\langle a', s \circ c'\rangle =_V s\langle a', c'\rangle\ ,$$
$$t\langle a, c\rangle =_V v\langle a, t \circ c\rangle =_V v\langle a', t \circ c'\rangle =_V t\langle a', c'\rangle\ ,$$

which means that t and s represent morphisms $Q \to V$. Similarly, using the third condition, it follows from $\langle a, c\rangle =_Q \langle a', c'\rangle$ that

$$s\langle a, c\rangle =_V v\langle a, s \circ c\rangle =_V v\langle a', t \circ c'\rangle =_V t\langle a', c'\rangle\ .$$

To show that $[t] =_{W \to V} [s]$, we demonstrate that $W \subseteq Q$ by proving that Q is a prefixed point of P_f. Suppose $\langle a, c\rangle =_{P_f Q} \langle a', c'\rangle$. Then $a =_A a'$, and for all $b, b' \in \mathbb{A}$ such that $b =_B b'$ and $f_0 \cdot b =_A a$ it is the case that $c \cdot b =_Q c' \cdot b'$. Because s and t represent the same morphism $Q \to V$ we get:

(1) $(s \circ c)b =_V s(cb) =_V s(c'b') =_V (s \circ c')b$,

(2) $(t \circ c)b =_V t(cb) =_V t(c'b') =_V (t \circ c')b$,

(3) $(s \circ c)b =_V s(cb) =_V t(c'b') =_V (t \circ c')b'$,

(4) $(s \circ c')b' =_V s(c'b') =_V t(cb) =_V (t \circ c)b$.

It now follows that $\langle a, c\rangle =_Q \langle a', c'\rangle$.

Lastly, we show the existence of morphisms defined by recursion. Let $[v]\colon P_f V \to V$ be an algebra for P_f. A morphism of algebras $[r]\colon W \to V$ must satisfy the recursive equation

$$r\langle a, c\rangle =_V v\langle a, r \circ c\rangle$$

for all $a, c \in \mathbb{A}$ such that $\langle a, c\rangle =_W \langle a, c\rangle$. We use the fixed point combinator Z to find such an r. Define

$$h = \lambda^* sp.\, v\langle \mathsf{fst}\, p, s \circ (\mathsf{snd}\, p)\rangle\ , \qquad\qquad r = \mathsf{Z}h\ .$$

Since $\mathsf{Z} \in \mathbb{A}_\sharp$ and $v \in \mathbb{A}_\sharp$, it is clear that $r \in \mathbb{A}_\sharp$. Because $(\mathsf{Z}h)z \simeq h(\mathsf{Z}h)z$ for all $z \in \mathbb{A}$, we get

$$r\langle a, c\rangle \simeq hr\langle a, c\rangle \simeq v\langle \mathsf{fst}\,\langle a, c\rangle, r \circ (\mathsf{snd}\,\langle a, c\rangle)\rangle \simeq v\langle a, r \circ c\rangle \ .$$

To establish that r represents an algebra morphism $[r]\colon W \to V$ we need to show that if $\langle a, c\rangle =_W \langle a', c'\rangle$ then $r\langle a, c\rangle =_V r\langle a', c'\rangle$. This is proved by a straightforward transfinite induction on the stage of the construction of W, at which $\langle a, c\rangle$ and $\langle a', c'\rangle$ enter into the least fixed point W. ∎

Coinductive Types

Coinductive types are the dual of inductive types. Let P_f be the polynomial functor associated with $f\colon B \to A$. A *P_f-coalgebra* is a morphism $c\colon C \to P_f C$. A P_f-coalgebra morphism is a morphism $g\colon C \to D$ such that the following diagram commutes:

$$\begin{tikzcd} C \arrow[r, "c"] \arrow[d, "g"'] & P_f C \arrow[d, "P_f g"] \\ D \arrow[r, "d"'] & P_f D \end{tikzcd}$$

Coalgebras for P_f and morphisms between them form a category. A *coinductive type* with signature f is a final coalgebra for P_f. If it exists, it is denoted by M_f and its *structure map* is denotes by $\mathsf{m}_f\colon \mathsf{M}_f \to P_f\mathsf{M}_f$.

The final coalgebra M_f is the terminal object in the category of P_f-coalgebras and so it has the following universal property: for every coalgebra $c\colon C \to P_f C$ there exists a unique morphism $r\colon C \to \mathsf{M}_f$ such that the following diagram commutes:

$$\begin{tikzcd} C \arrow[r, "c"] \arrow[d, "r"'] & P_f C \arrow[d, "P_f r"] \\ \mathsf{M}_f \arrow[r, "\mathsf{m}_f"'] & P_f \mathsf{M}_f \end{tikzcd}$$

We say that r is defined *by corecursion* on M_f. By Lambek's Theorem for final coalgebras, the structure map $\mathsf{m}_f\colon \mathsf{M}_f \to P_f\mathsf{M}_f$ of a final coalgebra is an isomorphism, hence M_f satisfies the corecursive equation

$$\mathsf{M}_f \cong P_f\mathsf{M}_f = \textstyle\sum_{x \in A} {\mathsf{M}_f}^{f^* x} \ .$$

Example 1.3.9 Consider the morphism $f = 1_A\colon A \to A$. The corresponding polynomial functor P_f is

$$P_f C = \textstyle\sum_{x \in A} C^{1_A^* x} = \sum_{x \in A} C = A \times C \ .$$

The coinductive type M_{1_A}, if it exists, is denoted by Stream_A. It is the modest set of infinite streams on A, and it satisfies the recursive equation

$$\mathsf{Stream}_A \cong A \times \mathsf{Stream}_A \ .$$

The structure map for Stream_A is a pair of morphisms $\mathsf{hd}\colon \mathsf{Stream}_A \to A$ and $\mathsf{tl}\colon \mathsf{Stream}_A \to \mathsf{Stream}_A$.

Theorem 1.3.10 *In modest sets all coinductive types exists.*

Proof. The proof goes much along the lines of the proof of Theorem 1.3.8, except that we take the *greatest* fixed point of the operator P_f, instead of the least one.

We adopt the same setup and notation as in the proof of Theorem 1.3.8. Let M be the greatest fixed point of P_f. As a structure map we take the identity morphism $1_M = [\mathsf{I}]\colon M \to P_f M$. Let us prove that $1_M\colon M \to M$ has the desired universal property.

In order to prove uniqueness of morphisms defined by corecursion, assume we have a coalgebra $[n]\colon N \to P_f N$ and coalgebra morphisms $[s], [t]\colon N \to M$. Let R_0 be the relation on $\mathbb{A}$ defined by

$$\begin{gathered}\langle a, u\rangle =_{R_0} \langle a', u'\rangle \\ \text{if and only if} \\ \exists\, x, x' \in \mathbb{A}\,.\,\bigl(x =_N x' \wedge \langle a, u\rangle =_M sx \wedge \langle a', u'\rangle \approx_M tx\bigr)\ ,\end{gathered}$$

and let R be the least per that contains R_0, i.e., R is the transitive closure of the symmetric closure of R_0. We show that R is a postfixed point of P_f, that is $R \subseteq P_f R$, from which it follows that $R \subseteq M$ because M is the greatest postfixed point of P_f. Then $[s] = [t]$ holds because R is defined so that $x =_N x'$ implies $sx =_R tx'$.

In order to show that $R \subseteq P_f R$, we only need to check that $R_0 \subseteq P_f R_0$. Suppose that for some $x, x' \in \mathbb{A}$ it is the case that $x =_N x'$, $\langle a, u\rangle =_M sx$ and $\langle a', u'\rangle =_M tx$. Taking into account that $[s]$ and $[t]$ are P_f-coalgebra morphisms, we see that

$$\begin{aligned}\langle a, u\rangle &=_M sx =_M \langle n_1 x, s \circ (n_2 x)\rangle\ ,\\ \langle a', u'\rangle &=_M tx' =_M \langle n_1 x', t \circ (n_2 x')\rangle\ ,\end{aligned}$$

where $n_1 = \mathsf{fst} \circ n$ and $n_2 = \mathsf{snd} \circ n$. Since M is a fixed point of P_f, it follows that

$$a =_A n_1 x =_A n_1 x' =_A a'\ . \tag{1.6}$$

Also, if $b =_B b'$, and $f_0 b =_A a$ then $n_2 x b =_N n_2 x' b'$, $ub =_M s(n_2 x b)$, and $u'b' =_M t(n_2 x' b')$. By definition of R_0, $ub =_{R_0} u'b'$ which together with (1.6) gives $\langle a, u\rangle =_{P_f R_0} \langle a', u'\rangle$ as required.

Lastly, we show the existence of morphisms defined by corecursion in much the same way as we showed existence of morphisms defined by recursion. Let $[n]\colon N \to P_f N$ be a P_f-coalgebra. A morphism of coalgebras $[r]\colon N \to M$ must satisfy the corecursive equation, for all $c \in N$,

$$rc =_M \langle n_1 c, r \circ (n_2 c)\rangle\ .$$

We use the fixed point combinator Z to find such an r. Let

$$h = \lambda^* sc.\, \langle n_1 c, s \circ (n_2 c)\rangle\ , \qquad r = \mathsf{Z}h\ .$$

Since $\mathsf{Z} \in \mathbb{A}_\sharp$ and $n \in \mathbb{A}_\sharp$, it is clear that $r \in \mathbb{A}_\sharp$. Because $(\mathsf{Z}h)z = h(\mathsf{Z}h)z$ for all $z \in \mathbb{A}$, we get

$$rc \simeq hrc \simeq \langle n_1 c, r \circ (n_2 c)\rangle\!\downarrow .$$

Therefore, r represents a coalgebra morphism $[r]\colon N \to M$. ∎

1.3.7 The Computability Operator

We can formalize the idea that $\mathbb{A}_\sharp$ represents the computable part of $\mathbb{A}$ by defining a functor $\#\colon \mathsf{Mod}(\mathbb{A}, \mathbb{A}_\sharp) \to \mathsf{Mod}(\mathbb{A}, \mathbb{A}_\sharp)$, called the *computability operator*, or the *operator "sharp"*. If S is a modest set, let $\#S$ be the modest set whose underlying set is

$$\#S = \{x \in S \mid (\mathsf{E}_S x) \cap \mathbb{A}_\sharp \neq \emptyset\}$$

and the existence predicate is $\mathsf{E}_{\#S} x = (\mathsf{E}_S x) \cap \mathbb{A}_\sharp$. The modest set $\#S$ is called the *computable part of* S, since its elements are those elements of S that have computable realizers. The functor $\#$ acts trivially on morphisms. If $f\colon S \to T$ is a map realized by $a \in \mathbb{A}_\sharp$ then let $\#f = f\colon \#S \to \#T$. The map $\#f$ is realized by a because if $b \Vdash_{\#S} x$ then $b \in \mathbb{A}_\sharp$, hence $ab\downarrow \in \mathbb{A}_\sharp$ and $ab \Vdash_T fx$.

The computable part $\#S$ is a subobject of S via the subset inclusion $i_S\colon |\#S| \subseteq |S|$ which is realized by the identity combinator I.

Because intersecting with $\mathbb{A}_\sharp$ is an idempotent operation $\#$ is an idempotent functor. Therefore, $\#$ is a comonad whose comultiplication is the identity natural transformation $1\colon \# \Longrightarrow \# \circ \#$, and the counit is the canonical subspace inclusion $i_\square\colon 1 \Longrightarrow \#$.

1.4 Applicative Morphisms

In Chapter 4 we are going to compare categories of modest sets on various PCAs. This is done most easily by using John Longley's theory of *applicative morphisms* between PCAs. We extend his definition to applicative morphisms between PCAs with subPCAs. We also recall the basic results about applicative morphisms and the induced functors between categories of modest sets. See [Lon94, Chapter 2] for further material on this topic. Let us first state the original definition.

Definition 1.4.1 (John Longley) [Lon94, Definition 2.1.1] Let $\mathbb{E}$ and $\mathbb{F}$ be PCAs. An *applicative morphism* $\rho\colon \mathbb{E} \xrightarrow{\mathsf{PCA}} \mathbb{F}$ is a total[15] relation $\rho \subseteq \mathbb{E} \times \mathbb{F}$ for which there exists $r \in \mathbb{F}$ such that, for all $u, v \in \mathbb{E}$, $x, y \in \mathbb{F}$, (a) if $\rho(u, x)$ then $r \cdot x\downarrow$, and (b) if $\rho(u, x)$ and $\rho(u, y)$ and $u \cdot v\downarrow$ then $r \cdot x \cdot y\downarrow$ and $\rho(u \cdot v, r \cdot x \cdot y)$.

We write $\rho(u) = \{x \in \mathbb{F} \mid \rho(u, x)\}$. We say that r *realizes* the morphism ρ. When ρ is an applicative morphism we write $\rho\colon \mathbb{E} \xrightarrow{\mathsf{PCA}} \mathbb{F}$.

One might expect that a morphism of PCAs ought to be a map $f\colon \mathbb{E} \to \mathbb{F}$ such that $f\mathsf{K}_\mathbb{E} = \mathsf{K}_\mathbb{F}$, $f\mathsf{S}_\mathbb{F} = \mathsf{S}_\mathbb{F}$, and $f(x \cdot y) \simeq (fx) \cdot (fy)$. This is how an algebraist would define a morphism of PCAs. However, we are interested in computational aspects of PCAs, not the algebraic ones. An applicative morphism $\mathbb{E} \xrightarrow{\mathsf{PCA}} \mathbb{F}$ is best viewed as an *implementation* of $\mathbb{E}$ within $\mathbb{F}$. Then $\rho(u, x)$ is understood as "x is a ρ-implementation of u". The realizer $r \in \mathbb{F}$ in the above definition implements the application of $\mathbb{E}$ in $\mathbb{F}$. An applicative morphism is allowed to be a relation rather than a function because there might be many implementations of $u \in \mathbb{E}$, and there might be no canonical way of choosing one of them.

For the purposes of studying computability, we need a notion of applicative morphisms between PCAs *with subPCAs*. We extend Longley's notion of applicative morphisms as follows.

[15] A relation ρ is total when $\forall\, x \,.\, \exists\, y \,.\, \rho(x, y)$.

Definition 1.4.2 Let $\mathbb{E}_\sharp \subseteq \mathbb{E}$ and $\mathbb{F}_\sharp \subseteq \mathbb{F}$ be PCAs with subPCAs. An *applicative morphism* $\rho\colon (\mathbb{E}, \mathbb{E}_\sharp) \xrightarrow{\mathsf{PCA}} (\mathbb{F}, \mathbb{F}_\sharp)$ is a total relation $\rho \subseteq \mathbb{E} \times \mathbb{F}$ such that: (a) for all $u \in \mathbb{E}_\sharp$ there exists $x \in \mathbb{F}_\sharp$ such that $\rho(u, x)$, and (b) there exists $r \in \mathbb{F}_\sharp$ such that, for all $u, v \in \mathbb{E}$, $x, y \in \mathbb{F}$, if $\rho(u, x)$ then $r \cdot x\downarrow$, and if $\rho(u, x)$, $\rho(u, y)$ and $u \cdot v\downarrow$ then $\rho(u \cdot v, r \cdot x \cdot y\downarrow)$.

If $\rho\colon (\mathbb{E}, \mathbb{E}_\sharp) \xrightarrow{\mathsf{PCA}} (\mathbb{F}, \mathbb{F}_\sharp)$ is an applicative morphism, then ρ is an applicative morphism from $\mathbb{E}$ to FF in the sense of Definition 1.4.1, and also the restriction of ρ to $\mathbb{E}_\sharp \times \mathbb{E}_\sharp$ is an applicative morphism from $\mathbb{E}_\sharp$ to $\mathbb{F}_\sharp$ in the sense of Definition 1.4.1.

Applicative morphisms can be composed in the usual way as relations. If $\rho\colon (\mathbb{E}, \mathbb{E}_\sharp) \xrightarrow{\mathsf{PCA}} (\mathbb{F}, \mathbb{F}_\sharp)$ and $\sigma\colon (\mathbb{F}, \mathbb{F}_\sharp) \xrightarrow{\mathsf{PCA}} (\mathbb{G}, \mathbb{G}_\sharp)$ then $\sigma \circ \rho\colon (\mathbb{E}, \mathbb{E}_\sharp) \xrightarrow{\mathsf{PCA}} (\mathbb{G}, \mathbb{G}_\sharp)$ is defined, for $u \in \mathbb{E}$, $s \in \mathbb{G}$, by

$$\sigma \circ \rho(u, s) \iff \exists\, x \in \mathbb{F}\,.\,(\rho(u, x) \wedge \sigma(x, s)) \ .$$

The identity applicative morphism $1_{(\mathbb{E},\mathbb{E}_\sharp)}\colon (\mathbb{E}, \mathbb{E}_\sharp) \to (\mathbb{E}, \mathbb{E}_\sharp)$ is the identity relation on $\mathbb{E}$.

If we take PCAs with subPCAs as objects and applicative morphisms as morphisms, we obtain a category that can be preorder-enriched as follows [Lon94, Proposition 2.1.6]. There is a preorder $\preceq$ on applicative morphisms $(\mathbb{E}, \mathbb{E}_\sharp) \xrightarrow{\mathsf{PCA}} (\mathbb{F}, \mathbb{F}_\sharp)$. If ρ and σ are such applicative morphisms, we define $\rho \preceq \sigma$ to hold if, and only if, there exists $t \in \mathbb{F}_\sharp$ such that, for all $u \in \mathbb{E}$, $x \in \mathbb{F}$, $\rho(u, x)$ implies $\sigma(u, t \cdot x\downarrow)$. We think of t as a *translation* of ρ-implementations into σ-implementations. We write $\rho \sim \sigma$ when $\rho \preceq \sigma$ and $\sigma \preceq \rho$. We say that $(\mathbb{E}, \mathbb{E}_\sharp)$ and $(\mathbb{F}, \mathbb{F}_\sharp)$ are *equivalent* when there exist applicative morphisms

$$\delta\colon (\mathbb{E}, \mathbb{E}_\sharp) \xrightarrow{\mathsf{PCA}} (\mathbb{F}, \mathbb{F}_\sharp) \ , \qquad\qquad \gamma\colon (\mathbb{F}, \mathbb{F}_\sharp) \xrightarrow{\mathsf{PCA}} (\mathbb{E}, \mathbb{E}_\sharp) \ ,$$

such that $\gamma \circ \delta \sim 1_{\mathbb{E}}$ and $\delta \circ \gamma \sim 1_{\mathbb{F}}$.

Definition 1.4.3 Let $\rho\colon (\mathbb{E}, \mathbb{E}_\sharp) \to (\mathbb{F}, \mathbb{F}_\sharp)$ be an applicative morphism:

(1) ρ is *discrete* when, for all $u, v \in \mathbb{E}$, $x \in \mathbb{F}$, if $\rho(u, x)$ and $\rho(v, x)$ then $u = v$.

(2) ρ is *projective* when there is a single-valued applicative morphism[16] ρ' such that $\rho' \sim \rho$.

(3) ρ is *decidable* when there is $d \in \mathbb{F}_\sharp$, called the *decider* for ρ, such that, for all $x \in \mathbb{F}$,

$$\rho(\mathsf{true}_{\mathbb{E}}, x) \Longrightarrow d \cdot x = \mathsf{true}_{\mathbb{F}} \ , \qquad\qquad \rho(\mathsf{false}_{\mathbb{E}}, x) \Longrightarrow d \cdot x = \mathsf{false}_{\mathbb{F}} \ .$$

Proposition 1.4.4 *A discrete applicative morphism $\rho\colon (\mathbb{E}, \mathbb{E}_\sharp) \xrightarrow{\mathsf{PCA}} (\mathbb{F}, \mathbb{F}_\sharp)$ induces a functor*

$$\widehat{\rho}\colon \mathsf{Mod}(\mathbb{E}, \mathbb{E}_\sharp) \longrightarrow \mathsf{Mod}(\mathbb{F}, \mathbb{F}_\sharp) \ ,$$

defined as follows. For $S \in \mathsf{Mod}(\mathbb{E}, \mathbb{E}_\sharp)$, let

$$|\widehat{\rho}S| = |S| \ , \qquad\qquad \mathsf{E}_{\widehat{\rho}S} t = \bigcup_{u \in \mathsf{E}_S t} \rho(u) \ .$$

A realized function $f\colon S \to T$ is mapped to $\widehat{\rho} f = f\colon |\widehat{\rho}S| \to |\widehat{\rho}T|$.

[16] In other words, ρ' is a function.

Proof. Because ρ is discrete $\widehat{\rho}S$ is indeed a modest set. Let r be a realizer for ρ. The function $\widehat{\rho}f : \widehat{\rho}S \to \widehat{\rho}T$ is realized because if $u \Vdash_{S \to T} f$ then there exists $x \in \mathbb{F}_\sharp$ such that $\rho(u, x)$, and $r \cdot x \Vdash_{\widehat{\rho}S \to \widehat{\rho}T} f$. ∎

All applicative morphism that we are going to consider are discrete.

Proposition 1.4.5 *Let* $\gamma\colon (\mathbb{E}, \mathbb{E}_\sharp) \xrightarrow{\mathsf{PCA}} (\mathbb{F}, \mathbb{F}_\sharp)$ *be a discrete applicative morphism.*

(1) The induced functor $\widehat{\gamma}\colon \mathsf{Mod}(\mathbb{E}, \mathbb{E}_\sharp) \to \mathsf{Mod}(\mathbb{F}, \mathbb{F}_\sharp)$ *is faithful, and it preserves finite limits, and coequalizers of kernel pairs.*

(2) If γ *is projective then* $\widehat{\gamma}$ *preserves projective objects.*

(3) If γ *is decidable then* $\widehat{\gamma}$ *preserves finite colimits and the natural numbers object.*

Proof. (1) $\widehat{\gamma}$ preserves finite limits by [Lon94, Proposition 2.2.2], and coequalizers of kernel pairs by [Lon94, Proposition 2.2.3]. It is faithful because it acts trivially on morphisms. (2) See [Lon94, Theorem 2.4.12]. (3) See [Lon94, Theorem 2.4.19]. ∎

Definition 1.4.6 [Lon94, Definition 2.5.1] An *adjoint pair* of applicative morphisms

$$(\gamma \dashv \delta)\colon (\mathbb{E}, \mathbb{E}_\sharp) \xrightarrow{\mathsf{PCA}} (\mathbb{F}, \mathbb{F}_\sharp)$$

consists of a pair of applicative morphisms

$$\delta\colon (\mathbb{E}, \mathbb{E}_\sharp) \xrightarrow{\mathsf{PCA}} (\mathbb{F}, \mathbb{F}_\sharp)\ , \qquad \gamma\colon (\mathbb{F}, \mathbb{F}_\sharp) \xrightarrow{\mathsf{PCA}} (\mathbb{E}, \mathbb{E}_\sharp)\ ,$$

such that $1_{\mathbb{F}} \preceq \delta \circ \gamma$ and $\gamma \circ \delta \preceq 1_{\mathbb{E}}$. We say that γ is *left adjoint* to δ, or that δ is *right adjoint* to γ.

Definition 1.4.7 [Lon94, Definition 2.5.2] Suppose $(\gamma \dashv \delta)\colon (\mathbb{E}, \mathbb{E}_\sharp) \xrightarrow{\mathsf{PCA}} (\mathbb{F}, \mathbb{F}_\sharp)$ is an adjoint pair. We say that (γ, δ) is an *applicative inclusion* when $\gamma \circ \delta \sim 1_{\mathbb{E}}$, and an *applicative retraction* when $\delta \circ \gamma \sim 1_{\mathbb{F}}$.

Theorem 1.4.8 [Lon94, Theorem 2.5.3] *Suppose* $\delta\colon (\mathbb{E}, \mathbb{E}_\sharp) \xrightarrow{\mathsf{PCA}} (\mathbb{F}, \mathbb{F}_\sharp)$ *and* $\gamma\colon (\mathbb{F}, \mathbb{F}_\sharp) \xrightarrow{\mathsf{PCA}} (\mathbb{E}, \mathbb{E}_\sharp)$ *are applicative morphisms.*

(1) If $\gamma \circ \delta \preceq 1_{\mathbb{E}}$ *then* δ *is discrete and* γ *is decidable.*

(2) If $\gamma \dashv \delta$ *then* γ *is also projective.*

Proof. The proof of [Lon94, Theorem 2.5.3] is stated for applicative morphisms, as defined by Definition 1.4.1, but it works just as well with Definition 1.4.2. ∎

Corollary 1.4.9 *If* $(\gamma \dashv \delta)$ *is a retraction then both* δ *and* γ *are discrete and decidable, and* γ *is projective.*

Proof. Immediate. This is [Lon94, Corollary 2.5.4]. ∎

Corollary 1.4.10 *If $(\mathbb{E}, \mathbb{E}_\sharp)$ and $(\mathbb{F}, \mathbb{F}_\sharp)$ are equivalent PCAs, then the there exist an equivalence*

$$\delta\colon (\mathbb{E}, \mathbb{E}_\sharp) \xrightarrow{\mathsf{PCA}} (\mathbb{F}, \mathbb{F}_\sharp)\ , \qquad\qquad \gamma\colon (\mathbb{F}, \mathbb{F}_\sharp) \xrightarrow{\mathsf{PCA}} (\mathbb{E}, \mathbb{E}_\sharp)\ ,$$

such that γ and δ are single-valued.

Proof. Both δ and γ are projective by Theorem 1.4.8. ∎

Theorem 1.4.11 *If $(\mathbb{E}, \mathbb{E}_\sharp)$ and $(\mathbb{F}, \mathbb{F}_\sharp)$ are equivalent via discrete applicative morphisms*

$$\delta\colon (\mathbb{E}, \mathbb{E}_\sharp) \xrightarrow{\mathsf{PCA}} (\mathbb{F}, \mathbb{F}_\sharp)\ , \qquad\qquad \gamma\colon (\mathbb{F}, \mathbb{F}_\sharp) \xrightarrow{\mathsf{PCA}} (\mathbb{E}, \mathbb{E}_\sharp)\ ,$$

then $\mathsf{Mod}(\mathbb{E}, \mathbb{E}_\sharp)$ and $\mathsf{Mod}(\mathbb{F}, \mathbb{F}_\sharp)$ are equivalent via the induced functors

$$\begin{tikzcd} \mathsf{Mod}(\mathbb{E}, \mathbb{E}_\sharp) \arrow[r, shift left, "\widehat{\delta}"] & \mathsf{Mod}(\mathbb{F}, \mathbb{F}_\sharp) \arrow[l, shift left, "\widehat{\gamma}"] \end{tikzcd}$$

Proof. By Corollary 1.4.10, there exist functions $\gamma\colon \mathbb{E} \to \mathbb{F}$ and $\delta\colon \mathbb{F} \to \mathbb{E}$ which are applicative morphisms. It is straightforward to check that $\widehat{\gamma}$ and $\widehat{\delta}$ constitute an equivalence of $\mathsf{Mod}(\mathbb{E}, \mathbb{E}_\sharp)$ and $\mathsf{Mod}(\mathbb{F}, \mathbb{F}_\sharp)$. See also [Lon94, Theorem 2.5.6]. ∎

Theorem 1.4.12 *Suppose we have discrete applicative morphisms*

$$\delta\colon (\mathbb{E}, \mathbb{E}_\sharp) \xrightarrow{\mathsf{PCA}} (\mathbb{F}, \mathbb{F}_\sharp)\ , \qquad\qquad \gamma\colon (\mathbb{F}, \mathbb{F}_\sharp) \xrightarrow{\mathsf{PCA}} (\mathbb{E}, \mathbb{E}_\sharp)\ .$$

(1) If $\gamma \dashv \delta$ is an adjoint pair, then $\widehat{\gamma} \dashv \widehat{\delta}$ is an adjunction. In addition, $\widehat{\gamma}$ preserves finite limits, and $\widehat{\delta}$ preserves regular epis.

(2) If $\gamma \dashv \delta$ is an inclusion then the counit $\widehat{\gamma} \circ \widehat{\delta} \Longrightarrow 1_{\mathsf{Mod}(\mathbb{E}, \mathbb{E}_\sharp)}$ is a natural isomorphism.

(3) If $\gamma \dashv \delta$ is a retraction then the unit $1_{\mathsf{Mod}(\mathbb{F}, \mathbb{F}_\sharp)} \Longrightarrow \widehat{\delta} \circ \widehat{\gamma}$ is a natural isomorphism.

Proof. This is the easy part of [Lon94, Proposition 2.5.9], except that we are using the extended notion of applicative morphism. Also, we are restricting attention to categories of modest sets, whereas [Lon94, Proposition 2.5.9] is stated for functors between realizability toposes. This is not a problem because by Corollary 1.4.9 the applicative morphisms γ and δ are discrete, therefore the induced functors between toposes restrict to functors between modest sets. ∎

Proposition 1.4.13 [Lon94, Proposition 2.5.11] *For discrete applicative morphisms $\gamma\colon \mathbb{E} \xrightarrow{\mathsf{PCA}} \mathbb{F}$ and $\delta\colon \mathbb{F} \xrightarrow{\mathsf{PCA}} \mathbb{E}$:*

(1) If $(\gamma \vdash \delta)$ is an applicative inclusion then $\widehat{\delta}$ is full and faithful and preserves exponentials.

(2) If $(\gamma \vdash \delta)$ is an applicative retraction then $\widehat{\delta}$ preserves finite colimits, and $\widehat{\gamma}$ reflects isomorphisms.

Proof. See [Lon94, Proposition 2.5.11]. ∎

Let us now look at examples of applicative morphisms between PCAs.

1.4.1 Applicative Adjunction between $\mathbb{N}$ and $\mathbb{P}_\sharp$

There is an applicative retraction $(\delta \dashv \gamma)\colon \mathbb{P}_\sharp \xrightarrow{\mathsf{PCA}} \mathbb{N}$. The inclusion $\delta\colon \mathbb{N} \to \mathbb{P}_\sharp$ is defined by

$$\delta n = \{n\} \ ,$$

and the retraction $\gamma\colon \mathbb{P}_\sharp \xrightarrow{\mathsf{PCA}} \mathbb{N}$ is defined by

$$\gamma(U, n) \iff U = \mathsf{im}(\varphi_n) \ ,$$

where $\varphi_\Box$ is a standard enumeration of partial recursive functions. See [Lon94, Proposition 3.3.7] for details.

1.4.2 Applicative Retraction from $(\mathbb{P}, \mathbb{P}_\sharp)$ to $(\mathbb{B}, \mathbb{B}_\sharp)$

Lietz [Lie99] compared realizability models over $\mathbb{P}$ and over $\mathbb{B}$ and observed that there is an applicative retraction $(\iota \dashv \delta)\colon PP \xrightarrow{\mathsf{PCA}} BB$. In this subsection we describe it explicitly and show that it is in fact a retraction from $(\mathbb{P}, \mathbb{P}_\sharp)$ to $(\mathbb{B}, \mathbb{B}_\sharp)$,

$$(\iota \dashv \delta)\colon (\mathbb{P}, \mathbb{P}_\sharp) \xrightarrow{\mathsf{PCA}} (\mathbb{B}, \mathbb{B}_\sharp) \ .$$

Given a finite sequence of natural numbers $a = [a_0, \dots, a_{k-1}]$, let $\mathsf{seq}\, a$ be the encoding of a as a natural number, as defined in Section 1.1.6. Define the embedding $\iota\colon \mathbb{B} \to \mathbb{P}$ by

$$\iota\alpha = \big\{\mathsf{seq}\, a \ \big|\ a \in \mathbb{N}^* \wedge a \sqsubseteq \alpha\big\} \ .$$

Observe that if $\alpha \in \mathbb{B}_\sharp$ then $\iota\alpha \in \mathbb{P}_\sharp$. Let $\mathbb{B}' = \mathsf{im}(\iota)$ and define $p\colon \mathbb{B}' \times \mathbb{B}' \to \mathbb{P}$ by

$$p\langle \iota\alpha, \iota\beta\rangle = \begin{cases} \iota(\alpha \mid \beta) & \text{if } \alpha \mid \beta \text{ defined} \\ \emptyset & \text{otherwise .} \end{cases}$$

The map p is continuous and it can be extended to an r.e. enumeration operator $p\colon \mathbb{P} \times \mathbb{P} \to \mathbb{P}$. Thus, p realizes ι, which is therefore an applicative morphism.

Let $\delta\colon \mathbb{P} \xrightarrow{\mathsf{PCA}} \mathbb{B}$ be the applicative morphism defined, for $x \in \mathbb{P}$, $\alpha \in \mathbb{B}$, by

$$\delta(x, \alpha) \iff x = \big\{n \in \mathbb{N} \ \big|\ \exists\, k \in \mathbb{N}\,.\, \alpha k = n + 1\big\} \ .$$

In words, α is a δ-implementation of x when it enumerates x. We added 1 to n in the above definition so that the empty set is enumerated as well. Clearly, if $\alpha \in \mathbb{B}_\sharp$ then $x \in \mathbb{P}_\sharp$. In order for δ to be an applicative morphism, it must have a realizer $\rho \in \mathbb{B}_\sharp$ such that

$$\delta(x, \alpha) \wedge \delta(y, \beta) \Longrightarrow \delta(x \cdot y, \rho \mid \alpha \mid \beta) \ .$$

Equivalently, we may require that $\delta(x \cdot y, \rho \mid \langle \alpha, \beta\rangle)$. Such a ρ can be obtained as follows. To determine the value $(\rho \,|\, \langle \alpha, \beta\rangle)(\langle\langle\rangle m, n\rangle)$, let $A = \{\alpha_0, \dots, \alpha_{m-1}\}$ and $B = \{\beta_0, \dots, \beta_{m-1}\}$. If there exists $k \in B$ such that $k = 1 + \langle\langle\rangle n, j\rangle$ and $\mathsf{finset}\, j \subseteq A$ then the value is $n+1$, otherwise it is 0.

Clearly, this is an effective procedure, therefore it is continuous and realized by an element $\rho \in \mathbb{B}_\sharp$. If we compare the definition of ρ to the definition of application in $\mathbb{P}$, we see that they match.

Let us show that $\iota \dashv \delta$ is an applicative retraction. Suppose $\alpha \in \mathbb{B}$, $x = \iota(\alpha)$, and $\delta(x, \beta)$. We can effectively reconstruct α from β, because β enumerates the initial segments of α. This shows that $\delta \circ \iota \preceq 1_{\mathbb{B}}$. Also, given α we can easily construct a sequence β which enumerates the initial segments of α, therefore $1_{\mathbb{B}} \preceq \delta \circ \iota$, and we conclude that $\delta \circ \iota \sim 1_{\mathbb{B}}$.

To see that $\iota \circ \delta \preceq 1_{\mathbb{P}}$, consider $x, y \in \mathbb{P}$ and $\alpha \in \mathbb{B}$ such that $\delta(x, \alpha)$ and $y = \iota(\alpha)$. The sequence α enumerates x, and y consists of the initial segments of α. Hence, we can effectively reconstruct x from y, by

$$m \in x \iff \exists\, n \in y \,.\, (n = 1 + \mathsf{seq}\, a \wedge \exists\, i < |a| \,.\, m = a_i) \ .$$

1.4.3 Applicative Inclusion from $(\mathbb{P}, \mathbb{P}_\sharp)$ to $(\mathbb{U}, \mathbb{U}_\sharp)$

We construct an applicative inclusion

$$(\eta \dashv \zeta) \colon (\mathbb{P}, \mathbb{P}_\sharp) \xrightarrow{\mathsf{PCA}} (\mathbb{U}, \mathbb{U}_\sharp) \ ,$$

with the additional property that η is discrete. The applicative morphism ζ is discrete by Theorem 1.4.8(1). By the Effective Embedding Theorem 4.1.12 there exists a computable embedding $\eta \colon \mathbb{U} \to \mathbb{P}$. Specifically, η is defined by

$$\eta S = \left\{ n \in \mathbb{N} \;\middle|\; B_n \subseteq S \right\} \ ,$$

where $\langle B_n \rangle_{n \in \mathbb{N}}$ is an effective enumeration of the compact elements of $\mathbb{U}$. For every $n \in \mathbb{N}$, let C_n be the clopen set

$$C_n = \left\{ \alpha \in \mathbb{N} \;\middle|\; \alpha 0 = \cdots = \alpha n = 0 \wedge \alpha(n+1) = 1 \right\} \ .$$

Note that whenever $n \neq m$ then C_n and C_m are disjoint. The family $\left\{ C_n \;\middle|\; n \in \mathbb{N} \right\}$ is a discrete subspace of $\mathbb{U}$. Because $\mathbb{U}$ is an effective universal domain and $\mathbb{P}$ is an effective domain, there exists a computable embedding-projection pair $(\zeta, \zeta^-) \colon \mathbb{U} \to \mathbb{P}$. In particular, we define $\zeta \colon \mathbb{P} \to \mathbb{U}$ and $\zeta^- \colon \mathbb{U} \to \mathbb{P}$ by

$$\zeta x = \bigcup_{n \in x} C_n \ , \qquad\qquad \zeta^- S = \left\{ n \in \mathbb{N} \;\middle|\; C_n \subseteq S \right\} \ .$$

It is obvious that $\zeta^- \circ \zeta = 1_{\mathbb{P}}$. Let us verify that η is an applicative morphism. Because it is computable, it is the case that $\eta x \in \mathbb{P}_\sharp$ whenever $x \in \mathbb{U}_\sharp$. The application on $\mathbb{U}$ is a computable map, therefore by the Effective Extension Theorem 4.1.13 there exists a computable map $\phi \colon \mathbb{P} \times \mathbb{P} \to \mathbb{P}$ such that, for all $S, T \in \mathbb{U}$,

$$\eta(S \cdot T) = \phi\langle \eta S, \eta T \rangle \ .$$

There exists $f \in \mathbb{P}_\sharp$ such that $\phi\langle \eta S, \eta T \rangle = f \cdot (\eta S) \cdot (\eta T)$ for all $S, T \in \mathbb{U}$. Therefore, η is an applicative morphism. We show next that ζ is an applicative morphism. Since it is computable, $\zeta a \in \mathbb{U}_\sharp$ whenever $a \in \mathbb{U}_\sharp$. Let $\gamma \colon \mathbb{U} \times \mathbb{U} \to \mathbb{U}$ be defined by

$$\gamma\langle S, T \rangle = \zeta((\zeta^- S) \cdot (\zeta^- T)) \ .$$

Because γ is a composition of computable maps, it is computable. There exists $g \in \mathbb{U}_\sharp$ such that $\gamma\langle S, T\rangle = g \cdot S \cdot T$ for all $S, T \in \mathbb{U}$. For all $x, y \in \mathbb{P}$,

$$\zeta(x \cdot y) = \zeta((\zeta^-(\zeta x)) \cdot (\zeta^-(\zeta x))) = \gamma\langle \zeta x, \zeta y\rangle = g \cdot (\zeta x) \cdot (\zeta y) \ .$$

Therefore, ζ is an applicative morphism.

The relations $\eta \circ \zeta \preceq 1_\mathbb{P}$ and $\zeta \circ \eta \preceq 1_\mathbb{U}$ hold because $\eta \circ \zeta$ and $\zeta \circ \eta$ are computable and realized in $\mathbb{P}$ and $\mathbb{U}$, respectively. Lastly, we need to verify that $1_\mathbb{P} \preceq \eta \circ \zeta$, which amounts to checking that there exists a computable map $\rho \colon \mathbb{P} \to \mathbb{P}$ such that $\rho \circ \eta \circ \zeta = 1_\mathbb{P}$. We define the graph of ρ to be

$$m \in \rho(\{n_1, \dots, n_k\}) \iff B_{n_1} \subseteq C_m \vee \dots \vee B_{n_k} \subseteq C_m \ .$$

The relation on the right-hand side is r.e. in $m, n_1, \dots, n_k$, hence ρ is computable. The map ρ is the left inverse of $\eta \circ \zeta$ because, for any $x \in \mathbb{P}$,

$$m \in \rho(\eta(\zeta x)) \iff \exists\, n \in \eta(\zeta x) \,.\, (B_n \subseteq C_m) \iff \\ \exists\, k \in x \,.\, \exists\, n \in \mathbb{N} \,.\, (B_n \subseteq C_k \wedge B_n \subseteq C_m) \iff m \in x \ .$$

Here we used the property that $\exists\, n \in \mathbb{N} \,.\, (B_n \subseteq C_k \wedge B_n \subseteq C_m)$ is equivalent to $C_k \cap C_m \neq \emptyset$, which is equivalent to $C_k = C_m$, which is equivalent to $k = m$.

1.4.4 Equivalence of $(\mathbb{P}, \mathbb{P}_\sharp)$ and $(\mathbb{V}, \mathbb{V}_\sharp)$

Recall that the PCAs $\mathbb{U}$ and $\mathbb{V}$ are both the universal domain, but with different applications. We show that when we replace $\mathbb{U}$ with $\mathbb{V}$, the applicative inclusion $\eta \dashv \zeta$ from Subsection 1.4.3 becomes an applicative equivalence. The embedding $\eta \colon \mathbb{V} \to \mathbb{P}$ extends to a computable embedding $\eta \colon \mathbb{V}^\top \to \mathbb{P}$, where the extended version is defined by the same expression as the original one. To see that $\eta \colon \mathbb{V} \to \mathbb{P}$ is an applicative morphism, observe that application on $\mathbb{V}$ is the restriction of a computable map

$$\Box \star \Box \colon \mathbb{V}^\top \times \mathbb{V}^\top \to \mathbb{V}^\top \ ,$$

hence by the Effective Extension Theorem 4.1.13, there exists a computable extension $\mathbb{P} \times \mathbb{P} \to \mathbb{P}$. Therefore, η is an applicative morphism. The embedding $\zeta \colon \mathbb{P} \to \mathbb{V}$ is an applicative morphism, too, which is proved exactly the same way as in Subsection 1.4.3.

It remains to show that $\eta \circ \zeta \sim 1_\mathbb{P}$ and $\zeta \circ \eta \sim 1_\mathbb{V}$. The proof of $\eta \circ \zeta \sim 1_\mathbb{P}$ and $1_\mathbb{V} \preceq \zeta \circ \eta$ is the same as the corresponding proof in Subsection 1.4.3. To prove $\zeta \circ \eta \preceq 1_\mathbb{V}$, we need to find $s \in \mathbb{V}_\sharp$ such that, for all $S \in \mathbb{V}$, $s \cdot \zeta(\eta S) = S$. This is equivalent to finding a computable map $\sigma \colon \mathbb{V} \to \mathbb{V}^\top$ such that $\sigma \circ \zeta \circ \eta = 1_\mathbb{V}$. Let

$$\sigma S = \bigcup \{B_n \mid n \in \mathbb{N} \wedge C_n \subseteq S\} \ ,$$

where C_n was defined in Subsection 1.4.3 as

$$C_n = \{\alpha \in \mathbb{N} \mid \alpha 0 = \dots = \alpha n = 0 \wedge \alpha(n+1) = 1\} \ .$$

The map σ is computable because the relation $B_m \subseteq \sigma(B_n)$ is equivalent to

$$\exists\, k_1, \dots, k_i \in \mathbb{N} \,.\, ((C_{k_1} \cup \dots \cup C_{k_i} \subseteq B_n) \wedge (B_m \subseteq B_{k_1} \cup \dots \cup B_{k_i})) \ ,$$

which is r.e. in $\langle m, n\rangle \in \mathbb{N} \times \mathbb{N}$. Note also that $\sigma\left(\bigcup_{n \in \mathbb{N}} C_n\right) = \top$, which is why this proof would have failed if we replaced $\mathbb{V}$ with $\mathbb{U}$. Finally, σ is a left inverse of $\zeta \circ \eta$ because, for all $S \in \mathbb{V}$,

$$\sigma(\zeta(\eta S)) = \sigma\left(\bigcup \{C_n \mid n \in \mathbb{N} \wedge B_n \subseteq S\}\right) = \bigcup \{B_n \mid B_n \subseteq S\} = S \ .$$

1.4.5 Equivalence of Reflexive Continuous Lattices

In Subsection 1.1.2 we saw that a reflexive CPO is a model of the untyped λ-calculus, hence a combinatory algebra. So far we have considered two reflexive CPOs, the graph model $\mathbb{P}$ and the universal domain $\mathbb{U}$. In this subsection we show that every countably based reflexive continuous lattice is equivalent to $\mathbb{P}$. Thus, as far as categories of modest sets on countably based reflexive continuous lattices are concerned, we do not lose any generality by considering only the graph model $\mathbb{P}$.

We only consider countably based continuous lattices. A continuous lattice L is *reflexive* if it contains at least two elements and its continuous function space L^L is a retract of L.

Proposition 1.4.14 *The graph model is a continuous retract of every reflexive continuous lattice.*

Proof. Let L be a reflexive continuous lattice. Then we have a section-retraction pair

$$L^L \underset{\Lambda}{\overset{\Gamma}{\rightleftarrows}} L \; .$$

The lattice L is a model of the untyped λ-calculus. The product $L \times L$ is a retract of L. The section $p^+ \colon L \times L \to L$ and the retraction $p^- \colon L \to L \times L$ can be most conveniently expressed as the untyped λ-terms as

$$p^+\langle x, y\rangle = \lambda z.\,(zxy) \;, \qquad\qquad p^- z = \langle \mathsf{fst}\, z, \mathsf{snd}\, z\rangle \;,$$

where $\mathsf{fst}\, z = z(\lambda xy.\, x)$ and $\mathsf{snd}\, z = z(\lambda xy.\, y)$. Let $p = p^+ \circ p^-$.

Let $\mathcal{R}(L)$ be the continuous lattice of retractions on L. There is a continuous pairing operation on $\mathcal{R}(L)$, defined by

$$A \times B = \lambda z \in L \,.\, p^+\langle A(\mathsf{fst}\,(pz)), B(\mathsf{snd}\,(pz))\rangle \; .$$

The Sierpinski space Σ is a retract of L, with the corresponding retraction $S \colon L \to L$

$$Sx = \begin{cases} \bot & \text{if } x = \bot \\ \top & \text{if } x \neq \bot \end{cases}$$

Let P be the least retraction on L satisfying the recursive equation

$$P = S \times P \; .$$

The retraction P is the directed supremum of the chain of retractions $P_0 \leq P_1 \leq \cdots$, defined by

$$P_0 = \bot \;, \qquad\qquad P_{k+1} = S \times P_k \; .$$

We abuse notation slightly and denote a retraction and its lattice of fixed points with the same letter. Clearly $P_k \cong \Sigma^k$ for every $k \in \mathbb{N}$. Thus, P is isomorphic to the limit/colimit of the chain

$$\bot \rightleftarrows \Sigma \rightleftarrows \Sigma^2 \rightleftarrows \Sigma^3 \rightleftarrows \cdots$$

where the pairs of arrows between the stages are the canonical section-retraction pairs between Σ^k and Σ^{k+1}. The limit/colimit is the lattice $\Sigma^{\mathbb{N}}$, which is isomorphic to $\mathbb{P}$. ∎

Corollary 1.4.15 *Every two countably based reflexive continuous lattices are retracts of each other, hence they are equivalent as combinatory algebras.*

Proof. If L and M are reflexive continuous lattices, then they are retracts of each other because each is a retract of $\mathbb{P}$, and $\mathbb{P}$ is a retract of each of them by Proposition 1.4.14. There are section-retraction pairs

$$L \underset{\lambda^-}{\overset{\lambda^+}{\rightleftarrows}} M \ , \qquad\qquad M \underset{\mu^-}{\overset{\mu^+}{\rightleftarrows}} L \ .$$

The applicative equivalence $L \sim M$ is witnessed by the sections λ^+ and μ^+. We omit the details.
∎

Chapter 2

A Logic for Modest Sets

In Chapter 1 we presented categories of modest sets, which provide a framework for the study of computable analysis and topology. We could now proceed with the construction of various mathematical structures, such as metric spaces and continuous functions, in categories of modest sets and study their computational content. This is the common path taken by various schools of computability.[1] However, there is an alternative, and quite often overlooked, approach to developing computable analysis and topology—we can use the *internal logic* of the categories of modest sets. The main advantage of using the internal logic is that we can avoid talking all the time in terms of realizability relations, partial equivalence relations, or representations of sets. As a result, we obtain an exposition of computable mathematics that parallels much more closely the usual presentations of classical mathematics.

The purpose of this chapter is to present the internal logic of modest sets. Since we intend to actually use the logic later on to develop analysis and topology, rather than to study the internal logic itself, we do not give formal inference rules and rules for derivations of types.[2] We now proceed with a rigorous axiomatic exposition of the internal logic of modest sets, skipping over some details about formation of dependent types, because those are best presented by formal inference rules, as in [Bir99, Appendix A]. In Chapter 3 we give an interpretation of the logic presented here in categories of modest sets.

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[1] Weihrauch and coworkers [Wei00, Wei95, Wei85, Wei87, BW99, KW85] built a theory of computable analysis in the setting of $\mathsf{Mod}(\mathbb{B}, \mathbb{B}_\sharp)$ by working directly with modest sets as representations. Spreen [Spr98] formulated a version of effective topology in the setting of numbered sets, which correspond to the category $\mathsf{Mod}(\mathbb{N})$. See Subsection 5.4.1 for details. Blanck [Bla97a, Bla97b, Bla99] studied computability on topological spaces via domain representations, which correspond to $\mathsf{Mod}(\mathbb{U}, \mathbb{U}_\sharp)$ and $\mathsf{Mod}(\mathbb{P}, \mathbb{P}_\sharp)$, see Subsection 4.1.5. Edalat and coworkers [Eda97, PEE97, ES99b, ES99a, EH98, EK99] worked with effectively presented continuous domains, which fit into $\mathsf{Mod}(\mathbb{P}, \mathbb{P}_\sharp)$, as explained in Subsection 4.1.3.

[2] This is not to say that formal logic is not important. In fact, without a thorough formal study of the internal logic of modest sets it would hardly be possible to come up with an informal one. This chapter is an informal version of the logic studied by Birkedal [Bir99, Appendix A].

2.1 The Logic of Simple Types

2.1.1 First-order Predicate Logic

The objects of discourse are *points* and *spaces*, also called *types*. Generally, spaces are denoted by capital letters A, B, C, ..., and points are denoted by lower case letters x, y, z, ...

A space *contains* points, and a point *belongs* to a space,[3] which we write as $x \in A$ or $x : A$. Every point is contained in exactly one space, which we call the *type* of the point. We say that a space A is *inhabited* when there exists a point $x \in A$. We abuse notation and use the same symbol for different points, e.g., $0 \in \mathbb{N}$, $0 \in \mathbb{R}$, and $0 \in \mathbb{C}$. If a variable x ranges over points, we must explicitly mention its type, unless it is clear from the context. The type of a variable can be mentioned inside an expression like this,

$$(x : \mathbb{N})^2 - x - 1 = 0 ,$$

or somewhat less confusingly like this,

$$x^2 + y^2 = z^2 . \qquad (x, y, z : \mathbb{N})$$

A space may *depend* on points. For example, we might consider the n-dimensional Euclidean space $\mathbb{R}^n$, where n is a point of the space of natural numbers $\mathbb{N}$.

The internal logic of modest sets is first-order *intuitionistic* logic with equality, satisfying some additional axioms. We do not repeat the exact definition and rules of inference here. They are readily available, see for example [TvD88a]. The logical connectives are conjunction $\land$, disjunction $\lor$, and implication $\longrightarrow$. Bi-implication $\longleftrightarrow$ is defined by

$$\phi \longleftrightarrow \psi \;\equiv\; (\phi \longrightarrow \psi) \land (\psi \longrightarrow \phi) .$$

Truth true and falsehood false are truth values.[4] Negation $\neg$ is defined by

$$\neg\phi \;\equiv\; \phi \longrightarrow \bot .$$

The logical quantifiers are the universal quantification $\forall$ and the existential quantification $\exists$.

Every space A has an equality relation $=_A$. We usually omit the subscript and write just $=$. It only makes sense to write $x =_A y$ when x and y belong to A. If $x \in A$ and $y \in B$, and A and B are not the same space, the formula $x = y$ is not false—it is meaningless. Inequality $x \neq y$ is an abbreviation for $\neg(x = y)$.[5] Equality satisfies all the usual axioms. It is reflexive, symmetric, and transitive, and the law of substitution of equals for equals is valid. In addition, in the logic of modest sets equality is a stable relation, which is expressed by the Axiom of Stability.

Axiom 2.1.1 (Axiom of Stability) *If not $x \neq y$ then $x = y$.*

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[3] We purposely avoid saying that a point is an element of a space, or that a space consists of points, in order to make it clear that we are not thinking of ordinary sets.

[4] We are not saying that truth and falsehood are the only truth values!

[5] In intuitionistic mathematics there is also a "positive" version of inequality, called the *apartness* relation $\lessgtr$, cf. Definition 5.5.4, which we denote by $\lessgtr$. Bishop [BB85] denotes the apartness relation by $\neq$. In the presence of Markov's Principle, cf. Axiom 2.3.2, the apartness relation and inequality often coincide.

We emphasize that the Axiom of Stability is a special feature of the logic of modest sets, and is not accepted in general intuitionistic logic.

The expression $\exists!\, x \in A \,.\, \phi(x)$ is read as "there exists a unique $x \in A$ such that $\phi(x)$", and is an abbreviation for

$$(\exists\, x \in A \,.\, \phi(x)) \;\wedge\; \forall\, x, y \in A \,.\, (\phi(x) \wedge \phi(y) \longrightarrow x = y) \;.$$

If we prove $\exists!\, x \in A \,.\, \phi(x)$ then we can denote the unique $x \in A$ for which $\phi(x)$ holds with the *description operator*:

$$\mathsf{the}\, x \in A \,.\, \phi(x) \;.$$

The expression $\mathsf{the}\, x \in A \,.\, \phi(x)$ is undefined if $\exists!\, x \in A \,.\, \phi(x)$ has not been proved.

Whenever a predicate or a relation is given, we must specify the types of all the variables that occur freely in it. For example, we may write $x : A, y : B \mid \phi(x, y)$ or $\phi(x : A, y : B)$ to indicate that x and y may occur freely in ϕ, and belong to spaces A and B, respectively. Unless we specifically state that the indicated variables are the only ones occurring freely, we allow for the possibility that there are additional parameters which are not mentioned explicitly.

We explain briefly how intuitionistic logic differs from classical logic. In order to prove a disjunction $\phi \vee \psi$ we must explicitly state which one of the disjuncts we are going to prove, and then prove it. The Law of Excluded Middle is not generally valid in intuitionistic logic, although it may happen that $\phi \vee \neg\phi$ holds for a particular predicate ϕ. In this case we say that ϕ is a *decidable* predicate.

It is not generally the case that $\neg\neg\phi$ implies ϕ. When this is the case, we say that ϕ is a *stable* predicate. The converse $\phi \longrightarrow \neg\neg\phi$ is always valid.

Some common valid rules of first-order intuitionistic logic are summarized by the following list [TvD88a, Introduction]:

$$\begin{aligned}
&\phi \longrightarrow \neg\neg\phi \;, \\
&\neg\phi \longleftrightarrow \neg\neg\neg\phi \;, \\
&\neg(\phi \vee \psi) \longleftrightarrow \neg\phi \wedge \neg\psi \;, \\
&\neg(\phi \wedge \psi) \longleftrightarrow (\phi \longrightarrow \neg\psi) \longleftrightarrow (\psi \longrightarrow \neg\phi) \;, \\
&\neg\neg(\phi \longrightarrow \psi) \longleftrightarrow (\neg\neg\phi \longrightarrow \neg\neg\psi) \longleftrightarrow (\phi \longrightarrow \neg\neg\psi) \longleftrightarrow \neg\neg(\neg\phi \vee \psi) \;, \\
&\neg\neg(\phi \wedge \psi) \longrightarrow \neg\neg\phi \wedge \neg\neg\psi \;, \\
&\neg(\phi \longrightarrow \psi) \longleftrightarrow \neg(\neg\phi \vee \psi) \;, \\
&(\phi \longrightarrow \psi) \longrightarrow (\neg\psi \longrightarrow \neg\phi) \;, \\
&\neg\exists\, x \in A \,.\, \phi(x) \longleftrightarrow \forall\, x \in A \,.\, \neg\phi(x) \;, \\
&\neg\neg\forall\, x \in A \,.\, \phi(x) \longrightarrow \forall\, x \in A \,.\, \neg\neg\phi(x) \;, \\
&\neg\neg\exists\, x \in A \,.\, \phi(x) \longleftrightarrow \neg\forall\, x \in A \,.\, \neg\phi(x) \;.
\end{aligned}$$

We now postulate ways of making new spaces out of the old ones. Whenever we define a new space, we must specify when two points in the space are considered equal.

2.1.2 Maps and Function Spaces

A *graph* is a relation $\gamma(x : A, y : B)$ that satisfies $\forall\, x \in A \,.\, \exists!\, y \in B \,.\, \gamma(x, y)$.

Axiom 2.1.2 (Function Spaces) *For any two spaces A and B there exists the* function space B^A *of* maps *from A to B. If $f \in B^A$ and $x \in A$, then fx, the* application *of f to x, is a point of B. For every graph $\gamma(x : A, y : B)$ there exists a* unique *map $f \in B^A$ such that $\forall\, x \in A \,.\, \gamma(x, fx)$.*

A function space is also called the *exponential space* of A and B. Another way to write B^A is $A \to B$. If $f \in B^A$ is a map, the space A is the *domain* of f and B is the *codomain* of f. Application fx of a map f to an argument x is also called *evaluation* of f at x. Application associates to the left, i.e., fxy is interpreted as $(fx)y$. The arrow notation $A \to B$ associates to the right, i.e., $A \to B \to C$ is interpreted as $A \to (B \to C)$.

The Axiom of Function Spaces states that a graph determines a unique map. This principle is known as *Unique Choice*:

Theorem 2.1.1 (Unique Choice) *Every graph determines a unique map.*

More precisely, the principle of Unique Choice states that

$$(\forall\, x \in A \,.\, \exists!\, y \in B \,.\, \gamma(x, y)) \longrightarrow \exists!\, f \in A^B \,.\, \forall\, x \in A \,.\, \gamma(x, fx) \ .$$

The following theorem characterizes equality of maps.

Theorem 2.1.2 (Extensionality) *Maps $f, g \colon A \to B$ are equal if, and only if, $fx = gx$ for all $x \in A$.*

Proof. Suppose $fx = gx$ for all $x \in A$. Since equality is transitive, it follows that, for all $x \in A$ and $y \in B$,

$$fx = y \longleftrightarrow gx = y \ .$$

But this means that f and g are determined by equivalent graphs, hence they are both choice maps for both graphs $fx = y$ and $gx = y$. Therefore they are the same map. The converse follows immediately from the law of substitution of equals for equals. ∎

Most frequently, when we define a map by Unique Choice, the graph $\rho(x, y)$ has the form of an equality $y = t(x)$ where $t(x)$ is an expression that may depend on x, but does not depend on y. In such cases we denote the map by

$$\lambda x : A \,.\, t(x)$$

instead of $\mathsf{the}\, f \in B^A \,.\, \forall\, x \in A \,.\, (fx = t(x))$. For every $y \in A$ we have

$$(\lambda x : A \,.\, t(x))y = t(y) \ .$$

This construction of maps is called λ-abstraction, and the equation above is called the *β-rule* of λ-calculus. It follows immediately from extensionality that, for any map $f \colon A \to B$,

$$\lambda x : A \,.\, fx = f \ .$$

This is known as the *η-rule* of λ-calculus. We demonstrate how λ-abstraction can be used to define some important maps.

The map given by the graph $x = x$ is $1_A = \lambda x : A \,.\, x$, called the *identity on A*, with the property that $1_A x = x$ for all $x \in A$.

Suppose $f\colon A \to B$ and $g\colon B \to C$. Then for every $x \in A$, $g(fx)$ is a point of C. By λ-abstraction we obtain a map $g \circ f = \lambda x : A \,.\, g(fx)$, called the *composition of* f *and* g, with the property that $(g \circ f)x = g(fx)$ for all $x \in A$. This property uniquely determines $g \circ f$, which follows from extensionality.

For $y \in B$ we obtain by λ-abstraction the map $\mathsf{K}_y = \lambda x : A \,.\, y$, called the *constant map* y.

An *inverse* of a map $f\colon A \to B$ is a map $f^{-1}\colon B \to A$ such that $f \circ f^{-1} = 1_B$ and $f^{-1} \circ f = 1_A$. Not every map has an inverse, and one that does is called an *isomorphism*. Two spaces are *isomorphic* if there exists an isomorphism between them. Normally, we study properties of spaces that are preserved by isomorphisms. We habitually switch between isomorphic versions of a space and use the one that is most convenient. If A and B are isomorphic we write $A \cong B$, or often just $A = B$.

2.1.3 Products

Axiom 2.1.3 (Products) *For every pair of spaces A and B there exists the* product space $A \times B$. *For any $x \in A$ and $y \in B$ the* ordered pair $\langle x, y\rangle$ *is a point of $A \times B$, and every point $p \in A \times B$ is equal to an ordered pair $p = \langle x, y\rangle$ for unique $x \in A$ and $y \in B$.*

By Unique Choice, there exist two canonical *projection maps* $\mathsf{fst}\colon A \times B \to A$ and $\mathsf{snd}\colon A \times B \to B$ which satisfy, for all $p \in A \times B$,

$$p = \langle \mathsf{fst}\, p, \mathsf{snd}\, p\rangle \ .$$

It also follows that, for all $x \in A$ and $y \in B$,

$$\mathsf{fst}\, \langle x, y\rangle = x \ , \qquad \mathsf{snd}\, \langle x, y\rangle = y \ .$$

The Axiom of Products characterizes equality on $A \times B$.

Theorem 2.1.3 *Ordered pairs $p, q \in A \times B$ are equal if, and only if,* $\mathsf{fst}\, p = \mathsf{fst}\, q$ *and* $\mathsf{snd}\, p = \mathsf{snd}\, q$.

Proof. If $\mathsf{fst}\, p = \mathsf{fst}\, q$ and $\mathsf{snd}\, p = \mathsf{snd}\, q$ then

$$p = \langle \mathsf{fst}\, p, \mathsf{snd}\, p\rangle = \langle \mathsf{fst}\, q, \mathsf{snd}\, q\rangle = q \ .$$

The converse is even more obvious. ■

For any two maps $f\colon C \to A$ and $g\colon C \to B$, we define the map $\langle f, g\rangle\colon C \to A \times B$ by

$$\langle f, g\rangle = \lambda x : A \,.\, \langle fx, gx\rangle \ .$$

For any two maps $f\colon A \to C$, $g\colon B \to D$ we define the map $f \times g\colon A \times B \to C \times D$ by

$$f \times g = \lambda p : A \times B \,.\, \langle f(\mathsf{fst}\, p), g(\mathsf{snd}\, p)\rangle.$$

Often we encounter λ-abstraction over a product type $A \times B$. In such cases we write $\lambda\langle x, y\rangle : A \times B \,.\, t$. For example, the definition of $f \times g$ can be written more simply as $f \times g = \lambda\langle x, y\rangle : A \times B \,.\, \langle fx, gx\rangle$.

Proposition 2.1.4 *The spaces $A \times B$ and $B \times A$ are isomorphic. The spaces $(A \times B) \times C$ and $A \times (B \times C)$ are isomorphic.*

Proof. We provide the isomorphism and leave the verification that they are inverses of each other as an exercise. For the first claim, take the maps

$$\lambda\langle x,y\rangle : A \times B \,.\, \langle y,x\rangle \;, \qquad \lambda\langle y,x\rangle : B \times A \,.\, \langle x,y\rangle \;,$$

and for the second claim take

$$\lambda\langle\langle x,y\rangle ,z\rangle : (A \times B) \times C \,.\, \langle x,\langle y,z\rangle\rangle \;, \qquad \lambda\langle x,\langle y,z\rangle\rangle : A \times (B \times C) \,.\, \langle\langle x,y\rangle ,z\rangle \;.$$

■

The following two propositions relate product spaces and function spaces.

Proposition 2.1.5 *The spaces $C^{A\times B}$ and $(C^A)^B$ are isomorphic.*

Proof. Suppose $f \in C^{A\times B}$. Then for all $x \in A$ and $y \in B$, $f\langle x,y\rangle$ is a point of C. Now if we λ-abstract three times in a row, over x, y, and f, we obtain a map, called the *Currying operation*,

$$\mathsf{curry} = \lambda f : C^{A\times B} \,.\, \lambda y : B \,.\, \lambda x : A \,.\, f\langle x,y\rangle \;:\; C^{A\times B} \to (C^A)^B \;.$$

We claim that curry is an isomorphism. To see this, consider the map

$$\mathsf{uncurry} = \lambda f : (C^A)^B \,.\, \lambda\langle x,y\rangle : A \times B \,.\, fxy \;.$$

Now an easy calculation shows that, for all $f \in C^{A\times B}$, $x \in A$, and $y \in B$,

$$(\mathsf{uncurry}(\mathsf{curry} f))\langle x,y\rangle = (\mathsf{curry} f)xy = f\langle x,y\rangle \;,$$

hence $\mathsf{uncurry} \circ \mathsf{curry} = 1_{C^{A\times B}}$. A similar calculation shows that $\mathsf{curry} \circ \mathsf{uncurry} = 1_{(C^A)^B}$. We usually denote the map $\mathsf{curry} f$ by $\widetilde{f}$. The map $\widetilde{f}$ is called the *transpose* of f. ■

Proposition 2.1.6 *The spaces $(A \times B)^C$ and $A^C \times B^C$ are isomorphic.*

Proof. The proof goes along the same lines as the previous one. The isomorphism are

$$\lambda f : (A \times B)^C \,.\, \langle \lambda x : C \,.\, \mathsf{fst}\,(fx), \lambda x : C \,.\, \mathsf{snd}\,(fx)\rangle$$

and

$$\lambda\langle g_1, g_2\rangle : A^C \times B^C \,.\, \lambda x : C \,.\, \langle g_1 x, g_2 x\rangle \;.$$

We leave the easy verification that these two maps are inverses of each other as an exercise. ■

2.1.4 Disjoint Sums

Axiom 2.1.4 (Disjoint Sums) *For every pair of spaces A and B there exists the* disjoint sum *$A + B$. For all $x \in A$ and $y \in B$, $\mathsf{inl}\, x$ and $\mathsf{inr}\, y$ are points of $A + B$. Every point $z \in A + B$ is equal to $z = \mathsf{inl}\, x$ for a unique $x \in A$, or to $z = \mathsf{inr}\, y$ for a unique $y \in B$. For all $x \in A$ and $y \in B$, $\mathsf{inl}\, x \neq \mathsf{inr}\, y$.*

By Unique Choice we obtain maps $\mathsf{inl} \colon A \to A + B$ and $\mathsf{inr} \colon B \to A + B$, called the *canonical inclusions.*

Theorem 2.1.7 *Points $z, w \in A + B$ are equal if, and only if, $z = w = \mathsf{inl}\, x$ for some $x \in A$, or $z = w = \mathsf{inr}\, y$ for some $y \in B$.*

Proof. If $z = w$ then by the Axiom of Disjoint Sums, $z = \mathsf{inl}\, x$ for some $x \in A$, and hence $z = w = \mathsf{inl}\, x$, or $z = \mathsf{inr}\, y$ for some $y \in A$, and hence $z = w = \mathsf{inr}\, y$. The converse is equally trivial. ∎

Suppose $f \colon A \to C$ and $g \colon B \to C$ are two maps. Consider any $z \in A + B$. Then $z = \mathsf{inl}\, x$ for a unique $x \in A$, or $z = \mathsf{inr}\, y$ for a unique $y \in B$. In the first case, there exists a unique $w \in C$ such that $w = fx$, namely $w = fx$. In the second case, there exists a unique $w \in C$ such that $w = gy$. So, by Unique Choice, there exists a map $[f, g] \colon A + B \to C$ such that, for all $x \in A$ and $y \in B$,

$$\begin{aligned} [f,g](\mathsf{inl}\, x) &= fx, \\ [f,g](\mathsf{inr}\, y) &= gy. \end{aligned}$$

In other words, we can define maps from $A + B$ *by cases*, like this:

$$[f,g]z = \begin{cases} fx & \text{if } z = \mathsf{inl}\, x \\ gy & \text{if } z = \mathsf{inr}\, y \ . \end{cases}$$

We usually abuse notation slightly and write

$$[f,g]z = \begin{cases} fz & \text{if } z \in A \\ gz & \text{if } z \in B \end{cases}$$

and sometimes we use the shorter notation

$$[f,g] = [\mathsf{inl}\,(x : A) \mapsto fx, \mathsf{inr}\,(y : B) \mapsto gy] \ .$$

For any two maps $f \colon A \to C$ and $g \colon B \to D$ we denote by $f + g \colon A + B \to C + D$ the map

$$f + g = [\mathsf{inl}\,(x : A) \mapsto \mathsf{inl}_C\,(fx), \mathsf{inr}\,(y : B) \mapsto \mathsf{inr}_D\,(gy)] \ .$$

Proposition 2.1.8 *The spaces $A + B$ and $B + A$ are isomorphic. The spaces $(A + B) + C$ and $A + (B + C)$ are isomorphic.*

Proof. The first claim holds because the maps

$$[\mathsf{inl}\,(x : A) \mapsto \mathsf{inr}\, x, \mathsf{inr}\,(y : B) \mapsto \mathsf{inl}\, y] \ , \qquad [\mathsf{inl}\,(y : B) \mapsto \mathsf{inr}\, y, \mathsf{inr}\,(x : A) \mapsto \mathsf{inl}\, x] \ .$$

are isomorphisms between $A + B$ and $B + A$. The second claim is left as an exercise. ∎

2.1.5 The Empty and the Unit Spaces

In the previous two subsections we postulated the existence of binary products and disjoint sums. Now we postulate that the existence of the empty and the unit spaces, which are the "neutral elements" for disjoint sums and products, respectively.

Axiom 2.1.5 (Empty Space) *There exists the* empty space $\mathsf{0}$ *that contains no points.*

More precisely, the axiom states that $\neg\exists\, x \in \mathsf{0}\,.\,\mathsf{true}$, or equivalently that $\forall\, x \in \mathsf{0}\,.\,\mathsf{false}$.

Proposition 2.1.9 *For every space A there exists exactly one map $\mathsf{0}_A \colon \mathsf{0} \to A$.*

Proof. The statement $\forall\, x \in 0\,.\, \exists!\, y \in A\,.\, \mathsf{true}$ holds trivially, therefore there exists at least one map $0_A\colon 0 \to A$. If $f\colon 0 \to A$ and $g\colon 0 \to A$ then for all $x \in 0$, $fx = gx$, hence $f = g$ by extensionality. ■

Axiom 2.1.6 (Unit Space) *There exists the* unit space 1, *that contains the* unit $\star \in 1$, *and every point of* 1 *is equal to* $\star$.

The unit space is also called the *terminal space* and the *singleton space*.

Proposition 2.1.10 *For every space A there exists exactly one map $!_A\colon A \to 1$.*

Proof. There exists at least one map, namely $\lambda x : A\,.\,\star$. If $f\colon A \to 1$ and $g\colon A \to 1$ then by the Axiom of Singleton $fx = \star = gx$ for all $x \in A$, therefore $f = g$ by extensionality. ■

Proposition 2.1.11 *For any space A, $A + 0$ is isomorphic to A, and A^0 is isomorphic to 1.*

Proof. The isomorphism between A and $A + 0$ is $\mathsf{inl}\colon A \to A + 0$. That A^0 is isomorphic to 1 follows from Proposition 2.1.9. ■

Proposition 2.1.12 *The spaces $A \times 1$ and A^1 are isomorphic to A.*

Proof. The isomorphism between $A \times 1$ and A is witnessed by $\mathsf{fst}\colon A \times 1 \to A$ and $\lambda x : A\,.\,\langle A, \star\rangle$. The isomorphism between A^1 and A is witnessed by $\lambda f : A^1\,.\, f\star$ and $\lambda x : A\,.\,\lambda y : 1\,.\,x$. ■

Since $0^0 \cong 1$, we could derive the Axiom of Singleton from the Axiom of Empty Space and the properties of function spaces.

Example 2.1.13 (Finite Discrete Spaces) In Example 2.2.3 we will construct the space of natural numbers $\mathbb{N}$. For every $n \in \mathbb{N}$ we obtain the *finite discrete space* consisting of n points

$$[n] = \{k \in \mathbb{N} \mid k < n\} = \underbrace{1 + \cdot + 1}_{n} .$$

The points of this space are denoted by $0, 1, \ldots, n-1$, and the space is written as $\{0, 1, \ldots, n-1\}$. The space $\{0, 1, \ldots, n-1\}$ is discrete in the sense that

$$\forall\, x \in \{0, 1, \ldots, n\}\,.\,(x = 0 \lor x = 1 \lor \cdots x = n) .$$

Thus we can define a map $f\colon [n] \to A$ by cases since it can be written uniquely as $f = [f_0, \ldots, f_{n-1}]$ where $f_k\colon 1 \to A$.

As a special case, we define $2 = 1 + 1$.

2.1.6 Subspaces

Axiom 2.1.7 (Subspaces) *Let A be a space and $\phi(x : A)$ a predicate on A. There exists the* subspace $\{x \in A \mid \phi(x)\}$. *If $y \in \{x \in A \mid \phi(x)\}$ then $\mathsf{i}_\phi y \in A$ and $\phi(\mathsf{i}_\phi y)$ hold. For every $x \in A$, if $\phi(x)$ holds then there is a unique $y \in \{x \in A \mid \phi(x)\}$ such that $x = \mathsf{i}_\phi y$.*

Theorem 2.1.14 *Points $y, z \in \{x \in A \mid \phi(x)\}$ are equal if, and only if, the points $\mathsf{i}_\phi y$ and $\mathsf{i}_\phi z$ are equal.*

Proof. Suppose $\mathsf{i}_\phi y = \mathsf{i}_\phi z$. Then $\phi(\mathsf{i}_\phi y)$ and $\mathsf{i}_\phi y \in A$, hence by the Axiom of Subspaces there exists a unique point $w \in \{x \in A \mid \phi(x)\}$ such that $\mathsf{i}_\phi y = \mathsf{i}_\phi w$. But both y and z can be taken as w, hence it must be the case that $y = z$. ∎

By Unique Choice, i_ϕ is a map $\mathsf{i}_\phi \colon \{x \in A \mid \phi(x)\} \to A$. It is called the *subspace inclusion.* Suppose $x \in A$ and $\phi(x)$. Then we denote the unique $y \in \{x \in A \mid \phi(x)\}$ for which $x = \mathsf{i}_\phi y$ by $\mathsf{o}_\phi x$. This notation can be misleading, and it is important to keep in mind that o_ϕ is *not* a map but an abbreviation for

$$\mathsf{the}\, y \in \{z \in A \mid \phi(z)\}\,.\,(x = \mathsf{i}_\phi y)\,.$$

Definition 2.1.15 An *injective map*, or an *injection*, is a map $f \colon A \to B$ that satisfies, for all $x, y \in A$,

$$fx = fy \longrightarrow x = y\,.$$

By Axiom of Stability, this condition is equivalent to $x \neq y \longrightarrow fx \neq fy$.

A subspace inclusion is always an injection, as follows immediately from the Axiom of Subspaces.

Theorem 2.1.16 *Every injection is isomorphic to a subspace inclusion.*[6]

Proof. Suppose $f \colon A \to B$ is an injection. Let $\phi(y : B)$ be the predicate $\exists\, x \in A\,.\,(fx = y)$, and let $B' = \{y \in B \mid \phi(y)\}$. For every $x \in A$ there exists a unique $z \in B'$ such that $fx = \mathsf{i}_\phi z$, namely $z = \mathsf{o}_\phi(fx)$. Uniqueness of z holds because $fx = \mathsf{i}_\phi z'$ implies $\mathsf{i}_\phi z' = \mathsf{i}_\phi z$, therefore $z = z'$. By Unique Choice there is a map $h \colon A \to B'$ such that $fx = \mathsf{i}_\phi(hx)$ for all $x \in A$, and so $f = \mathsf{i}_\phi \circ h$ by extensionality. It only remains to be seen that h is an isomorphism.

For every $z \in B'$ there exists a unique $x \in A$ such that $fx = \mathsf{i}_\phi z$. Existence of x follows from the definition of B'. Suppose $x' \in A$ also satisfies $fx' = \mathsf{i}_\phi z$. Then $fx = fx'$ and because f is an injection $x = x'$. By Unique Choice, there exists a map $k \colon B' \to A$ such that $f(kz) = \mathsf{i}_\phi z$ for all $z' \in B$.

The maps h and k are inverses of each other. Indeed, for every $x \in A$ we have $f(k(hx)) = \mathsf{i}_\phi(hx) = fx$, hence $k(hx) = x$ by injectivity of f, and so $k \circ h = 1_A$. The other way, for every $z \in B'$, $\mathsf{i}_\phi z = f(kz) = \mathsf{i}_\phi(h(kz))$, therefore $h \circ k = 1_{B'}$ by injectivity of i_ϕ. ∎

When $e \colon A \to B$ is an inclusion we say that A is a *subspace* of B via e, and write $A \subseteq_e B$. Often it is implicitly clear which embedding we have in mind. We loosely speak of A being a subspace of B without reference to an embedding, and write $A \subseteq B$. It is important to remember that a space can be a subspace of another one in many different ways. If $A \subseteq_e B$, then A is isomorphic to the subspace $\{y \in B \mid \exists\, x \in A\,.\,ex = y\}$. We call the predicate $\exists\, x \in A\,.\,ex = y$ the *defining predicate for* A. It is easy to see that defining predicates are equivalent if, and only if, they define canonically isomorphic subspaces.

Theorem 2.1.17 *Let $f \colon A \to B$ be a map and $\phi(y : B)$ a predicate on B. If $\phi(fx)$ holds for all $x \in A$, then there exists a unique map $\overline{f} \colon A \to \{y \in B \mid \phi(y)\}$ such that $f = \mathsf{i}_\phi \circ \overline{f}$.*

[6] Whenever we talk of two maps $f \colon A \to C$ and $g \colon B \to D$ being isomorphic, we mean to say that there exist isomorphisms $h \colon A \to B$ and $k \colon C \to D$ such that $k \circ f = g \circ h$.

Proof. By Unique Choice, it suffices to show that for every $x \in A$ there exists exactly one $z \in \{y \in B \mid \phi(y)\}$ such that $fx = \mathsf{i}_\phi z$. We assumed that such a z exists, namely $z = \mathsf{o}_\phi(fx)$. To see uniqueness of z, observe that $fx = \mathsf{i}_\phi w$ implies $\mathsf{i}_\phi z = fx = \mathsf{i}_\phi w$, hence $z = w$. ∎

It quickly becomes very tedious to use i_ϕ and o_ϕ explicitly at all times. We abuse notation and omit explicit conversions from and to a subspace. Strictly speaking, however, i_ϕ and o_ϕ cannot be dispensed with. It is very important to keep a clear distinction between saying that $\mathsf{o}_\phi y \in \{x \in A \mid \phi(x)\}$ and that $y \in A$ such that $\phi(y)$. The points $\mathsf{o}_\phi y$ and y are *not* equal. It does not even make sense to compare them because they belong to different spaces. It is useful to think of $\mathsf{o}_\phi y$ as y together with *evidence* that $\phi(y)$ holds. Similarly, if $y \in \{x \in A \mid \phi(x)\}$ then $\mathsf{i}_\phi y$ is a point in A, but *without* the evidence that $\phi(y)$ holds. As long as we are aware of this we can omit i_ϕ and o_ϕ, and we usually do so.

If $A \subseteq_e B$ and $x \in B$ we abuse notation slightly and write $x \in A$, or $x \in_e A$, instead of $\exists\, y \in A \,.\, ey = x$.

Recall that a predicate ϕ is *stable* if $\neg\neg\phi \longrightarrow \phi$.

Definition 2.1.18 A subspace $\{x \in A \mid \phi(x)\}$ is a *regular subspace* of A when ϕ is a stable predicate.

Definition 2.1.19 A map $f\colon A \to B$ is an *embedding* when it is isomorphic to an inclusion of a regular subspace.

Proposition 2.1.20 *A map $f\colon A \to B$ is an embedding if, and only if, it is an injection and the defining predicate $\exists\, x \in A \,.\, fx = y$ is stable, i.e., for all $y \in B$,*

$$(\neg\neg\exists\, x \in A \,.\, fx = y) \longrightarrow \exists\, x \in A \,.\, fx = y \ .$$

Proof. The 'if' part follows from Theorem 2.1.16. The 'only if' part follows from the observation that if two inclusions are isomorphic and one of them is an embedding, then so is the other. ∎

See Corollary 2.1.29 for another characterization of embeddings.

2.1.7 Quotient Spaces

Recall that an *equivalence relation* on a space A is a binary relation ρ on A that is reflexive, symmetric, and transitive, i.e., for all $x, y, z \in A$,

$$\begin{aligned}
&\rho(x, x) \ , \\
&\rho(x, y) \longrightarrow \rho(y, x) \ , \\
&\rho(x, y) \wedge \rho(y, z) \longrightarrow \rho(x, z) \ .
\end{aligned}$$

A common way of defining an equivalence relation on a space A from a map $f\colon A \to B$ is by $\rho(x, y) \longleftrightarrow fx = fy$. Observe that such an equivalence relation is always stable.

We say that a binary relation ρ on A *contains* a relation σ on A, written $\sigma \subseteq \rho$, when $\sigma(x, y)$ implies $\rho(x, y)$.

Axiom 2.1.8 (Quotient Spaces) *If ρ is a binary relation on a space A, there exists the* quotient space *A/ρ and the* canonical quotient map *$\mathsf{q}_\rho \colon A \to A/\rho$ such that $\rho(x,y)$ implies $\mathsf{q}_\rho\, x = \mathsf{q}_\rho\, y$. For every $\xi \in A/\rho$ there exists $x \in A$ such that $\xi = \mathsf{q}_\rho\, x$. Every* stable *equivalence relation on A that contains ρ also contains the relation $\mathsf{q}_\rho\, x = \mathsf{q}_\rho\, y$, where $x, y \in A$.*

We usually write $[x]_\rho$ or just $[x]$ instead of $\mathsf{q}_\rho\, x$. In effect, the Axiom of Quotient Spaces states that any binary relation ρ on A generates a smallest $\neg\neg$-stable equivalence relation σ that contains ρ, and that we can quotient A by σ. It also tells us that σ can be recovered from the canonical quotient map $\mathsf{q}_\rho \colon A \to A/\rho$ by defining

$$\sigma(x,y) \longleftrightarrow \mathsf{q}_\rho\, x = \mathsf{q}_\rho\, y \ . \qquad (x, y : A)$$

Definition 2.1.21 A map $f \colon A \to B$ is a *surjective map*, or a *surjection*, when for every $y \in B$ there not-not exists $x \in A$ such that $fx = y$.

Definition 2.1.22 A map $f \colon A \to B$ is a *quotient map* when for every $y \in B$ there exists $x \in A$ such that $fx = y$.

The Axiom of Quotient Spaces states that every canonical quotient map is indeed a quotient map.

Theorem 2.1.23 *Let ρ be a binary relation on A and $f \colon A \to B$ a map such that, for all $x, y \in A$, $\rho(x,y)$ implies $fx = fy$. Then there exists a unique map $\overline{f} \colon A/\rho \to B$ such that $f = \overline{f} \circ \mathsf{q}_\rho$.*

Proof. By Unique Choice, it suffices to show that for every $\xi \in A/\rho$ there exists a unique $y \in B$ for which there exists $x \in A$ such that $fx = y$ and $[x] = \xi$.

For every $\xi \in A/\rho$ there exists $x \in A$ such that $\xi = [x]$. Hence, there exists $y \in B$ such that $fx = y$ and $\xi = [x]$, namely $y = fx$. We claim that this is the unique such y. Indeed, suppose that $y' \in B$ and there exists $x' \in A$ such that $y' = fx'$ and $\xi = [x']$. Then $[x] = [x']$ hence $y = fx = fx' = y'$ by the Axiom of Quotient Spaces. ■

The points of A/ρ are called the *equivalence classes* and are denoted with Greek letters $\xi, \zeta, \ldots$ If $\xi \in A/\rho$ and $[x] = \xi$ then we say that x is a *representative* of the equivalence class ξ. Equivalence classes ξ and ζ are equal if, and only if, for all $x \in A$,

$$[x] = \xi \longleftrightarrow [x] = \zeta.$$

The quotient spaces described here are the *stable quotients*, because the equivalence relation $[x]_\rho = [y]_\rho$, $x, y \in A$, is the smallest *stable* equivalence relation generated by ρ. If we wanted more general equivalence relations we would have to proceed to the larger categories of realizability toposes. It turns out that most equivalence relations needed for analysis are stable anyway, so we do not need the topos-theoretic machinery.

Proposition 2.1.24 *A map $f \colon A \to B$ is a quotient map if, and only if, there is an isomorphism $h \colon A/\rho \to B$ such that $f = h \circ \mathsf{q}_\rho$, where $\mathsf{q}_\rho \colon A \to A/\rho$ is the canonical quotient map for the relation ρ defined by*

$$\rho(x,y) \longleftrightarrow fx = fy \ . \qquad (x, y : A)$$

Proof. Suppose there exists an isomorphism $h \colon A/\rho \to B$ such that $f = h \circ \mathsf{q}_\rho$. For every $y \in B$ there exists $x \in A$ such that $\mathsf{q}_\rho x = h^{-1} y$ because q_ρ is a quotient map. Apply h on both sides to get $fx = h(\mathsf{q}_\rho\, x) = h(h^{-1} y) = y$. We showed that for every $y \in B$ there exists $x \in A$ such that $fx = y$, therefore f is a quotient map.

Conversely, suppose f is a quotient map. By Theorem 2.1.23 there exists a unique map $h \colon A/\rho \to B$ such that $f = h \circ \mathsf{q}_\rho$. We show that h is an injective quotient map. By Proposition 2.1.27, which we are going to prove without relying on this proposition, it follows that h is an isomorphism. If $h[x]_\rho = h[y]_\rho$ then $fx = fy$, therefore $\rho(x, y)$ and $[x]_\rho = [y]_\rho$. Hence h is injective. Because f is a quotient map for any $y \in B$ there exists $x \in A$ such that $fx = y$, but then $h[x]_\rho = y$, therefore h is a quotient map. ∎

2.1.8 Factorization of Maps

Definition 2.1.25 A *bijective map*, or a *bijection*, is a map that is injective and surjective.

Note that a bijection need not be an isomorphism.

Proposition 2.1.26 *For any map $f \colon A \to B$:*

(1) f is an isomorphism if, and only if, $\forall\, y \in B \,.\, \exists!\, x \in A \,.\, fx = y$,

(2) f is a bijection if, and only if, $\forall\, y \in B \,.\, \neg\neg\exists!\, x \in A \,.\, fx = y$.

Proof. First we prove (1). If f is an isomorphism, then for every $y \in B$ there exists $x \in A$ such that $fx = y$ because f is surjective, and the choice of x is unique because f is injective.

Conversely, suppose that for every $y \in B$ there exists a unique $x \in A$ such that $fx = y$. By Unique Choice there exists a map $g \colon B \to A$ such that $f(gy) = y$ for all $y \in B$. To see that $g(fx) = x$ for all $x \in A$, note that f is injective, hence from $f(g(fx)) = fx$ we may conclude $g(fx) = x$.

Now we prove (2). Suppose f is a bijection. Because it is surjective, for every $y \in B$ there not-not exists $x \in A$ such that $fx = y$. Suppose that for some $x' \in A$, $fx' = y$. Then $fx = y = fx'$ and since f is injective, $x = x'$. Hence the choice of x is unique.

Conversely, suppose that for every $y \in B$ there not-not exists a unique $x \in A$ such that $fx = y$. Clearly, f is surjective. To see that it is injective, suppose $fx = fx'$. Then $\neg\neg(x = x')$, and by the Axiom of Stability, $x = x'$. ∎

Proposition 2.1.27 *For a map $f \colon A \to B$ the following are equivalent:*

(1) f is an isomorphism,

(2) f is a surjective embedding,

(3) f is an injective quotient map.

Proof. It is obvious that (1) implies (2) and (3).

Suppose f is a surjective embedding. For every $y \in B$ there not-not exists $x \in A$ such that $fx = y$. But since f is also an embedding, it follows by Proposition 2.1.20 that there exists $x \in A$ such that $fx = y$. This x is unique, for if $fx' = y$ for some $x' \in A$ then $fx = fx'$ and since f is injective $x = x'$. By Unique Choice there exists a map $g \colon B \to A$ such that $f(gy) = y$ for all $y \in B$. It is also the case that $g(fx) = x$ for all $x \in A$ because $f(g(fx)) = fx$ for all $x \in A$ and f is injective. This proves that (2) implies (1).

Suppose f is an injective quotient map. For every $y \in B$ there exists $x \in A$ such that $fx = y$. The choice of x is unique, since $fx' = y$ implies $fx' = fx$, and so $x' = x$ because f is injective. By Unique Choice there exists a map $g \colon B \to A$ such that $f(gy) = y$ for all $y \in B$. Because f is injective, it follows like before that $x = g(fx)$ for all $x \in A$. Thus (3) implies (1). ∎

Theorem 2.1.28 *Every map $f \colon A \to B$ can be factored uniquely up to isomorphism as*

$$\begin{tikzcd}
A \arrow[rrr, "f"] \arrow[dr, "q"', two heads] & & & B \\
& A' \arrow[r, tail, two heads, "b"'] & B' \arrow[ur, hook, "i"'] &
\end{tikzcd}$$

such that q is a quotient map, b is a bijection, and i is an embedding. The space B' is called the image *of f and is denoted by* $\mathsf{im}(f)$.

Proof. Let $\sim$ be the equivalence relation on A defined by

$$x \sim x' \longleftrightarrow fx = fx'.$$

Let $A' = A/{\sim}$ and let $B' = \{y \in B \mid \neg\neg\exists\, x \in A \,.\, fx = y\}$. Define $q \colon A \to A'$ to be the canonical quotient map, and $i \colon B' \to B$ to be the canonical subspace inclusion.

By Theorem 2.1.23, there exists a unique map $\overline{f} \colon A' \to B$ such that $f = \overline{f} \circ q$. Observe that $\overline{f}$ is injective. Because $\exists\, \xi \in A' \,.\, f\xi = y$ implies $\neg\neg\exists\, \xi \in A' \,.\, f\xi = y$, it follows by Theorem 2.1.17 that there exists a unique $b \colon A' \to B'$ such that $\overline{f} = i \circ b$. We check that b is a bijection. If $b[y] = b[z]$ then there not-not exists $x \in A$ such that $\overline{f}[y] = i(b[y]) = fx = i(b[z]) = \overline{f}[z]$. Thus, $\neg\neg([y] = [z])$ and by the Axiom of Stability, $[y] = [z]$. This proves that b is injective. To see that b is surjective, suppose $y \in B'$. Then there not-not exists $x \in A$ such that $fx = iy$, hence $b[x] = y$.

We now prove that the factorization $f = i \circ b \circ q$ is unique up to isomorphism. Suppose that $q' \colon A \to A''$ is a quotient map, $b' \colon A'' \to B''$ is a bijection, $i \colon B'' \to B$ is an embedding, and $f = i' \circ b' \circ q'$. It is sufficient to find isomorphisms $h \colon A'' \to A'$ and $k \colon B'' \to B'$ such that $q = h \circ q'$ and $i' = i \circ k$. Since q' is surjective and i' is injective it then follows easily that $b \circ h = k \circ b'$, as required.

For any $x, y \in A$, $fx = fy$ implies $q'x = q'y$ because $(i' \circ b')$ is injective. By Theorem 2.1.23, there exists a map $h \colon A' \to A''$ such that $q' = h \circ q$. We show that h is an isomorphism by proving that for every $z \in A''$ there exists a unique $\xi \in A'$ such that $h\xi = z$. Because q' is a quotient map, for any $z \in A''$ there exist $x \in A$ such that $q'x = z$, so we can take $\xi = [x]$. If $q'x' = z$ as well, then $fx' = fx$ and so $[x'] = [x] = \xi$, hence the choice of ξ is unique.

To show that $i\colon B' \to B$ and $i'\colon B'' \to B$ are isomorphic, it is sufficient to show that the defining predicates for B' and B'' are equivalent, i.e., for all $y \in B$,

$$(\neg\neg\exists\, x \in A \,.\, fx = y) \longleftrightarrow \exists\, z \in B'' \,.\, i'z = y \ .$$

Suppose there exists $z \in B''$ such that $y = i'z$. Because $b' \circ q'$ is surjective, there not-not exists $x \in A$ such that $b'(q'x) = z$, hence $y = i'z = i'(b'(q'x)) = fx$. Conversely, suppose there not-not exists $x \in A$ such that $fx = y$. Then there not-not exists $z \in B''$, namely $z = b'(q'x)$, such that $i'z = y$. Since i' is an embedding it follows by Proposition 2.1.20 that there exists $z \in B''$ such that $i'z = y$. ■

Corollary 2.1.29 *A map $f\colon A \to B$ is an embedding if, and only if, $f\colon A \to \mathsf{im}(f)$ is an isomorphism.*

Proof. More precisely, $f\colon A \to \mathsf{im}(f)$ is the map $b \circ q$ in the canonical factorization $f = b \circ q \circ i$, but we do not bother to use different notation for it. The corollary holds because it is equivalent to the statement that $b \circ q$ is an isomorphism. ■

2.2 The Logic of Complex Types

The axioms presented in Section 2.1 provide us with enough structure to construct many important spaces, such as real numbers, but they are not sufficient for everything that we would like to talk about. We need types that are *parametrized* by other spaces, which leads to *dependent types*. We also want to be able to handle *recursive* constructions of types, which leads to *inductive types*, and their corecursive version, the *coinductive types*. Categories of modest sets also support *parametric polymorphism* [Rey83, Rey98], but we do not consider that here.

2.2.1 Dependent Sums and Products

A *dependent type* is a space that depends on one or more parameters.[7] When we want to emphasize that a space A depends on parameter $x \in B$ we indicate this by writing $A(x)$.

Dependent types are common in everyday mathematics. For example, we might consider the closed interval $[-a, a]$ where $a > 0$ is a parameter. Another common example is the *inverse image space* f^*y defined by

$$f^*y = \{x \in A \mid fx = y\} \ . \qquad (f : B^A, y : B)$$

Axiom 2.2.1 (Dependent Sums) *If $A(x : B)$ is a dependent type, then there exists the* dependent sum $\sum_{x:B} A(x)$. *For every $u \in A(x)$, the* dependent pair $\langle x, u\rangle$ *is a point of $\sum_{x:B} A(x)$. Every point $p \in \sum_{x:B} A(x)$ is equal to a dependent pair $p = \langle x, u\rangle$ for unique $x \in B$ and unique $u \in A(x)$.*

Just like with product spaces, there are two projections

$$\mathsf{fst} : \textstyle\sum_{x:B} A(x) \to B \ , \qquad \mathsf{snd} : \textstyle\sum_{x:B} A(x) \to B \ ,$$

[7] This discussion of dependent types is not very rigorous. The precise formal rules for formation of dependent types can readily be found in Birkedal [Bir99, Appendix A].

which satisfy, for all $x \in B$, $u \in A(x)$,

$$\mathsf{fst}\,\langle x,u\rangle = x\ , \qquad \mathsf{snd}\,\langle x,u\rangle = u\ , \qquad p = \langle \mathsf{fst}\,p, \mathsf{snd}\,p\rangle\ .$$

From this it follows that points $p, q \in \sum_{x:B} A(x)$ are equal, if and only if, $\mathsf{fst}\,p = \mathsf{fst}\,q$ and $\mathsf{snd}\,p = \mathsf{snd}\,q$.

Proposition 2.2.1 *For every map $f\colon A \to B$, the space A is isomorphic to $\sum_{y:B} f^*y$.*

Proof. Let $g\colon A \to \sum_{y:B} f^*y$ be the map defined by

$$gx = \langle fx, x\rangle\ .$$

Clearly, this is well defined because $x \in f^*(fx)$ for every $x \in A$. Define a map $h\colon \sum_{y:B} f^*y \to A$ by

$$h\langle y, x\rangle = x\ .$$

Then we have, for every $x \in A$, $h(gx) = h\langle fx, x\rangle = x$, and for every $\langle y, x\rangle \in \sum_{y:B} f^*y$, $g(h\langle y, x\rangle) = gx = \langle fx, x\rangle = \langle y, x\rangle$ since $x \in f^*y$ means that $fx = y$. Thus g and h are inverses of each other. ∎

Suppose $A(x{:}B)$ is a dependent type. A *dependent graph* on $A(x)$ is a relation $\rho \subseteq \sum_{x:B} A(x)$ such that

$$\forall\, x \in B\,.\,\exists!\, u \in A(x)\,.\,\rho(x,u)\ .$$

Axiom 2.2.2 (Dependent Products) *If $A(x{:}B)$ is a dependent type, then there exists the* dependent product $\prod_{x:B} A(x)$. *The points of $\prod_{x:B} A(x)$ are called* maps. *If $x \in B$ and $f \in \prod_{x:B} A(x)$ then $fx \in A(x)$. For every dependent graph ρ on $A(x)$ there exists a unique map $f \in \prod_{x:B} A(x)$ such that, for all $x \in B$,*

$$\forall\, x \in B\,.\,\rho(x, fx)\ .$$

Just like in the case of function spaces, we introduce λ-abstraction to define maps that are elements of dependent products. Dependent maps are extensional: for $f, g \in \prod_{x:B} A(x)$,

$$f = g \longleftrightarrow \forall\, x \in B\,.\,fx = gx\ .$$

2.2.2 Inductive Spaces

Let $f\colon B \to A$ be a map. For a space X, let $P_f X$ be the space

$$P_f X = \textstyle\sum_{x:A} X^{f^*x}\ .$$

Every point of $P_f X$ is a dependent pair $\langle x, u\rangle$ where $x \in A$ and $u\colon f^*x \to X$. If $g\colon X \to Y$ is a map, let $P_f g\colon P_f X \to P_f Y$ be the map

$$(P_f g)\langle x, u\rangle = \langle x, g \circ u\rangle\ .$$

Axiom 2.2.3 (Inductive Spaces) *Let $f\colon B \to A$ be a map. There exists the* inductive space W_f *and a* structure map $\mathsf{w}_f\colon P_f\mathsf{W}_f \to \mathsf{W}_f$ *such that, for any space C and a map $c\colon P_fC \to C$, there exists a unique map $h\colon \mathsf{W}_f \to C$ satisfying the recursive equation*

$$h(\mathsf{w}_f\langle x,u\rangle) = c\,\langle x, h\circ u\rangle$$

for all $\langle x,u\rangle \in P_f$. We say that h is defined by recursion on W_f.

Theorem 2.2.2 (Induction Principle) *For any relation ρ on an inductive space W_f, the following induction principle holds:*

$$\big(\forall\,\langle x,u\rangle \in P_f\mathsf{W}_f\,.\;(\forall\, y\in f^*x\,.\,\rho(uy)) \longrightarrow \rho(\mathsf{w}_f\langle x,u\rangle)\big) \longrightarrow \forall\, t\in\mathsf{W}_f\,.\,\rho(t)\ .$$

Proof. Suppose ρ is a relation in W_f that satisfies

$$\forall\,\langle x,u\rangle \in P_f\mathsf{W}_f\,.\;(\forall\, y\in f^*x\,.\,\rho(uy)) \longrightarrow \rho(\mathsf{w}_f\langle x,u\rangle)\ . \tag{2.1}$$

Let $C = \{t\in\mathsf{W}_f \mid \rho(t)\}$. It is sufficient to show that the inclusion $\mathsf{i}_\rho\colon C\to\mathsf{W}_f$ is an isomorphism. Define a map $c\colon P_fC \to C$ by

$$c\,\langle x,u\rangle = \mathsf{o}_\rho(\mathsf{w}_f\langle x, \mathsf{i}_\rho\circ u\rangle)\ .$$

For c to be well defined, we must show that $\rho(\mathsf{w}_f\langle x,\mathsf{i}_\rho\circ u\rangle)$ for every $\langle x,u\rangle\in P_fC$. Since $u\colon f^*x\to C$, it follows that, for all $y\in f^*x$, $uy\in C$, therefore $\rho(\mathsf{i}_\rho(uy))$ and by (2.1) we get $\rho(\mathsf{w}_f\langle x,\mathsf{i}_\rho\circ u\rangle)$, as required.

By the Axiom of Inductive Spaces there exists a unique map $h\colon \mathsf{W}_f\to C$ such that, for all $\langle x,u\rangle\in\mathsf{W}_f$,

$$h(\mathsf{w}_f\langle x,u\rangle) = \mathsf{o}_\rho(\mathsf{w}_f\langle x,\mathsf{i}_\rho\circ h\circ u\rangle)\ .$$

The maps $\mathsf{i}_\rho\circ h$ and 1_{W_f} are equal because they both satisfy the same recursive equation, namely,

$$\begin{aligned}(\mathsf{i}_\rho\circ h)(\mathsf{w}_f\langle x,u\rangle) &= \mathsf{w}_f\langle x,(\mathsf{i}_\rho\circ h)\circ u\rangle\ ,\\ 1_{\mathsf{W}_f}(\mathsf{w}_f\langle x,u\rangle) &= \mathsf{w}_f\langle x, 1_{\mathsf{W}_f}\circ u\rangle\ .\end{aligned}$$

That $h\circ\mathsf{i}_\rho = 1_C$ follows immediately from injectivity of i_ρ. ∎

Example 2.2.3 (Natural Numbers) Consider the map $f\colon 1\to 2$, defined by $f\star = 1$. It is not hard to see that

$$P_fX = \textstyle\sum_{x:2}X^{f^*x} \cong X^{f^*0} + X^{f^*1} \cong 1+X\ .$$

Let $\mathbb{N}$ be the inductive space W_f. The structure map $\mathsf{w}_f\colon 1+\mathbb{N}\to\mathbb{N}$ has the form $\mathsf{w}_f = [\mathsf{z},\mathsf{s}]\colon 1+\mathbb{N}\to\mathbb{N}$ where $\mathsf{z}\colon 1\to\mathbb{N}$ and $\mathsf{s}\colon\mathbb{N}\to\mathbb{N}$.

The induction principle for $\mathbb{N}$ simplifies as follows. The outermost universal quantifier in the antecedent of the induction principle breaks up into two cases, one for each summand in the disjoint sum $1+\mathbb{N}\cong P_f\mathbb{N}$. The first case simplifies to $\rho(\mathsf{z}\star)$ and the second case simplifies to $\forall\, n\in\mathbb{N}\,.\,(\rho(n)\longrightarrow\rho(\mathsf{s}n))$. By putting this all together and writing $0 = \mathsf{z}\star$, we get the induction principle

$$(\rho(0)\wedge\forall\, n\in\mathbb{N}\,.\,(\rho(n)\longrightarrow\rho(\mathsf{s}n)))\longrightarrow\forall\, n\in\mathbb{N}\,.\,\rho(n)\ .$$

This is the well known induction principle for natural numbers. The space $\mathbb{N}$ is the space of *natural numbers*, 0 is the first natural number, and s is the successor map.

Definition by recursion reduces to the usual definition of maps on $\mathbb{N}$ by simple recursion: if $c_0 \in C$ and $c\colon C \to C$, then there exists a unique map $h\colon \mathbb{N} \to C$ such that $h(0) = c_0$ and $h(\mathsf{s}n) = c(hn)$ for all $n \in \mathbb{N}$.

Example 2.2.4 For a space A consider the canonical inclusion $\mathsf{inr}\colon A \to 1 + A$. It is not hard to see that $P_{\mathsf{inr}} X = 1 + A \times X$. The space of *finite sequences over* A is the inductive type $\mathsf{List}(A) = \mathsf{W}_{\mathsf{inr}\colon A \to 1+A}$. Every point $l \in \mathsf{List}(A)$ is either the *empty sequence* $[\,]$, or it can be written uniquely in the form $l = h{::}t$ where $h \in A$ and $t \in \mathsf{List}(A)$. We say that h is the *head* and t is the *tail* of l.

The space $\mathsf{List}^*(A)$ of *finite non-empty sequences over* A can be defined similarly as the inductive type for the canonical inclusion $\mathsf{inr}\colon A \to A + A$.

Proposition 2.2.5 *The structure map of an inductive space is an isomorphism.*

Proof. Define $h\colon \mathsf{W}_f \to P_f \mathsf{W}_f$, by recursion on W_f, to be the map for which

$$h(\mathsf{w}_f\langle x, u\rangle) = \langle x, \mathsf{w}_f \circ h \circ u\rangle \ .$$

Observe that both $1_{P_f \mathsf{W}_f}$ and $\mathsf{w}_f \circ h$ satisfy the same recursive definition, namely

$$\begin{aligned} 1_{P_f \mathsf{W}_f}(\mathsf{w}_f\langle x, u\rangle) &= \mathsf{w}_f\langle x, 1_{P_f \mathsf{W}_f} \circ u\rangle \ , \\ (\mathsf{w}_f \circ h)(\mathsf{w}_f\langle x, u\rangle) &= \mathsf{w}_f\langle x, (\mathsf{w}_f \circ h) \circ u\rangle \ . \end{aligned}$$

This means that they must be the same map, $\mathsf{w}_f \circ h = 1_{P_f \mathsf{W}_f}$. Now it also follows that

$$h(\mathsf{w}_f\langle x, u\rangle) = \langle x, (\mathsf{w}_f \circ h) \circ u\rangle = \langle x, u\rangle \ ,$$

therefore $h \circ \mathsf{w}_f = 1_{\mathsf{W}_f}$. The structure map w_f is an isomorphism because h is its inverse. ∎

By Proposition 2.2.5, every point $t \in \mathsf{W}_f$ can be written as $t = \mathsf{w}_f\langle x, u\rangle$ for a unique $\langle x, u\rangle \in P_f \mathsf{W}_f$. Thus, in W_f equality is characterized by

$$\mathsf{w}_f\langle x, u\rangle = \mathsf{w}_f\langle y, v\rangle \longleftrightarrow x = y \land u = v \ .$$

2.2.3 Coinductive Spaces

Recall that $P_f X = \sum_{x \in A} X^{f^* x}$, thus a map $h\colon X \to P_f X$ can be decomposed as $h = \langle h_0, h_1\rangle$ where $h_0 = \mathsf{fst} \circ h$ and $h_1 = \mathsf{snd} \circ h$.

Axiom 2.2.4 (Coinductive Spaces) *Let $f\colon B \to A$ be a map. There exists the* coinductive space M_f *and a structure map $\mathsf{m}_f\colon \mathsf{M}_f \to P_f \mathsf{M}_f$ such that for every map $\langle c_0, c_1\rangle\colon C \to P_f C$ there exists a unique map $h\colon C \to \mathsf{M}_f$ satisfying the corecursive equation*

$$\mathsf{m}_f(hx) = \langle c_0 x, h(c_1 x)\rangle$$

for all $x \in C$. We say that h is defined by corecursion on M_f.

Theorem 2.2.6 (Coinduction Principle) *For any relation ρ on a coinductive space M_f, the following* coinduction principle *holds:*

$$(\forall\, x, y \in \mathsf{M}_f \,.\, (\rho(x,y) \longrightarrow \mathsf{m}_{f,0}\, x = \mathsf{m}_{f,0}\, y \wedge \forall\, z \in f^*(\mathsf{m}_{f,0}\, x) \,.\, \rho((\mathsf{m}_{f,1}\, x)z, (\mathsf{m}_{f,1}\, y)z))) \longrightarrow \\ \forall\, x, y \in \mathsf{M}_f \,.\, (\rho(x,y) \longrightarrow x = y) \enspace .$$

Proof. Suppose ρ is a relation on M_f such that

$$\forall\, x, y \in \mathsf{M}_f \,.\, (\rho(x,y) \longrightarrow \mathsf{m}_{f,0}\, x = \mathsf{m}_{f,0}\, y \wedge \forall\, z \in f^*(\mathsf{m}_{f,0}\, x) \,.\, \rho((\mathsf{m}_{f,1}\, x)z, (\mathsf{m}_{f,1}\, y)z)) \enspace . \tag{2.2}$$

Let $C = \mathsf{M}_f/\rho$. Let $c\colon C \to P_f C$ be the map

$$c[t] = \langle \mathsf{m}_{f,0}\, t, \mathsf{q}_\rho \circ (\mathsf{m}_{f,1}\, t)\rangle \enspace .$$

Let $\sigma(u,v)$ be the binary relation on M_f, defined by

$$\sigma(u,v) \longleftrightarrow (\mathsf{m}_{f,0}\, u = \mathsf{m}_{f,0}\, v \wedge \mathsf{q}_\rho \circ (\mathsf{m}_{f,1}\, u) = \mathsf{q}_\rho \circ (\mathsf{m}_{f,1}\, v)) \enspace .$$

For c to be well defined we must show that $[u] = [v]$ implies $\sigma(u,v)$. Because σ is a stable equivalence relation, we only need to check that $\rho(u,v)$ implies $\sigma(u,v)$. So, if $\rho(u,v)$ then it follows from by (2.2) that $\mathsf{m}_{f,0}\, u = \mathsf{m}_{f,0}\, v$. Furthermore, $\rho((\mathsf{m}_{f,1}\, u)z, (\mathsf{m}_{f,1}\, v)z)$ holds for all $z \in f^*(\mathsf{m}_{f,0}\, u)$, therefore

$$[(\mathsf{m}_{f,1}\, u)z] = [(\mathsf{m}_{f,1}\, v)z] \enspace ,$$

for all $z \in f^*(\mathsf{m}_{f,0}\, u)$, from which we conclude that $\mathsf{q}_\rho \circ (\mathsf{m}_{f,1}\, u) = \mathsf{q}_\rho \circ (\mathsf{m}_{f,1}\, v)$, as required.

Let $h\colon C \to \mathsf{M}_f$ be defined from c by corecursion. It satisfies the corecursive equation

$$\mathsf{m}_f(h[t]) = \langle \mathsf{m}_{f,0}\, t, h \circ \mathsf{q}_\rho \circ (\mathsf{m}_{f,1}\, t)\rangle \enspace .$$

Both maps $h \circ \mathsf{q}_\rho$ and 1_{M_f} satisfy the corecursive equation, for all $t \in \mathsf{M}_f$,

$$\mathsf{m}_f((h \circ \mathsf{q}_\rho)t) = \langle \mathsf{m}_{f,0}\, t, (h \circ \mathsf{q}_\rho) \circ (\mathsf{m}_{f,1}\, t)\rangle \enspace , \\ \mathsf{m}_f(1_{\mathsf{M}_f} t) = \langle \mathsf{m}_{f,0}\, t, 1_{\mathsf{M}_f} \circ (\mathsf{m}_{f,1}\, t)\rangle \enspace ,$$

therefore they must be the same map $h \circ \mathsf{q}_\rho = 1_{\mathsf{M}_f}$. It is also the case that $\mathsf{q}_\rho \circ h = 1_{\mathsf{M}_f}$, because q_ρ is a surjection. Now for any $x, y \in \mathsf{M}_f$, if $\rho(x,y)$ then $[x] = [y]$, therefore $x = h[x] = h[y] = y$. ■

Proposition 2.2.7 *The structure map of a coinductive space is an isomorphism.*

Proof. The proof is left as an exercise. It is dual to the proof of Proposition 2.2.5. ■

Example 2.2.8 Let A be a space and consider the identity map $1_A\colon A \to A$. It is easy to see that $P_{1_A} X = A \times X$. Let $\mathsf{Stream}(A)$ be the coinductive type M_{1_A}. The structure map decomposes into the *head* map $\mathsf{hd}\colon \mathsf{Stream}(A) \to A$ and the *tail* map $\mathsf{tl}\colon \mathsf{Stream}(A) \to \mathsf{Stream}(A)$.

The coinduction principle simplifies as follows:

$$\forall\, x, y \in \mathsf{Stream}(A)\,.\,(\rho(x,y) \longrightarrow (\mathsf{hd}\, x = \mathsf{hd}\, y \wedge \rho(\mathsf{tl}\, x, \mathsf{tl}\, y))) \longrightarrow \\ \forall\, x, y \in \mathsf{Stream}(A)\,.\,(\rho(x,y) \longrightarrow x = y)\ .$$

Definition by corecursion gives us the common way of defining maps on an infinite stream. Given maps $c\colon C \to A$ and $d\colon C \to C$ there exists a unique map $f\colon C \to \mathsf{Stream}(A)$ such that, for all $x \in C$,

$$\mathsf{hd}(fx) = cx\ , \qquad \mathsf{tl}(fx) = f(dx)\ .$$

Let $\square :: \square\colon A \times \mathsf{Stream}(A) \to \mathsf{Stream}(A)$ be the inverse of the structure map. The corecursive definition of f can be written more succinctly as $fx = cx :: f(dx)$.

Example 2.2.9 For every natural number $n \geq 1$ let $[n] = \{k \in \mathbb{N} \mid k < n\} = \{0, 1, \ldots, n-1\}$, and let $B = \sum_{n \geq 1} [n]$. The space of *finitely branching spreads*, or *fans*, is the coinductive type $\mathsf{Fan} = \mathsf{M}_{\mathsf{fst}}$, where $\mathsf{fst}\colon B \to \mathbb{N}$ is the first projection. A fan can be pictured as a finitely branching tree that keeps branching on every path, as in Figure 2.1. The space Fan satisfies the recursive equation

$$\mathsf{Fan} = \textstyle\sum_{n \geq 1} \mathsf{Fan}^n = \mathsf{List}^*(\mathsf{Fan})\ ,$$

where $\mathsf{List}^*(A)$ is the space of non-empty finite sequences over A, as in Example 2.2.4. Thus every

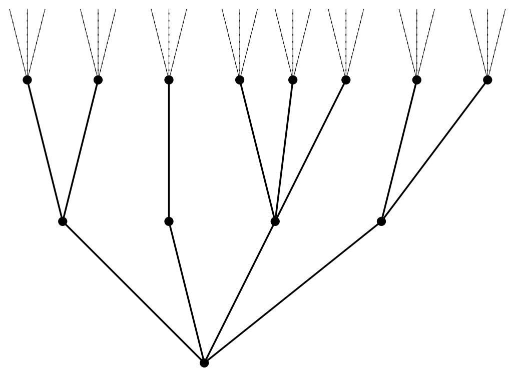

Figure 2.1: A Fan

fan $F \in \mathsf{Fan}$ can be uniquely written as a finite non-empty sequence $[F_0, \ldots, F_n]$ of fans, where $n+1$ is the *width* of F and is denoted by $\mathsf{wd}(F)$. We say that F_i is the *i-th subfan* of F. When $i \geq \mathsf{wd}(F)$ we define $F_i = F_{\mathsf{wd}(F)-1}$. This notation can be extended so that, for every finite sequence $[i_0, \ldots, i_{n-1}] \in \mathsf{List}(\mathbb{N})$, $F_{[i_0, \ldots, i_{n-1}]}$ is a fan, defined inductively by

$$F_{[\,]} = F\ , \qquad F_{[i_0, \ldots, i_n]} = (F_{i_0})_{[i_1, \ldots, i_n]}\ .$$

A *path* in F is an infinite sequence of numbers that tells which branch to choose at each level. More precisely, $p\colon \mathbb{N} \to \mathbb{N}$ is a path in F when $pn < \mathsf{wd}(F_{[p0, \ldots, p(n-1)]})$ for every $n \in \mathbb{N}$. For any $F \in \mathsf{Fan}$

we can define the space $\mathsf{Path}(F)$ of *paths* in F to be

$$\mathsf{Path}(F) = \left\{ p \in \mathbb{N}^{\mathbb{N}} \mid \forall n \in \mathbb{N} \,.\, pn < \mathsf{wd}(F_{[p0,\dots,p(n-1)]}) \right\} \ .$$

The space Fan is isomorphic to $\mathbb{N}^{\mathbb{N}}$, which can be seen as follows. The isomorphism $\Phi \colon \mathsf{Fan} \to \mathbb{N}^{\mathbb{N}}$ can be defined by

$$\begin{aligned} (\Phi F)0 &= \mathsf{wd}(F) - 1 \ , \\ (\Phi F)(n+1) &= (\Phi F_{n \bmod \mathsf{wd}(F)})(n \operatorname{div} \mathsf{wd}(F)) \ , \end{aligned}$$

where div and mod are the integer division and the remainder operations. We leave the details as exercise.

Proposition 2.2.10 *There is a unique map $e \colon \mathsf{W}_f \to \mathsf{M}_f$ such that, for all $\langle x, u \rangle \in P_f \mathsf{W}_f$,*

$$\langle x, e \circ u \rangle = \mathsf{m}_f(e(\mathsf{w}_f \langle x, u \rangle)) \ .$$

The map e is injective.

Proof. If we apply m_f^{-1} to both sides of (2.2.10), we see that it is simply a recursive definition of e:

$$e(\mathsf{w}_f \langle x, u \rangle) = \mathsf{m}_f^{-1} \langle x, e \circ u \rangle \ ,$$

Let us prove that e is injective by induction. Let $\langle x, u \rangle, \langle x', u' \rangle \in P_f \mathsf{W}_f$ and suppose $e(\mathsf{w}_f \langle x, u \rangle) = e(\mathsf{w}_f \langle x', u' \rangle)$. Then $\langle x, e \circ u \rangle = \langle x', e \circ u' \rangle$, hence $x = x'$ and $e \circ u = e \circ u'$. By induction hypothesis, for every $y \in f^* x$, $e(uy) = e(u'y)$ implies $uy = u'y$, or in other words, $e \circ u = e \circ u'$ implies $u = u'$. Since $e \circ u = e \circ u'$ by assumption, it follows that $u = u'$, as required.

Remark: we could have defined e corecursively like this, as well:

$$\mathsf{m}_f(et) = \langle \mathsf{fst}\,(\mathsf{w}_f^{(-1)} t), e \circ (\mathsf{snd}\,(\mathsf{w}_f^{(-1)} t)) \rangle \ .$$

∎

2.3 Computability, Decidability, and Choice

2.3.1 Computability

For every space A there is the *computability predicate* $\#_A$ on A. We read the statement $\#_A(x)$ as "x is computable" or "sharp x". We define the *computable part of A* to be the subspace

$$\#A = \left\{ x \in A \mid \#_A(x) \right\} \ .$$

Axiom 2.3.1 (Axiom of Computability)

(1) Every point of a computable part is computable:

$$\#\#A = \#A \ .$$

(2) *A computable map applied to a computable point gives a computable point:*

$$\#_{B^A}(f) \wedge \#_A(x) \longrightarrow \#_B(fx) \,. \qquad (f : B^A, x : A)$$

(3) *Let $\phi(x : A)$ be a predicate on A that does not depend on any parameters other than x. Suppose it has been proved that $\exists!\, x \in A \,.\, \phi(x)$. Then the point $\mathsf{the}\, x \in A \,.\, \phi(x)$ is computable:*

$$\text{if} \quad \exists!\, x \in A \,.\, \phi(x) \quad \text{is proved then} \quad \#_A(\mathsf{the}\, x \in A \,.\, \phi(x)) \,.$$

(4) *The following are computable: evaluation, pairing $\langle \Box, \Box \rangle$, canonical projections fst and snd, canonical inclusions inl and inr, subspace inclusions i_ρ, canonical quotient maps q_ρ, and the structure maps w_f and m_f.*

The third clause requires an explanation. It should be read as a meta-logical statement, i.e., if we construct a proof of a specific formula $\exists!\, x \in A \,.\, \phi(x)$, then the point $\mathsf{the}\, x \in A \,.\, \phi(x)$ is computable. The third clause is *not* the internal statement[8]

$$(\exists!\, x \in A \,.\, \phi(x)) \longrightarrow \#_A(\mathsf{the}\, x \in A \,.\, \phi(x)) \,.$$

It may seem that every point of every space is computable, by the following argument: if $x \in A$ then $\exists!\, y \in A \,.\, x = y$, therefore $\mathsf{the}\, y \in A \,.\, x = y$ is computable by the third clause, but $x = (\mathsf{the}\, y \in A \,.\, x = y)$ and so x is computable. However, we cannot use the third clause because x appears as a parameter in the formula $x = y$.

The spirit of the Axiom of Computability is captured by the slogan

"Definability implies computability."

Note, however, that in general computability is a more extensive notion than definability—there might be computable points that are not definable. The Axiom of Computability does not imply anything about the existence of *non-computable* points. In fact, there are categories of modest sets where everything is computable—those of the form $\mathsf{Mod}(\mathbb{A}, \mathbb{A})$. The non-computable points have the status of existing "potentially", as we cannot explicitly construct one in the internal logic. This is similar to the status of infinitesimals in synthetic differential geometry, where we know that there are infinitesimals but we cannot explicitly exhibit one in the internal logic.

Proposition 2.3.1 *Let $\rho(x : A, y : B)$ be a relation that does not depend on any parameters other than x and y. Suppose we have proved that $\forall\, x \in A \,.\, \exists!\, y \in B \,.\, \rho(x, y)$. Then the unique map determined by ρ is computable.*

Proof. If we have a proof of $\forall\, x \in A \,.\, \exists!\, y \in B \,.\, \rho(x, y)$ then we can also prove that

$$\exists!\, f \in B^A \,.\, \forall\, x \in A \,.\, \rho(x, fx) \,.$$

Now we can apply the third clause of the Axiom of Computability to conclude that the unique map $f \colon A \to B$ that satisfies $\forall\, x \in A \,.\, \rho(x, fx)$ is computable. ■

[8] In formal logic, we would express the third clause by saying: if $\vdash \exists!\, x \in A \,.\, \phi(x)$ then $\vdash \#_A(\mathsf{the}\, x \in A \,.\, \phi(x))$. This is different from claiming that $\vdash (\exists!\, x \in A \,.\, \phi(x)) \longrightarrow \#_A(\mathsf{the}\, x \in A \,.\, \phi(x))$.

Corollary 2.3.2 *Every map defined by λ-abstraction that does not depend on any free parameters is computable.*

Proof. Recall that λ-abstraction is derived as a special form of the Principle of Unique Choice. The corollary follows from Proposition 2.3.1 because of the assumption that there are no additional free parameters involved. ∎

For example, the map $\lambda x : A . x$ is computable. We cannot claim that if $y \in B$ then the map $\lambda x : A . \langle x, y \rangle$ is computable because y is a free parameter. This makes sense, because we do not know whether y could take on a non-computable value. However, by a λ-abstraction over y we obtain a computable map $\lambda y : B . \lambda x : A . \langle x, y \rangle$.

Proposition 2.3.3 *If $f : A \to B$ is computable and an isomorphism, then its inverse is computable.*

Proof. By Proposition 2.1.26, for every $y \in B$ there exists a unique $x \in A$ such that $fx = y$. By Proposition 2.3.1, the unique map $g : B \to A$ for which $f(gy) = y$ for all $y \in B$ is computable. But this map is the inverse of f. ∎

When A and B are isomorphic via a computable isomorphism, we say that they are *computably isomorphic*. Whenever we explicitly construct an isomorphism which does not depend on any free parameters it is automatically computable ("Definability implies computability").

We state some basic results about computability of maps at higher types. For this purpose we define several important maps of a higher types.

Let W_f be an inductive space for $f : B \to A$, and let C be a space. For every $c \in P_f C \to C$ there exists a unique $r_c : \mathsf{W}_f \to C$ such that $r_c(\mathsf{w}_f\langle x, u \rangle) = c\,\langle x, r_c \circ u \rangle$ for all $\langle x, u \rangle \in P_f \mathsf{W}_f$. We obtain a map $r_\Box : (P_f C \to C) \to (\mathsf{W}_f \to C)$, which is the higher-order "operation of defining a map by recursion". There is a similar operation of defining a map by corecursion.

There is the operation of factoring a map through a subspace, as in Theorem 2.1.17. Consider spaces A and B, and a subspace $A' = \{x \in A \mid \phi(x)\}$. Define the space

$$F = \{f \in B \to A \mid \forall y \in B . \phi(fy)\} \ .$$

By Theorem 2.1.17, for every $f \in F$ there exists a unique $\overline{f} : B \to A'$ such that $f = \mathsf{i}_\phi \circ \overline{f}$. Thus, we can define a map $\overline{\Box} : F \to (B \to A')$ which corresponds to the operation of factoring a function through a subspace. There is a similar map that corresponds to factoring of a function through a quotient space, as in Theorem 2.1.23.

Let $\mathsf{Iso}(A, B) = \{f \in B^A \mid \exists g \in A^B . (f \circ g = 1_B \wedge g \circ f = 1_A)\}$ be the space of isomorphisms between A and B. Because an isomorphism has a unique inverse there is the operation of taking an inverse $\Box^{-1} : \mathsf{Iso}(A, B) \to \mathsf{Iso}(B, A)$.

Proposition 2.3.4 *Computability of maps:*

(1) The operation of taking the inverse of an isomorphism is computable.

(2) The operation of defining a map by recursion is computable.

(3) The operation of defining a map by corecursion is computable.

(4) The operation of factoring a map through a subspace, as in Theorem 2.1.17, is computable.

(5) *The operation of factoring a map through a quotient space, as in Theorem 2.1.23, is computable.*

Proof. These maps are defined by Unique Choice, therefore computable by Proposition 2.3.1. ∎

Note that Proposition 2.3.4 is *not* of the form "if x is computable then fx is computable", but rather "f is computable". It then follows immediately that for a computable x also fx is computable.

Theorem 2.3.5 *The computable part of* 1 *is computably isomorphic to* 1*:*

$$\#1 = 1 \; .$$

The computable part of a product is computably isomorphic to the product of computable parts:

$$\#(A \times B) = \#A \times \#B \; , \qquad \#\textstyle\sum_{x \in B} A(x) = \sum_{x \in \#B} \#A(x) \; .$$

The computable part of a disjoint sum is computably isomorphic to the disjoint sum of the computable parts:

$$\#(A + B) = \#A + \#B \; .$$

Proof. Every point of 1 is equal to $(\mathsf{the}\, x \in 1 \,.\, (x = x))$, which is computable, therefore $\#1 = 1$.

Let $i_A \colon \#A \to A$ and $i_B \colon \#B \to B$ be the canonical subspace inclusions. Define a map $f \colon \#A \times \#B \to \#(A \times B)$ to be the unique factorization of the map $\langle i_A, i_B \rangle$ through $i_{A \times B} \colon \#(A \times B) \to A \times B$, i.e., for all $x \in \#A$, $y \in \#B$,

$$i_{A \times B}(f\langle x, y \rangle) = \langle i_A x, i_B y \rangle \; .$$

The map f is well defined because i_A, i_B, and pairing are computable, so that $\langle i_A x, i_B y \rangle$ is computable for all $x \in \#A$, $y \in \#B$. It is easy to check that f is an isomorphism whose inverse is the map $g \colon \#(A \times B) \to \#A \times \#B$, defined by

$$gz = \langle \mathsf{fst}\,(i_{A \times B} z), \mathsf{snd}\,(i_{A \times B} z) \rangle \; .$$

The map f is computable by Proposition 2.3.4(4). In the case of dependent sums we proceed similarly, except that we use the dependent projections instead.

To see that $\#A + \#B$ and $\#(A + B)$ are isomorphic, note that the map $i_A + i_B \colon \#A + \#B \to A + B$ factors through $i_{A+B} \colon \#(A + B) \to A + B$ as a map $f \colon \#A + \#B \to \#(A + B)$ satisfying, for all $z \in \#A + \#B$,

$$i_{A+B}(fz) = \begin{cases} i_A z & \text{if } z \in \#A \; , \\ i_B z & \text{if } z \in \#B \; . \end{cases}$$

The map f is an isomorphism whose inverse is the factorization of the inclusion $i_{A+B} \colon \#(A + B) \to A + B$ through the injective map $i_A + i_B$. It is computable by Proposition 2.3.4(4). ∎

Definition 2.3.6 A *computable space* is a space that is equal to its computable part.

For example, the empty space $\mathsf{0}$ is computable because the only subspace of $\mathsf{0}$ is $\mathsf{0}$ itself. More interesting is the next proposition.

Proposition 2.3.7 *The natural numbers are computable.*

Proof. The natural numbers $\mathbb{N}$ are an inductive space W_f for the map $f\colon 1 \to 2$, $f\star = 1$. Since f is computable, the zero element $\mathsf{z}\colon \mathsf{1} \to \mathbb{N}$ and the successor map $\mathsf{s}\colon \mathbb{N} \to \mathbb{N}$ are both computable because they comprise the structure map $[\mathsf{z}, \mathsf{s}]\colon \mathsf{1} + \mathbb{N} \to \mathbb{N}$, which is computable by the Axiom of Computability. We prove by induction that $\#_{\mathbb{N}}(n)$ for every $n \in \mathbb{N}$. The base case holds because z is computable, $\star$ is computable, and so $0 = \mathsf{z}\star$ is computable. The induction step is also easy: if n is computable, then $\mathsf{s}n$ is computable as well, because s is computable. ∎

2.3.2 Decidable Spaces

Definition 2.3.8 A predicate $\phi(x : A)$ is *decidable* when, for all $x \in A$, $\phi(x)$ or not $\phi(x)$.

Proposition 2.3.9 *The following are equivalent:*

(1) $\phi(x : A)$ *is a decidable predicate.*

(2) $A = \{x \in A \mid \phi(x)\} + \{x \in A \mid \neg\phi(x)\}$.

(3) There exists a map $c\colon A \to 2$ *such that,* $\forall\, x \in A\,.\,(\phi(x) \longleftrightarrow cx = 1)$.

Proof. Let $A_1 = \{x \in A \mid \phi(x)\}$ and $A_2 = \{x \in A \mid \neg\phi(x)\}$. Suppose $\phi(x : A)$ is a decidable predicate. The map $f = [\mathsf{i}_\phi, \mathsf{i}_{\neg\phi}]\colon A_1 + A_2 \to A$ is an isomorphism. It is injective because $fx = fy$ implies $x = y \in A_1$ or $x = y \in A_2$, and it is a quotient map because $\phi(x) \vee \neg\phi(x)$ holds by assumption. Thus f is an isomorphism, which shows that (1) implies (2).

To see that (2) implies (3), define the map $c\colon A_1 + A_2 \to 2$ to be $c = [\mathsf{inl}\, x \mapsto 1, \mathsf{inr}\, y \mapsto 0]$. It is obvious that, for all $x \in A_1 + A_2$, $cx = 1$ if, and only if, $x \in A_1$.

Finally, suppose (3) holds. Then we have, for all $x \in A$, $cx = 1 \longleftrightarrow \phi(x)$. But since $cx \neq 1$ is equivalent to $cx = 0$, we also get $cx = 0 \longleftrightarrow \neg\phi(x)$. Now (1) holds because $cx = 0 \vee cx = 1$ for all $x \in A$. ∎

Proposition 2.3.10 *The space* 2 *classifies decidable predicates. More precisely, there is a bijective correspondence between decidable predicates on a space* A *and maps* $A \to 2$*. If* $\phi(x : A)$ *is a decidable predicate, then the corresponding map* $f\colon A \to 2$ *satisfies* $\forall\, x \in A\,.\,(\phi(x) \longleftrightarrow fx = 1)$*. A map* $f\colon A \to 2$ *corresponds to the decidable predicate* $\phi(x) \equiv (fx = 1)$.

Proof. Suppose $\phi(x : A)$ is a decidable predicate. By Proposition 2.3.9, there exists a map $f\colon A \to 2$ such that, for all $x \in A$, $\phi(x) \longleftrightarrow fx = 1$. Uniqueness of f is a consequence of the fact that, for all $g, h\colon A \to 2$,

$$g = h \longleftrightarrow \forall\, x \in A\,.\,(gx = 1 \longleftrightarrow hx = 1)\ .$$

Given any $f\colon A \to 2$, the predicate $\phi(x) \cong (fx = 1)$ is decidable because equality on 2 is decidable. ∎

Definition 2.3.11 A *decidable space* is a space whose equality is a decidable relation.

Corollary 2.3.12 *The following are equivalent:*

(1) A is a decidable space.

(2) $A \times A = \{\langle x, y\rangle \in A \times A \mid x = y\} + \{\langle x, y\rangle \in A \times A \mid x \neq y\}$.

(3) Equality on A has a characteristic map $\mathsf{eq}_A \colon A \times A \to 2$, *which satisfies*

$$\mathsf{eq}_A\langle x, y\rangle = \begin{cases} 1 & \text{if } x = y\ , \\ 0 & \text{if } x \neq y\ . \end{cases}$$

Proof. This is Proposition 2.3.9 applied to equality on A. ∎

Proposition 2.3.13 *The space of natural numbers is decidable.*

Proof. Since the structure map $[\mathsf{z}, \mathsf{s}] \colon 1 + \mathbb{N} \to \mathbb{N}$ is an isomorphism, every natural number is either equal to 0, or is a successor of a natural number.

We prove by induction on $n \in \mathbb{N}$ that, for all $n, k \in \mathbb{N}$, $n = k$ or $n \neq k$. For the base case, consider any $k \in \mathbb{N}$. If $k = 0$ then $0 = k$, and if $k = \mathsf{s}k'$ for some $k' \in \mathbb{N}$ then $0 \neq k$. For the induction step, assume that for all $k \in \mathbb{N}$, $n = k$ or $n \neq k$. Consider any $k \in \mathbb{N}$. If $k = 0$ then clearly $\mathsf{s}n \neq k$, and if $k = \mathsf{s}k'$, then $\mathsf{s}n = \mathsf{s}k'$ if, and only if $n = k'$. By induction hypothesis we can decide whether $n = k'$ or $n \neq k'$. ∎

2.3.3 Choice Principles

Markov's Principle

Markov's Principle is *not* universally valid in intuitionistic logic. It is a rather special axiom of constructive mathematics that is not even accepted by all constructivists. We accept Markov's Principle simply because its interpretation in $\mathsf{Mod}(\mathbb{A}, \mathbb{A}_\sharp)$ is valid.

Axiom 2.3.2 (Markov's Principle) *For every* $f \colon \mathbb{N} \to 2$,

$$\neg\neg(\exists\, n \in \mathbb{N} \,.\, fn = 0) \longrightarrow \exists\, n \in \mathbb{N} \,.\, fn = 0\ .$$

Note: the map f may depend on free parameters.

Proposition 2.3.14 *The following are equivalent:*

(1) Markov's principle

(2) $(\neg\forall\, n \in \mathbb{N} \,.\, fn = 1) \longrightarrow \exists\, n \in \mathbb{N} \,.\, fn = 0$,

(3) If $\phi(x : \mathbb{N})$ is a decidable predicate,[9] *then* $(\neg\neg\exists\, n \in \mathbb{N} \,.\, \phi(x)) \longrightarrow \exists\, n \in \mathbb{N} \,.\, \phi(x)$.

[9] Note that we allow additional parameters to appear in the predicate ϕ.

Proof. The equivalence of (1) and (2) follows easily from the observation that $fn = 1$ is equivalent to $fn \neq 0$. The equivalence of (1) and (3) holds because 2 classifies decidable predicates, as explained in Proposition 2.3.10. ∎

Markov's principle simplifies several constructions in analysis. For example, the apartness relation and inequality on the space of real numbers coincide, cf. Proposition 5.5.19.

Projective Spaces

Definition 2.3.15 A space A is *projective* when for every binary relation ρ on B and for every map $f\colon A \to B/\rho$ there exists a map $\overline{f}\colon A \to B$ such that $fx = [\overline{f}x]_\rho$ for all $x \in A$.

Axiom 2.3.3 (Projective Spaces) *A binary product of projective spaces is projective. A dependent sum of projective spaces indexed by a projective space is projective. A regular subspace of a projective space is projective.*

Proposition 2.3.16 *The disjoint sum of projective spaces is a projective space.*

Proof. This is very easy and is left as an exercise. ∎

Definition 2.3.17 For spaces A and B, the *choice principle* $AC_{A,B}$ holds when, for every relation $\rho(x : A, y : B)$,

$$(\forall\, x \in A \,.\, \exists\, y \in B \,.\, \rho(x, y)) \longrightarrow \exists\, c \in B^A \,.\, \forall\, x \in A \,.\, \rho(x, cx)\ .$$

The map $c \in B^A$ above is called *a choice map* for ρ. The *choice principle* AC_A holds when $AC_{A,B}$ holds for every space B.

Theorem 2.3.18 *A space A is projective if, and only if,* AC_A *holds.*

Proof. Suppose AC_A holds. Let B be a space, ρ a binary relation on B, and $f\colon A \to B/\rho$ a map. For every $x \in A$ there exists $y \in B$ such that $fx = [y]$, because $\mathsf{q}_\rho\colon B \to B/\rho$ is a quotient map. By AC_A there exists a map $c\colon A \to B$ such that $fx = [cx]$ for all $x \in A$. Therefore, A is projective.

Conversely, suppose A is projective and for every $x \in A$ there exists $y \in B$ such that $\phi(x, y)$ holds. Define the space C by

$$C = \big\{ \langle x, y\rangle \in A \times B \bigm| \phi(x, y) \big\}\ .$$

Let $\sim$ be the equivalence relation on C, defined by

$$\langle x, y\rangle \sim \langle x', y'\rangle \longleftrightarrow x = x'\ .$$

Let $p\colon C/{\sim} \to A$ be the unique map for which $p[\langle x, y\rangle] = x$ for all $\langle x, y\rangle \in C$. For every $x \in A$ there is a unique $\xi \in C/{\sim}$ such that $x = p\xi$, namely $\xi = [\langle x, y\rangle]$ where y is such that $\phi(x, y)$. Therefore, there is a map $f\colon A \to C/{\sim}$ such that $x = p(fx)$ for all $x \in A$. Because A is projective, there exists a map $\overline{f}\colon A \to C$ such that $x = p[\overline{f}x]$ for all $x \in A$. This means that for all $x \in A$, $\mathsf{fst}\,(\overline{f}x) = x$ and $\phi(\mathsf{fst}\,(\overline{f}x), \mathsf{snd}\,(\overline{f}x))$. The map $\mathsf{snd} \circ \overline{f}$ is a choice map for ϕ. This proves AC_A. ∎

What projective spaces are there? Certainly the empty and the unit spaces are projective, and so is a finite disjoint sum $1 + \cdots + 1$ of copies of the unit space. The most important projective space is the space of natural numbers.

Axiom 2.3.4 (Number Choice) *The space of natural numbers* $\mathbb{N}$ *is projective.*

By Theorem 2.3.18, Number Choice is equivalent to the statement that for every relation $\rho(n : \mathbb{N}, x : A)$, if for all $n \in \mathbb{N}$ there exists $x \in A$ such that $\rho(n, x)$ holds, then there exists a choice map $c \in A^{\mathbb{N}}$ such that $\rho(n, cn)$ holds for every $n \in \mathbb{N}$.

Chapter 3

The Realizability Interpretation of the Logic in Modest Sets

3.1 The Interpretation of Logic

We present an interpretation, in a category of modest sets $\mathsf{Mod}(\mathbb{A}, \mathbb{A}_\sharp)$, of the language and the logic that were presented in Chapter 2. See Birkedal [Bir99, Appendix A] for a formal and detailed specification of the interpretation.

If X is an entity in the language of modest sets, such as a space, a map, or a formula, we denote its interpretation by $[\![X]\!]$.

A space A is interpreted as a modest set $[\![A]\!]$. We denote the underlying set of $[\![A]\!]$ by $|A|$.

Suppose $t(x_1 : A_1, \ldots, x_n : A_n) \in B$ is a point of B, where $x_1, \ldots, x_n$ are all the free parameters that t depends on. Then we interpret t as a morphism between modest sets

$$[\![t]\!] \colon [\![A_1]\!] \times \cdots \times [\![A_n]\!] \longrightarrow [\![B]\!] \; .$$

If $t \in B$ does not depend on any parameters then it is interpreted as a morphism $[\![t]\!] : 1 \to [\![B]\!]$.

A dependent type $A(x : B)$ is interpreted as a family of modest sets $\left\{[\![A(t)]\!] \;\middle|\; t \in |B|\right\}$. A point $t(x)$ of a dependent type $B(x : A)$ is interpreted as a *uniformly realizable* family of maps

$$\left\{[\![t(u)]\!] : [\![A]\!] \to [\![B(u)]\!] \;\middle|\; u \in |A|\right\} \; ,$$

which means that there exists a single realizer $a \in \mathbb{A}_\sharp$ such that for all $u \in |A|$ and $b \Vdash_A u$, $ab\!\downarrow \Vdash_{A \to B(u)} [\![t_u]\!]$.

We adopt the *realizability interpretation* of logic. An n-ary relation $\phi(x_1 : A_1, \ldots, x_n : A_n)$ is interpreted as a function $[\![\phi]\!] \colon [\![A_1]\!] \times \cdots [\![A_n]\!] \to \mathcal{P}\mathbb{A}$. In order to keep the notation simple we only show how unary relations $\phi(x : A)$ are interpreted. The generalization to n-ary relations is straightforward—just replace $[\![A]\!]$ by the product $[\![A_1 \times \cdots \times A_n]\!]$ and the variable x with a tuple of variables $\langle x_1, \ldots, x_n \rangle$.

If $\phi(x : A)$ is a predicate on $[\![A]\!]$ then its interpretation is a function $[\![\phi]\!] \colon |A| \to \mathcal{P}\mathbb{A}$. For $t \in |A|$ we write $[\![\phi(t)]\!]$ instead of $[\![\phi]\!]t$. When $a \in [\![\phi(t)]\!]$ we say that *a realizes* $\phi(t)$ and write $a \Vdash_\phi \phi(t)$, or usually just $a \Vdash \phi(t)$. The function $[\![\phi]\!]$ and the realizability relation $\Vdash_\phi$ can be defined in terms of each other one via the equivalence

$$a \in [\![\phi(t)]\!] \iff a \Vdash_\phi \phi(t) \; . \qquad ((t \in |A|, a \in \mathbb{A}))$$

We prefer to specify the interpretation in terms of realizability relations.

A statement $\phi(x : A)$ is *valid* in $\mathsf{Mod}(\mathbb{A}, \mathbb{A}_\sharp)$, written $\mathsf{Mod}(\mathbb{A}, \mathbb{A}_\sharp) \models \phi(x)$, when

$$\mathbb{A}_\sharp \cap [\![\forall x \in A \,.\, \phi(x)]\!] \neq \emptyset \;.$$

In other words, there exists a *computable* realizer $a \in \mathbb{A}_\sharp$ which realizes $\phi(x)$ uniformly in x.

The realizability interpretation is equivalent to the standard categorical interpretation where predicates are interpreted as subobjects. To see this, we show that there is a natural bijective correspondence between the subobjects of a modest set A and (equivalence classes of) functions $|A| \to \mathcal{P}\mathbb{A}$. Here functions $\phi, \psi \colon |A| \to \mathcal{P}A$ are considered equivalent if there exist $a, b \in \mathbb{A}_\sharp$ such that

$$\forall x \in |A| \,.\, (c \in \phi(x) \Longrightarrow ac{\downarrow} \in \psi(x)) \;,$$
$$\forall x \in |A| \,.\, (c \in \psi(x) \Longrightarrow bc{\downarrow} \in \phi(x)) \;.$$

A subobject represented by a monomorphism $i \colon B \rightarrowtail [\![A]\!]$ corresponds to the equivalence class of the function $\phi_B \colon |A| \to \mathcal{P}\mathbb{A}$ defined by

$$\phi_B t = \bigcup_{u \in i^* t} \mathsf{E}_B u \;,$$

and an equivalence class represented by a function $\phi \colon |A| \to \mathcal{P}\mathbb{A}$ corresponds to the subobject $i \colon B_\phi \rightarrowtail A$ where B_ϕ is the modest set defined by

$$|B_\phi| = \{t \in |A| \mid \phi(t) \neq \emptyset\} \;, \qquad \langle a, b\rangle \Vdash_{B_\phi} t \iff a \Vdash_A t \wedge b \Vdash_\phi \phi(t) \;.$$

The monomorphism $i \colon B_\phi \rightarrowtail A$ is the canonical subset inclusion. It is realized by the combinator fst. We omit the verification that this establishes the desired one-one correspondence.

The interpretation of formulas is given by an inductive definition on the structure of a formula. Let $\phi(x : A)$ and $\psi(x : A)$ be predicates on a space A and $t \in |A|$.

1. $a \Vdash \mathsf{true}$ for every $a \in \mathbb{A}$.
2. $a \Vdash \mathsf{false}$ for no $a \in \mathbb{A}$.
3. $\langle a, b\rangle \Vdash \phi(t) \wedge \psi(t)$ if, and only if, $a \Vdash \phi(t)$ and $b \Vdash \psi(t)$.
4. $a \Vdash \phi(t) \longrightarrow \psi(t)$ if, and only if, whenever $b \Vdash \phi(t)$ then $ab{\downarrow} \Vdash \psi(t)$.
5. $\langle a, b\rangle \Vdash \phi(t) \vee \psi(t)$ if, and only if, $a = \mathsf{false}$ and $b \Vdash \phi(t)$, or $a = \mathsf{true}$ and $b \Vdash \psi(t)$.

Suppose $\phi(x : A, y : B)$ is a relation on $A \times B$ and $t \in |A|$. The realizability interpretation of the existential and universal quantifiers is as follows:

6. $\langle a, b\rangle \Vdash \exists\, y \in B \,.\, \phi(t, y)$ if, and only if, there exists $u \in |B|$ such that $a \Vdash_B u$ and $b \Vdash \phi(t, u)$.
7. $a \Vdash \forall\, y \in B \,.\, \phi(t, y)$ if, and only, if for all $u \in |B|$, whenever $b \Vdash_B u$ then $ab{\downarrow} \Vdash \phi(t, u)$.

If A is a space and $u, t \in |A|$ then equality $=_A$ on A is interpreted as

8. $\langle a, b\rangle \Vdash t = u$ if, and only if, $t = u$, $a \Vdash_A t$ and $b \Vdash_A u$.

The above definition is given in terms of realizability relations. An equivalent interpretation in terms of functions into $\mathcal{P}\mathbb{A}$ is given below:

$$
\begin{aligned}
[\![\mathsf{true}]\!] &= \mathbb{A}\ , \\
[\![\mathsf{false}]\!] &= \emptyset\ , \\
[\![\phi(t) \wedge \psi(t)]\!] &= \left\{\langle a, b\rangle \in \mathbb{A} \;\middle|\; a \in [\![\phi(t)]\!] \wedge b \in [\![\psi(t)]\!]\right\}\ , \\
[\![\phi(t) \longrightarrow \psi(t)]\!] &= \left\{a \in \mathbb{A} \;\middle|\; \forall\, b \in [\![\phi(t)]\!] \,.\, (ab{\downarrow} \in [\![\psi(t)]\!])\right\}\ , \\
[\![\phi(t) \vee \psi(t)]\!] &= \left\{\langle \mathsf{false}, a\rangle \in \mathbb{A} \;\middle|\; a \in \phi(t)\right\} \cup \left\{\langle \mathsf{true}, b\rangle \in \mathbb{A} \;\middle|\; b \in \psi(t)\right\}\ , \\
[\![\exists\, y \in B \,.\, \phi(t, y)]\!] &= \left\{\langle a, b\rangle \in \mathbb{A} \;\middle|\; \exists\, u \in |B| \,.\, (a \Vdash_B u \wedge b \in [\![\phi(t, u)]\!])\right\}\ , \\
[\![\forall\, y \in B \,.\, \phi(t, y)]\!] &= \left\{a \in \mathbb{A} \;\middle|\; \forall\, u \in |B| \,.\, \forall\, b \in \mathbb{A} \,.\, (b \Vdash_B u \Longrightarrow ab{\downarrow} \in [\![\phi(t, u)]\!])\right\}\ , \\
[\![t =_A u]\!] &= \left\{\langle a, b\rangle \in \mathbb{A} \;\middle|\; u = t \wedge a \Vdash_A t \wedge b \Vdash_A u\right\}
\end{aligned}
$$

Since $\neg\phi$ is defined as $\phi \longrightarrow \mathsf{false}$, the realizability of negation comes to the following: $a \Vdash \neg\phi(t)$ if, and only if, whenever $b \Vdash \phi(t)$ then $ab{\downarrow} \Vdash \mathsf{false}$. We can write this more clearly in terms of a function $[\![\neg\phi]\!] \colon |A| \to \mathcal{P}\mathbb{A}$ as

$$
[\![\neg\phi(t)]\!] = \begin{cases} \mathbb{A} & \text{if } [\![\phi(t)]\!] = \emptyset\ , \\ \emptyset & \text{if } [\![\phi(t)]\!] \neq \emptyset\ . \end{cases}
$$

If $\phi(x : A)$ is a stable predicate, then it is equivalent to $\neg\neg\phi(x)$, which means that for a fixed $t \in |A|$, $\phi(t)$ is either realized by every element of $\mathbb{A}$, or by none at all. This often simplifies matters.

As an example of how this works, let us verify validity of the Axiom of Stability, i.e., we must exhibit a realizer $e \in \mathbb{A}_\sharp$ for the statement

$$
\forall\, \langle x, y\rangle \in A \times A \,.\, (\neg\neg(x = y) \longrightarrow x = y)\ .
$$

Such a realizer e takes as an argument a pair $\langle a, b\rangle$ of realizers for $\langle u, t\rangle \in |A| \times |A|$. If $u = t$, $e\langle u, t\rangle$ applied to *any* realizer must give a realizer for $u = t$. If $u \neq t$, then there is no further condition on e because in this case there are no realizers for $\neg\neg(u = t)$. A moment's thought shows that $e = \lambda^* u\, v \,.\, u$ does the job.

It is worthwhile spelling out the interpretation of unique existence. Recall that the definition of $\exists!\, x \in A \,.\, \phi(x)$ is

$$
(\exists\, x \in A \,.\, \phi(x)) \wedge \forall\, x, y \in A \,.\, (\phi(x) \wedge \phi(y) \longrightarrow x = y)\ .
$$

A realizer for this statement is a pair $\langle a, b\rangle$ such that

$$
a \Vdash \exists\, x \in A \,.\, \phi(x)\ , \qquad b \Vdash \forall\, x, y \in A \,.\, (\phi(x) \wedge \phi(y) \longrightarrow x = y)\ .
$$

Thus $a = \langle a_1, a_2\rangle$ where $a_1 \Vdash_A t$ for some $t \in |A|$ and $a_2 \Vdash \phi(t)$. Since $\phi(x) \wedge \phi(y) \longrightarrow x = y$ is a stable predicate, the realizer for b is just a dummy that witnesses uniqueness. This means that we can think of a realizer for the statement $\exists!\, x \in A \,.\, \phi(x)$ as a triple $\langle a_1, a_2, b\rangle$ where $a_1 \Vdash_A t$ for some $t \in |A|$, $a_2 \Vdash \phi(t)$, and b is a dummy witness of the uniqueness of t. The description operator $\mathsf{the}\, x \in A \,.\, \phi(x)$ is interpreted as the function $t \colon 1 \to [\![A]\!]$. It is realized by $\mathsf{K}(\mathsf{fst}\, a)$, where a is a realizer for $\exists\, x \in A \,.\, \phi(x)$. If additional parameters are present, say $\exists!\, x \in A \,.\, \phi(x, y : B)$ is provable, then $\mathsf{the}\, x \in A \,.\, \phi(x, y)$ is interpreted as the function $[\![B]\!] \to [\![A]\!]$ that maps every $u \in |B|$ to the unique $t \in |A|$ for which $\phi(t, u)$ is valid.

Consider a realizer $a \in \mathbb{A}$ for a $\forall\exists$ statement

$$\forall\, x \in A \,.\, \exists\, y \in B \,.\, \phi(x, y) \;.$$

Applied to any realizer $b \Vdash_A t$ for any $t \in |A|$, ab is a pair $\langle c_1, c_2 \rangle$ such that $c_1 \Vdash_B u$ for some $u \in |B|$ and $c_2 \Vdash \phi(t, u)$. We see that a is described by a pair of realizers $a_1 = \lambda^* b.\,(\mathsf{fst}\,(ab))$ and $a_1 = \lambda^* b.\,(\mathsf{fst}\,(ab))$. Then for every $t \in |A|$ and $b \Vdash_A t$ we get a realizer $a_1 b \Vdash u$ for some $u \in |B|$ and a realizer $a_2 b \Vdash \phi(t, u)$.

A logical formula ϕ built from equality, conjunction, implication, negation and universal quantification is called a *negative formula*. By induction on the structure of the formula we can prove that a negative formula is stable. Indeed, if $\phi \equiv t = u$ then ϕ is stable by the Axiom of Stability; if $\phi \equiv \psi \wedge \rho$ then $\neg\neg(\psi \wedge \rho) \longrightarrow \neg\neg\psi \wedge \neg\neg\rho \longrightarrow \psi \wedge \rho$, where we used the induction hypothesis in the last implication; if $\phi \equiv (\psi \longrightarrow \rho)$ the proof is similar; the case $\phi \equiv \neg\psi$ is obvious; if $\phi \equiv \forall\, x \in A \,.\, \psi(x)$ then $\neg\neg(\forall\, x \in A \,.\, \psi(x)) \longrightarrow \forall\, x \in A \,.\, (\neg\neg\psi(x)) \longrightarrow \forall\, x \in A \,.\, \psi(x)$ where the last step follows by the induction hypothesis.

Now suppose $\phi(x : A)$ is a negative formula. Then it has a realizability interpretation $[\![\phi]\!] : |A| \to \mathcal{P}\mathbb{A}$, and it can also be interpreted in classical logic as a statement about the underlying set $|A|$, i.e., as a set-theoretic statement. For example, the formula

$$\forall\, x, y \in A \,.\, (fx = fy \longrightarrow x = y)$$

interpreted set-theoretically says that the function $f \colon |A| \to |B|$ is injective.

Theorem 3.1.1 *A negative formula is valid in the realizability interpretation if, and only if, it is true when interpreted set-theoretically.*

Proof. Suppose $\phi(x : A)$ is a negative formula. Its realizability interpretation is a function $[\![\phi]\!] \colon |A| \to \mathcal{P}\mathbb{A}$, whereas the set-theoretic interpretation can be thought of as a function $S(\phi) \colon |A| \to \{\mathsf{false}, \mathsf{true}\}$. In the realizability interpretation ϕ is valid when there exists $a \in \mathbb{A}_\sharp$ such that for all $x \in |A|$ and $b \Vdash_A x$,

$$ab{\downarrow} \in [\![\phi(x)]\!] \;,$$

and it is set-theoretically valid when

$$\forall\, x \in |A| \,.\, (S(\phi)x = \mathsf{true}) \;.$$

It is generally the case that validity in $\mathsf{Mod}(\mathbb{A}, \mathbb{A}_\sharp)$ implies set-theoretic validity. We show the converse by induction on the structure of ϕ.

Suppose ϕ is the formula $t(x) = u(x)$ where $t(x), u(x) \in B$. The terms t and u are interpreted as morphisms $[\![t]\!], [\![u]\!] \colon [\![A]\!] \to [\![B]\!]$. Let $t_0, u_0 \in \mathbb{A}_\sharp$ be realizers for $[\![t]\!]$ and $[\![u]\!]$, respectively. If $S(t(x) = u(x)) = \mathsf{true}$ then $[\![t]\!]$ and $[\![u]\!]$ are the same function $|A| \to |B|$ and it is easy to see that $t(x) = u(x)$ is realized by $\lambda^* a.\, \langle t_0 a, u_0 a \rangle$.

Suppose ϕ is the formula $\psi(x) \wedge \rho(x)$. If $S(\psi(x) \wedge \rho(x)) = \mathsf{true}$ then $S(\psi(x)) = \mathsf{true}$ and $S(\rho(x)) = \mathsf{true}$. By induction hypothesis this implies validity of $\psi(x)$ and $\rho(x)$, which in turn implies validity of $\psi(x) \wedge \rho(x)$.

Consider the case when ϕ is the formula $\psi(x) \longrightarrow \rho(x)$. Suppose $S(\psi(x) \to \rho(x)) = \mathsf{true}$. We claim that $\neg\neg\psi(x) \longrightarrow \neg\neg\phi(x)$ is valid because it is realized by, say, $a = \mathsf{K}(\mathsf{KK})$. Any combinator will do, as long as it has the property that $abc{\downarrow}$ for all $b, c \in \mathbb{A}$. Indeed, suppose $b \Vdash_A x$ and

$c \Vdash \neg\neg\psi(x)$. This implies $S(\neg\neg\psi(x)) = \mathsf{true}$, therefore $S(\neg\neg\rho(x)) = \mathsf{true}$ and $\neg\neg\rho(x)$ is valid, which means that abc realizes $\neg\neg\rho(x)$, as required.

If ϕ is the formula $\neg\psi(x)$ then we proceed as follows. Suppose $S(\neg(\phi(x))) = \mathsf{true}$. To establish that $\neg(\phi(x))$ is valid we just need to show that for any given $t \in A$, $\phi(t)$ does not have any realizers. But if it did, that would imply $S(\phi(t)) = \mathsf{true}$, which contradicts $S(\neg\phi(t)) = \mathsf{true}$.

Finally, consider the case when ϕ is the formula $\forall\, x \in A \,.\, \psi(x)$. Observe that $\forall\, x \in A \,.\, \psi(x)$ is valid if, and only if, $\psi(x)$ is valid. Now use the induction hypothesis. ■

3.2 Simple Types

3.2.1 Function Spaces

A function space B^A is interpreted by the exponential $[\![B]\!]^{[\![A]\!]}$. Suppose the map $f \colon A \to B$ and the point $x \in A$ are interpreted as a pair of morphisms $[\![f]\!] \colon \cdot \to [\![B^A]\!]$ and $[\![x]\!] \colon \cdot \to [\![A]\!]$.[1] The application of f to x is interpreted by the composition

$$\cdot \xrightarrow{\langle f, x\rangle} [\![B]\!]^{[\![A]\!]} \times [\![A]\!] \xrightarrow{\quad \mathsf{ev} \quad} [\![B]\!] \;,$$

where ev is the evaluation morphism for $[\![B]\!]^{[\![A]\!]}$.

Let us verify the validity of the principle of Unique Choice,

$$(\forall\, x \in A \,.\, \exists!\, y \in B \,.\, \phi(x,y)) \longrightarrow \exists!\, f \in B^A \,.\, \forall\, x \in A \,.\, \phi(x, fx) \;. \tag{3.1}$$

A realizer for the antecedent of (3.1) can be thought of as a pair $\langle a_1, a_2\rangle$ such that, for all $t \in |A|$ and $b \Vdash_A t$, $a_1 b \Vdash u$ for some $u \in |B|$ and $a_2 b \Vdash \phi(t,u)$. In addition, the existence of such a realizer also witnesses the uniqueness of u, i.e., if b' and b realize the same element of $|A|$, then $a_1 b$ and $a_1 b'$ realize the same element of $|B|$. This is exactly the condition that a_1 must satisfy in order to track a function $f \colon |A| \to |B|$. The function f can be recovered from a_1 by

$$ft = u \iff \exists\, b \in \mathsf{E}_A t \,.\, (a_1 b \Vdash_B u) \;.$$

It follows that $\lambda^*\langle a_1, a_2\rangle .\, (\langle a_1, \lambda^* b .\, (a_2(a_1 b))\rangle)$ is a realizer for (3.1). It remains to be seen that the choice function f is unique, provided that the antecedent has a realizer. We leave the easy verification as an exercise.

3.2.2 Products

The product space $A \times B$ is interpreted as the binary product of $[\![A]\!]$ and $[\![B]\!]$,

$$[\![A \times B]\!] = [\![A]\!] \times [\![B]\!] \;.$$

The canonical projections $\mathsf{fst} \colon A \times B \to A$ and $\mathsf{snd} \colon A \times B \to B$ are interpreted as the first and second projection from $[\![A]\!] \times [\![B]\!]$, respectively. The Axiom of Products is valid because its interpretation just states the universal properties of binary products.

[1] For the purposes of this argument it is not important what the domain of $[\![f]\!]$ and $[\![x]\!]$ is. In such cases we just use a dot instead of an arbitrary letter.

3.2.3 Disjoint Sums

The disjoint sum $A+B$ is interpreted as the binary coproduct $[\![A]\!]+[\![B]\!]$,

$$[\![A+B]\!] = [\![A]\!]+[\![B]\!] \ .$$

The canonical inclusion maps $\mathsf{inl}\colon A \to A+B$ and $\mathsf{inr}\colon A \to A+B$ are interpreted as the canonical inclusions. The Axiom of Disjoint Sums is valid because its interpretation just states the universal properties of binary coproducts.

3.2.4 The Empty and the Unit Spaces

The empty space $\mathsf{0}$ is interpreted as the initial object of $\mathsf{Mod}(\mathbb{A}, \mathbb{A}_\sharp)$, which is the empty modest set $\emptyset$.

The unit space $\mathsf{1}$ is interpreted as the terminal object of $\mathsf{Mod}(\mathbb{A}, \mathbb{A}_\sharp)$, and the unit $\star$ is interpreted as the unique morphism $1 \to 1$.

3.2.5 Subspaces

A subspace $\{x \in A \mid \phi(x)\}$ is interpreted as the modest set

$$\bigl|\{x \in A \mid \phi(x)\}\bigr| = \{t \in |A| \mid \exists\, b \in \mathbb{A} \,.\, (b \Vdash \phi(t))\} \ ,$$

with the realizability relation

$$\langle a, b\rangle \Vdash x \iff a \Vdash_A x \wedge b \Vdash \phi(x) \ . \tag{3.2}$$

The canonical subspace inclusion i_ϕ is interpreted as the canonical subset inclusion

$$\bigl|\{x \in A \mid \phi(x)\}\bigr| \hookrightarrow |A| \ ,$$

and is realized by the first projection combinator fst.

Proposition 3.2.1 *A map $f\colon A \to B$ is injective if, and only if, its interpretation is a mono.*

Proof. The statement $\forall\, x, y \in A \,.\, (fx = fy \longrightarrow x = y)$ is a negative formula, therefore we can interpret it set-theoretically by Theorem 3.1.1. It says that $[\![f]\!]\colon |A| \to |B|$ is an injective function, which is exactly what it takes for it to be monic. ∎

Proposition 3.2.2 *A map $f\colon A \to B$ is an embedding if, and only if, it is interpreted as a regular mono.*

Proof. It is sufficient to show that a canonical subspace inclusion i_ϕ is interpreted as a regular mono if, and only if, $\phi(x)$ is stable. If $\phi(x)$ is stable then the realizability relation (3.2) for $\bigl|\{x \in A \mid \phi(x)\}\bigr|$ can be replaced by one that is defined by

$$a \Vdash x \iff a \Vdash_A x$$

because if any element of $\mathbb{A}$ realizes $\phi(x)$ then every element of $\mathbb{A}$ does. This shows that $[\![\mathsf{i}_\phi]\!]$ is a regular mono. For the converse, suppose $[\![\mathsf{i}_\phi]\!]$ is a regular mono. Then it is isomorphic to a morphism $i\colon S \to T$ where $i\colon |S| \to |T|$ is a subset inclusion and E_S is the restriction of E_T to $|S|$. We only need to verify that the predicate $\exists\, s \in S \,.\, is = t$ is stable. This is indeed the case, since for every $s \in |S|$, $a \Vdash_S s$ if, and only if, $a \Vdash_T s$. ∎

3.2.6 Quotient Spaces

Let A be a space and $\rho(x : A, y : A)$ a binary relation on A. The relation ρ can be interpreted as a function $[\![\rho]\!] \colon |A| \times |A| \to \mathcal{P}\mathbb{A}$, or as a subobject $r \colon R \rightarrowtail [\![A]\!] \times [\![A]\!]$. The subobject R is the modest set

$$|R| = \left\{ \langle x, y \rangle \in |A| \times |A| \;\middle|\; [\![\rho(x, y)]\!] \neq \emptyset \right\}$$

with the realizability relation

$$\langle a, b, c \rangle \Vdash \langle x, y \rangle \iff a \Vdash_A x \wedge b \Vdash_A y \wedge c \Vdash \rho(x, y) \ .$$

The map $r \colon R \to [\![A]\!] \times [\![A]\!]$ is the canonical subset inclusion and is realized by $\lambda^* \langle a, b, c \rangle . \langle a, b \rangle$. Let $r_1 = \mathsf{fst} \circ r$ and $r_2 = \mathsf{snd} \circ r$. The quotient space A/ρ and the canonical quotient map $\mathsf{q}_\rho \colon A \to A/\rho$ are interpreted as shown in the following coequalizer diagram:

$$R \underset{r_2}{\overset{r_1}{\rightrightarrows}} [\![A]\!] \xrightarrow{[\![\mathsf{q}_\rho]\!]} [\![A/\rho]\!] \ .$$

We can describe $[\![A/\rho]\!]$ explicitly as follows. Let $\sim$ be the smallest equivalence relation on $|A|$ that satisfies, for all $x, y \in |A|$,

$$[\![\rho(x, y)]\!] \neq \emptyset \quad \Longrightarrow \quad x \sim y \ .$$

Then $|A/\rho| = |A|/{\sim}$, and the existence predicate on $|A/\rho|$ is

$$\mathsf{E}_{A/\rho}[x] = \bigcup_{y \sim x} \mathsf{E}_A y \ .$$

The quotient map $[\![\mathsf{q}_\rho]\!] \colon |A| \to |A/\rho|$ is the canonical quotient map $x \mapsto [x]_\sim$. It is realized by the combinator I.

Proposition 3.2.3 *A map $f \colon A \to B$ is surjective if, and only if, its interpretation is an epi.*

Proof. The statement $\forall\, y \in B \,.\, \neg\neg\exists\, x \in A \,.\, fx = y$ is equivalent to the statement

$$\forall\, y \in B \,.\, \neg\forall\, x \in A \,.\, fx \neq y \ ,$$

which is a negative formula, therefore we can interpret it set-theoretically. It says that $[\![f]\!] \colon |A| \to |B|$ is a surjective function, which is exactly what it takes for it to be epi. ∎

Proposition 3.2.4 *A map $f \colon A \to B$ is a quotient map if, and only if, it is interpreted as a regular epi.*

Proof. By Proposition 2.1.24, f is a quotient map if, and only if, it is isomorphic to a canonical quotient map $\mathsf{q}_\rho \colon A \to A/\rho$, so it suffices to check that every canonical quotient map is interpreted as a regular epi, and that every regular epi arises as the interpretations of a canonical open map. The first part is trivial because the interpretation of a canonical quotient map was defined to be a coequalizer. Conversely, suppose $q \colon A \to B$ is a regular epi. Then it is the coequalizer of its kernel pair

$$\begin{tikzcd} R \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] & A \arrow[r, "q"] & B \end{tikzcd} .$$

By construction of pullbacks in $\mathsf{Mod}(\mathbb{A}, \mathbb{A}_\sharp)$, as described in Section 1.3, the map $\langle r_1, r_2 \rangle \colon R \to A \times A$ is a regular mono. In fact, the subobject $\langle r_1, r_2 \rangle \colon R \to A \times A$ is an equivalence relation, which is not hard to see from the explicit construction of the kernel pair $\langle r_1, r_2 \rangle$. Thus, the map q is isomorphic to the interpretation of the canonical quotient map $\mathsf{q}_R \colon A \to A/R$. ∎

3.3 Complex Types

3.3.1 Dependent Sums and Products

Recall that a dependent type $A(i : I)$ is interpreted as a family of modest sets $\{ [\![A(i)]\!] \mid i \in |I| \}$. The interpretation of dependent sums and products suggests itself—the dependent sums and products are interpreted by their categorical versions:

$$[\![\textstyle\sum_{i : I} A(i)]\!] = \textstyle\sum_{i \in |I|} [\![A(i)]\!] , \qquad [\![\textstyle\prod_{i : I} A(i)]\!] = \textstyle\prod_{i \in |I|} [\![A(i)]\!] .$$

The two lines above are of course just an outline of how the dependent types are supposed to be interpreted. For a detailed explanation we would have to introduce quite a bit of formal type-theoretic machinery. This would obfuscate the main idea of dependent types, namely that they are just families of modest sets and that their interpretation really is quite natural. In case there should be any doubts about the interpretation, we use [Bir99, Appendix A] as a reference.

3.3.2 Inductive and Coinductive Spaces

Inductive and coinductive types are interpreted by their categorical versions:

$$[\![\mathsf{W}_f]\!] = \mathsf{W}_{[\![f]\!]} , \qquad [\![\mathsf{M}_f]\!] = \mathsf{M}_{[\![f]\!]} .$$

The Axiom of Inductive Types is valid because its interpretation states that $[\![\mathsf{W}_f]\!]$ is an initial algebra for the polynomial functor $P_{[\![f]\!]}$. The Axiom of Coinductive Types is valid for the dual reason.

In Section 1.1 we defined the Curry numerals and in Example 2.2.3 we defined the space of natural numbers $\mathbb{N}_I$ as an inductive type.[2] Let us show that $\mathbb{N}_I$ is isomorphic to the modest set of Curry numerals $\mathbb{N}_C$. Here $\mathbb{N}_C$ is the modest set whose underlying set is the set of natural numbers, $|\mathbb{N}_C| = \mathbb{N}$, and the existence predicate is defined by $\mathsf{E}_{\mathbb{N}_C} n = \{\overline{n}\}$. Recall that $\overline{n}$ is the n-th Curry numeral. In order to show that $[\![\mathbb{N}_I]\!]$ is isomorphic $\mathbb{N}_C$, we only need to show that $\mathbb{N}_C$ is an initial algebra for the polynomial functor $P_{1 \to 2} X = 1 + X$.

[2] We are using the subscript I to denote the inductively defined natural numbers, and C to denote the modest set of Curry numerals.

First of all, we need to specify the structure morphism for $\mathbb{N}_C$. It is the function $s\colon 1 + \mathbb{N} \to \mathbb{N}$ defined by

$$s(\mathsf{inl}\,\star) = 0\ , \qquad\qquad s(\mathsf{inr}\,n) = n + 1\ .$$

The function s is realized by $\lambda^* a.\,(\mathsf{if}\ (\mathsf{fst}\,a)\ (\mathsf{succ}(\mathsf{snd}\,a))\ \overline{0})$. Now suppose $[t, f]\colon 1 + A \to A$ is an algebra for $P_{1\to 2}$. Let $a, b \in \mathbb{A}_\sharp$ be realizers for t and f, respectively. Because the underlying set of $\mathbb{N}_C$ is the set of natural numbers there exists a unique function $g\colon |\mathbb{N}_C| \to |A|$ such that

$$g0 = t\star\ , \qquad\qquad g(n+1) = f(gn)\ .$$

We only need to show that g is realized. This is a simple matter of programming—the function g is realized by $\mathsf{rec}\ (a\mathsf{K})\ b$, where rec was defined in Section 1.1. A proof by induction shows that this realizer tracks g.

3.4 The Computability Predicate

The computability predicate $\#_A$ on a space A is interpreted as

$$a \Vdash \#_A(x) \iff a \in \mathbb{A}_\sharp \wedge (a \Vdash_A x)\ .$$

In words, the realizers for the statement "x is computable" are the computable realizers for x. In terms of a function $[\![\#_A(\Box)]\!]\colon |A| \to \mathcal{P}\mathbb{A}$ the computability predicate is interpreted as $[\![\#_A(x)]\!] = (\mathsf{E}_A x) \cap \mathbb{A}_\sharp$. The subspace $\#A$ turns out to be the set

$$|\#A| = \big\{x \in |A| \bigm| (\mathsf{E}_A x) \cap \mathbb{A}_\sharp \neq \emptyset\big\}$$

with the existence predicate $\mathsf{E}_{\#A} x = (\mathsf{E}_A x) \cap \mathbb{A}_\sharp$.

Let us verify the validity of the Axiom of Computability. The first clause, $\#\#A = \#A$ holds because intersecting with $\mathbb{A}_\sharp$ twice is the same as intersecting with $\mathbb{A}_\sharp$ once.

The second clause of the axiom states that

$$\#_{B^A}(f) \wedge \#_A(x) \longrightarrow \#_B(fx)\ . \tag{3.3}$$

Essentially, this is valid because $\mathbb{A}_\sharp$ is closed under application. More precisely, if $a \Vdash \#_{B^A}(f)$ and $b \Vdash \#_A(x)$, then $a \Vdash_{B^A} f$, $b \Vdash_A x$, hence $ab\!\downarrow \Vdash_B fx$, and since a and b are both elements of $\mathbb{A}_\sharp$ so is ab, hence $ab \Vdash \#_B(fx)$. Therefore, (3.3) is realized by $\lambda^* uvw.\,(\mathsf{fst}\,w)(\mathsf{snd}\,w)$.

To show the validity of the third clause, suppose

$$\mathsf{Mod}(\mathbb{A}, \mathbb{A}_\sharp) \models \exists!\, x \in A\,.\,\phi(x)\ .$$

Then there exists a realizer $\langle a_1, a_2\rangle \in \mathbb{A}_\sharp$ for $\exists!\, x \in A\,.\,\phi(x)$, where $a_1, a_2 \in \mathbb{A}_\sharp$, $a_1 \Vdash_A t$ for some $t \in |A|$ and $a_2 \Vdash \phi(t)$. Since a_2 is a computable realizer for t and $\phi(t)$ is valid, this means that

$$a_1 \Vdash \#_A(\mathsf{the}\,x \in A\,.\,\phi(x))\ ,$$

as required. The fourth clause of the Axiom of computability is seen to be valid by inspection: all of the maps claimed to be computable are realized by computable realizers.

3.5 Choice Principles

3.5.1 Markov's Principle

To validate Markov's Principle we use the fact that natural numbers are interpreted by Curry numerals, as was just proved in Subsection 3.3.2. Suppose $f(x : A) \colon \mathbb{N} \to 2$ is a map that depends on a parameter $x \in A$. Markov's principle is the statement

$$\neg\neg(\exists\, n \in \mathbb{N} .\, fxn = 0) \longrightarrow \exists\, n \in \mathbb{N} .\, fxn = 0\ .$$

We need to find a realizer $m \in \mathbb{A}_\sharp$ such that if a realizes f, b realizes x, and c realizes the antecedent of the above implication, then $mabc\!\downarrow$ realizes the consequent. We claim that the following is such a realizer:

$$m = \lambda^* abc.\, ((\mathsf{Z}(\lambda^* rn.\, (\mathsf{if}\ (\mathsf{iszero}(abn))\ n\ (r(\mathsf{succ}\, n)))))\, \overline{0})\ .$$

Note that m ignores the realizer c of the antecedent, as it should since the antecedent is a stable formula. It is easier to understand what m does by looking at an equivalent program written in the style of SML:

$$
\begin{array}{l}
\texttt{let}\ m\ a\ b\ c = \\
\quad \texttt{let}\ \mathit{search}\ n = \texttt{if}\ (a\ b\ n = 0)\ \texttt{then}\ n\ \texttt{else}\ (\mathit{search}\ (n+1)) \\
\quad \texttt{in} \\
\quad\quad \mathit{search}\ 0 \\
\quad \texttt{end}
\end{array}
$$

We see that the only issue is whether the program $mabc$ terminates. The antecedent is equivalent to the negative formula $\neg\forall\, n \in \mathbb{N} .\, fn = 1$, therefore by Theorem 3.1.1 its set-theoretic interpretation holds if, and only if, it is realized. But we assumed that c is a realizer for the antecedent, therefore the classical reading of it is true: there exists $n \in \mathbb{N}$ such that $fn = 0$. Therefore $mabc$ terminates after n recursive calls, if not before.

Strictly speaking, $mabc$ does not realize the consequent. It realizes a natural number n such that $fn = 0$ holds. The consequent is realized by the pair $\langle mabc, \overline{0}\rangle$.

3.5.2 Choice Principles

Definition 2.3.15 of projective spaces is the internal logic version of Definition 1.3.1 of projective modest sets. The Axiom of Projective Spaces is valid if, and only if, projective modest sets are closed under binary products, dependent sums, and regular subobjects. This is indeed the case, as can be verified easily by using Theorem 1.3.4, which characterizes projective modest sets as those that are separated.

Number Choice is valid because the natural numbers $\mathbb{N}$ are interpreted as the modest set of Curry numerals, which is a canonically separated, therefore projective by Theorem 1.3.4.

3.6 The Realizability Operator

In this section we add to the logic of modest sets a *realizability operator* r which assigns to each space and to each formula the corresponding space of realizers. This is the internal version of

Proposition 1.3.3. This way we make it possible to talk about realizers in the language of modest sets, which proves to be useful for computing representations of mathematical structures.

The Realizability Operator

Let A be a space. Its interpretation is a modest set $[\![A]\!] = (|A|, \Vdash_A)$. Let A_0 be the modest set whose underlying set is

$$|A_0| = \bigcup_{x \in |A|} \mathsf{E}_A x$$

and the realizability relation is the identity: $a \Vdash_{A_0} b$ if, and only if $a = b$. We call A_0 the *modest set of realizers* of A. In the logic of modest sets we postulate that for each A there is the *space* $\mathsf{r}A$ *of realizers* of A. The interpretation of $\mathsf{r}A$ is $[\![\mathsf{r}A]\!] = A_0$. This definition can be extended to dependent types in a straightforward fashion. If $A(i : I)$ is a dependent type, then $\mathsf{r}A(i : I)$ is a dependent type such that $\mathsf{r}A(i)$ is the space of realizers for $A(i)$, for every $i \in I$.

Let $\phi(x : A)$ be a formula whose only freely occurring variable is $x \in A$. Recall that the realizability interpretation of ϕ is a function $[\![\phi]\!] \colon |A| \to \mathcal{P}\mathbb{A}$. For each $t \in |A|$, we can view the set $[\![\phi(t)]\!] \subseteq \mathbb{A}$ as a modest set whose realizability relation is the identity. This way, $[\![\phi(x : A)]\!]$ is construed as a dependent type, i.e., it is a family of modest sets $\{[\![\phi(t)]\!] \mid t \in |A|\}$. We bring this idea into the logic of modest sets by postulating that for each formula $\phi(x : A)$, there is a dependent type $\mathsf{r}[\phi(x : A)]$ of *realizers of* ϕ. The interpretation of $\mathsf{r}[\phi(x : A)]$ is the family of modest sets

$$[\![\mathsf{r}[\phi(x : A)]]\!] = \{[\![\phi(t)]\!] \mid t \in |A|\} \ ,$$

where $[\![\phi(x : A)]\!]$ is viewed as the modest set, as explained before. In order for the realizability operator r to be of any use, we need to describe its properties in the logic of modest sets. Since r is essentially just the internal version of the interpretation function $[\![\Box]\!]$, we obtain the properties of r by following the definition of $[\![\Box]\!]$.

For any formula $\phi(x : A)$ and for every $x \in A$, the space of realizers $\mathsf{r}[\phi(x : A)]$ is projective. For any space A the space of realizers $\mathsf{r}A$ is projective. The realizability operator is idempotent, i.e.,

$$\mathsf{r}(\mathsf{r}A) = \mathsf{r}A \ , \qquad \mathsf{r}(\mathsf{r}[\phi]) = \mathsf{r}[\phi] \ .$$

There is a canonical quotient map $[\Box]_\mathsf{r} \colon \mathsf{r}A \to A$. When we are dealing with many spaces of realizers at once, we write $[\Box]_\mathsf{r}^A$ to indicate which quotient map we have in mind.

The following Theorem, which really ought to be an axiom, is the internal version of Proposition 1.3.3.

Theorem 3.6.1 (Presentation Principle) *For every space A there is a projective space $\mathsf{r}A$, called the* canonical cover *of A, and a quotient map $[\Box]_\mathsf{r} \colon \mathsf{r}A \to A$.*

We now list a number of equations that relate the realizability operator to the basic constructions of spaces. The equations are just the internal version of similar equations describing the relationships between canonical covers and the basic categorical constructions of modest sets.

The singleton space 1 and the empty spaces are their own spaces of realizers, i.e.,

$$\mathsf{r}1 = 1 \ , \qquad\qquad \mathsf{r}0 = 0 \ .$$

It follows that the maps $[\Box]_{\mathsf{r}}^{1}$ and $[\Box]_{\mathsf{r}}^{0}$ are the identity maps. For any spaces A and B,

$$\mathsf{r}(A \times B) = \mathsf{r}A \times \mathsf{r}B \ ,$$

and $[\langle x, y\rangle]_{\mathsf{r}}^{A\times B} = \langle [x]_{\mathsf{r}}^{A}, [y]_{\mathsf{r}}^{B}\rangle$ for all $x \in \mathsf{r}A$, $y \in \mathsf{r}B$. Similarly,

$$\mathsf{r}(A + B) = \mathsf{r}A + \mathsf{r}B \ ,$$

and $[\mathsf{inl}\, x]_{\mathsf{r}}^{A+B} = \mathsf{inl}\, [x]_{\mathsf{r}}^{A}$, $[\mathsf{inr}\, y]_{\mathsf{r}}^{A+B} = \mathsf{inr}\, [y]_{\mathsf{r}}^{B}$, for all $x \in \mathsf{r}A$, $y \in \mathsf{r}B$. For function spaces we have

$$\mathsf{r}\left(B^{A}\right) = \mathsf{r}\left(B^{\mathsf{r}A}\right) \ ,$$

and for all $f \in \mathsf{r}\left(B^{\mathsf{r}A}\right)$ and all $x \in \mathsf{r}A$,

$$[f]_{\mathsf{r}}^{B^{A}}\,[x]_{\mathsf{r}}^{A} = [f]_{\mathsf{r}}^{B^{\mathsf{r}A}}\,x \ .$$

For a space A and a predicate $\phi(x : A)$,

$$\mathsf{r}\left\{x \in A \mid \phi(x)\right\} = \textstyle\sum_{x \in \mathsf{r}A} \mathsf{r}[\phi([x]_{\mathsf{r}})] \ ,$$

and $[\langle x, a\rangle]_{\mathsf{r}}^{\{x\in A \mid \phi(x)\}} = \mathsf{o}_{\phi}([x]_{\mathsf{r}}^{A})$ for all $x \in \mathsf{r}A$ such that $\phi([x]_{\mathsf{r}})$, and for all $a \in \mathsf{r}[\phi([x]_{\mathsf{r}})]$. For a space A and a binary relation ρ on A,

$$\mathsf{r}(A/\rho) = \mathsf{r}A \ ,$$

and $[x]_{\mathsf{r}}^{A/\rho} = [[x]_{\mathsf{r}}^{A}]_{\rho}$ for all $x \in \mathsf{r}A$. For a dependent type $A(i : I)$,

$$\mathsf{r}\left(\textstyle\sum_{i\in I} A(i)\right) = \textstyle\sum_{j \in \mathsf{r}I} \mathsf{r}(A([j]_{\mathsf{r}})) \ ,$$
$$\mathsf{r}\left(\textstyle\prod_{i\in I} A(x)\right) = \mathsf{r}\left(\textstyle\prod_{j \in \mathsf{r}I} A([j]_{\mathsf{r}})\right) \ .$$

For all $j \in \mathsf{r}I$ and all $a \in \mathsf{r}(A([j]_{\mathsf{r}}))$,

$$[\langle j, a\rangle]_{\mathsf{r}}^{\sum_{i\in I} A(i)} = \langle [j]_{\mathsf{r}}^{I}, [a]_{\mathsf{r}}^{A([j]_{\mathsf{r}}^{I})}\rangle \ .$$

For all $f \in \mathsf{r}\left(\prod_{j\in \mathsf{r}I} A([j]_{\mathsf{r}})\right)$ and for all $k \in \mathsf{r}I$,

$$[f]_{\mathsf{r}}^{\prod_{i\in I} A(x)}[k]_{\mathsf{r}}^{I} = [f]_{\mathsf{r}}^{\prod_{j\in \mathsf{r}I} A([j]_{\mathsf{r}})} k \ .$$

For formulas $\phi(x : A)$ and $\psi(x : A)$ we have, for all $x \in A$,

$$\mathsf{r}[\mathsf{true}] = 1 \qquad \mathsf{r}[\mathsf{false}] = 0$$
$$\mathsf{r}[\phi(x) \wedge \psi(x)] = \mathsf{r}[\phi(x)] \times \mathsf{r}[\psi(x)] \qquad \mathsf{r}[\phi(x) \vee \psi(x)] = \mathsf{r}[\phi(x)] + \mathsf{r}[\psi(x)]$$
$$\mathsf{r}[\phi(x) \longrightarrow \psi(x)] = \mathsf{r}\left(\mathsf{r}[\psi(x)]^{\mathsf{r}[\phi(x)]}\right) \qquad \mathsf{r}[\neg\phi(x)] = \left\{u \in 1 \mid \neg\phi(x)\right\}$$

For a formula $\phi(x : A, y : B)$, and for all $x \in A$, we have

$$\mathsf{r}[\exists\, y \in B\,.\,\phi(x, y)] = \textstyle\sum_{b\in \mathsf{r}B} \mathsf{r}[\phi(x, [b]_{\mathsf{r}})] \ ,$$
$$\mathsf{r}[\forall\, y \in B\,.\,\phi(x, y)] = \mathsf{r}\left(\textstyle\prod_{b\in \mathsf{r}B} \mathsf{r}[\phi(x, [b]_{\mathsf{r}})]\right) \ .$$

The realizability interpretation of equality gives, for all $x, y \in A$,

$$\mathsf{r}[x = y] = \left\{a \in \mathsf{r}A \mid x = [a]_{\mathsf{r}} = y\right\} \ .$$

Finally, the relationship between the realizability operator and computability is described by

$$\mathsf{r}(\#A) = \#(\mathsf{r}A) \ .$$

Intensional Maps and Intensional Choice

Let us look more closely at the space of realizers for a "for all–exists" formula

$$\forall\, x \in A \,.\, \exists\, y \in B \,.\, \phi(x, y) \ .$$

According to the equations describing r, we get

$$\mathsf{r}[\forall\, x \in A \,.\, \exists\, y \in B \,.\, \phi(x, y)] = \mathsf{r}\left(\prod_{a \in \mathsf{r}A} \sum_{b \in \mathsf{r}B} \mathsf{r}[\phi([a]_\mathsf{r}, [b]_\mathsf{r})]\right) \ .$$

A point $f \in \prod_{a \in \mathsf{r}A} \sum_{b \in \mathsf{r}B} \mathsf{r}[\phi(x, y)]$ can be thought of as a pair of maps

$$f_1 \colon \mathsf{r}A \to \mathsf{r}B \ , \qquad f_2 \colon \prod_{a \in \mathsf{r}A} \mathsf{r}[\phi([a]_\mathsf{r}, [f_1 a]_\mathsf{r})] \ .$$

The map $[\square]_\mathsf{r}^B \circ f_1$ is a choice map for the statement

$$\forall\, a \in \mathsf{r}A \,.\, \exists\, y \in B \,.\, \phi([a]_\mathsf{r}, y) \ .$$

Therefore, in the logic of modest sets a weak version of the axiom of choice is valid. We say that a relation $\rho(x \colon A, y \colon B)$ is *total* when for all $x \in A$ there exists $y \in B$ such that $\rho(x, y)$.

Theorem 3.6.2 (Intensional Choice Principle) *For any relation $\phi(x \colon A, y \colon B)$,*

$$\forall\, x \in A \,.\, \exists\, y \in B \,.\, \phi(x, y) \longleftrightarrow \exists\, f \in B^{\mathsf{r}A} \,.\, \forall\, a \in \mathsf{r}A \,.\, \phi([a]_\mathsf{r}, f a) \ .$$

Proof. As we have just seen, a realizer for the left-hand side realizes a pair of maps $\langle f_1, f_2 \rangle$, and so its first component realizes the choice map $[\square]_\mathsf{r}^B \circ f_1$ for the right-hand side. The converse is straightforward, as well. ∎

This choice principle is called "intensional" because a map $f \colon \mathsf{r}A \to B$ can be thought of as an *intensional* map $A \to B$, i.e., the value of f at a point $x \in A$ depends on the particular realizer $a \in \mathsf{r}A$ of x.

In computable analysis intensional maps are inescapable. For example, a program which computes a real number to a given precision is intensional. Another important instance of an intensional map comes up in Gaussian elimination method for solving a system of linear equations, where the pivot needs to be chosen from a list of real number which are not all zero. This can be done, but only with an intensional choice map.

Intensional Choice implies the so-called Dependent Choice.

Theorem 3.6.3 (Dependent Choice) *Let $\rho(x \colon A, y \colon A)$ be a total relation and $x_0 \in A$. There exists $f \colon \mathbb{N} \to A$ such that $f0 = x_0$ and $\rho(fn, f(n+1))$ for all $n \in \mathbb{N}$.*

Proof. Define the relation $\sigma(x \colon A, b \colon \mathsf{r}A)$ to mean $\rho(a, [b]_\mathsf{r})$. Then σ is a total relation because $\mathsf{r}B$ covers B and ρ is a total relation. By Intensional Choice there exists a choice map $g \colon \mathsf{r}A \to \mathsf{r}A$ such that, for all $a \in \mathsf{r}A$, $\rho([a]_\mathsf{r}, [ga]_\mathsf{r})$. There exists $a_0 \in \mathsf{r}A$ such that $x_0 = [a_0]_\mathsf{r}$. Define a map $h \colon \mathbb{N} \to \mathsf{r}A$ inductively by

$$h0 = a_0 \ , \qquad\qquad h(n+1) = g(hn) \ .$$

Finally, let $fn = [hn]_\mathsf{r}$. Now we have $f0 = [h0] = [a_0] = x_0$. For every $n \in \mathbb{N}$, $\rho(fn, f(n+1))$ is equivalent to $\rho([hn], [g(hn)])$, which holds since g is an intensional choice map. ∎

Realizers and Subspaces

We prove three propositions that are useful for computing representations of subspaces.

Proposition 3.6.4 *A subspace* $\{x \in A \mid \phi(x)\}$ *is isomorphic to* $\big(\sum_{x \in A} \mathsf{r}[\phi(x)]\big)/{\sim}$, *where* $\sim$ *is defined by*

$$\langle x, a\rangle \sim \langle y, b\rangle \longleftrightarrow x = y \ .$$

Proof. Both spaces, $\{x \in A \mid \phi(x)\}$ and $\big(\sum_{x \in A} \mathsf{r}[\phi(x)]\big)/{\sim}$ are quotients of $\sum_{a \in \mathsf{r}A} \mathsf{r}[\phi([a]_\mathsf{r})]$:

$$\begin{aligned}\{x \in A \mid \phi(x)\} &= \big(\textstyle\sum_{a \in \mathsf{r}A} \mathsf{r}[\phi([a]_\mathsf{r})]\big)/{\approx_1} \ ,\\ \big(\textstyle\sum_{x \in A} \mathsf{r}[\phi(x)]\big)/{\sim} &= \big(\textstyle\sum_{a \in \mathsf{r}A} \mathsf{r}[\phi([a]_\mathsf{r})]\big)/{\approx_2} \ ,\end{aligned}$$

where, for all $a, a' \in \mathsf{r}A$, $b \in \mathsf{r}[\phi([a]_\mathsf{r})]$, $b' \in \mathsf{r}[\phi([a']_\mathsf{r})]$,

$$\begin{aligned}\langle a, b\rangle &\approx_1 \langle a', b'\rangle \longleftrightarrow [a]_\mathsf{r} = [a']_\mathsf{r} \ ,\\ \langle a, b\rangle &\approx_2 \langle a', b'\rangle \longleftrightarrow \langle [a]_\mathsf{r}, b\rangle \sim \langle [a']_\mathsf{r}, b'\rangle \longleftrightarrow [a]_\mathsf{r} = [a']_\mathsf{r} \ .\end{aligned}$$

Therefore, $\approx_1$ and $\approx_2$ agree. ∎

Proposition 3.6.5 *Suppose* $\phi(x : A)$ *is a proposition and* $F(x : A)$ *is a dependent type, such that there exist two families of maps* $\{f_x \colon \mathsf{r}[\phi(x)] \to F(x) \mid x \in A\}$ *and* $\{g_x \colon F(x) \to \mathsf{r}[\phi(x)] \mid x \in A\}$. *Then*

$$\{x \in A \mid \phi(x)\} = \big(\textstyle\sum_{x \in A} F(x)\big)/{\sim} \ ,$$

where $\langle x, u\rangle \sim \langle y, v\rangle$ *if, and only if,* $x = y$.

Proof. Define the maps $f \colon \sum_{x \in A} \mathsf{r}[\phi(x)] \to \sum_{x \in A} F(x)$ and $g \colon \sum_{x \in A} F(x) \to \sum_{x \in A} \mathsf{r}[\phi(x)]$ by

$$f\langle x, a\rangle = \langle x, f_x a\rangle \ , \qquad g\langle x, u\rangle = \langle x, g_x u\rangle \ .$$

By Proposition 3.6.4, the subspace $\{x \in A \mid \phi(x)\}$ can we written as the quotient

$$\{x \in A \mid \phi(x)\} = \big(\textstyle\sum_{x \in A} \mathsf{r}[\phi(x)]\big)/{\approx}$$

where

$$\langle x, a\rangle \approx \langle y, b\rangle \longleftrightarrow x = y \ .$$

The maps f and g preserve $\sim$ and $\approx$, hence they induce a pair of maps $f' \colon \{x \in A \mid \phi(x)\} \to \big(\sum_{x \in A} F(x)\big)/{\sim}$ and $g' \colon \big(\sum_{x \in A} F(x)\big)/{\sim} \to \{x \in A \mid \phi(x)\}$, and it is easy to check that f' and g' are inverses of each other. ∎

The following proposition demonstrates how we can compute representations by using the realizability operator.

Proposition 3.6.6 *Suppose* $\phi(x : A, y : B, z : C)$ *is a predicate. The subspace*

$$\{x \in A \mid \forall\, y \in B \,.\, \exists\, z \in C \,.\, \phi(x, y, z)\}$$

is isomorphic to the quotient

$$\{\langle x, f\rangle \colon A \times (\mathsf{r}B \to C) \mid \forall\, b \in \mathsf{r}B \,.\, \phi(x, [b]_\mathsf{r}, f b)\}/{\sim} \ ,$$

where $\langle x, f\rangle \sim \langle y, g\rangle$ *if, and only if,* $x = y$.

Proof. Compute:

$$\begin{aligned}
&\mathsf{r}\left\{\langle x,f\rangle\colon A\times(\mathsf{r}B\to C)\;\middle|\;\forall\,b\in\mathsf{r}B\,.\,\phi(x,[b]_{\mathsf{r}},fb)\right\} = \\
&\qquad\textstyle\sum_{\langle a,g\rangle\in\mathsf{r}A\times\mathsf{r}(\mathsf{r}B\to C)}\mathsf{r}[\forall\,b\in\mathsf{r}B\,.\,\phi([a]_{\mathsf{r}},[b]_{\mathsf{r}},[gb]_{\mathsf{r}})] = \\
&\qquad\textstyle\sum_{a\in\mathsf{r}A}\sum_{g\in\mathsf{r}(\mathsf{r}B\to C)}\mathsf{r}[\forall\,b\in\mathsf{r}B\,.\,\phi([a]_{\mathsf{r}},[b]_{\mathsf{r}},[gb]_{\mathsf{r}})] = \\
&\qquad\textstyle\sum_{a\in\mathsf{r}A}\mathsf{r}[\exists\,f\in C^{\mathsf{r}B}\,.\,\forall\,b\in\mathsf{r}B\,.\,\phi([a]_{\mathsf{r}},[b]_{\mathsf{r}},fb)] = \\
&\qquad\textstyle\sum_{a\in\mathsf{r}A}\mathsf{r}[\forall\,y\in B\,.\,\exists\,z\in C\,.\,\phi([a]_{\mathsf{r}},y,c)] = \\
&\qquad\qquad\mathsf{r}\left\{x\in A\;\middle|\;\forall\,y\in B\,.\,\exists\,z\in C\,.\,\phi(x,y,z)\right\}\,.
\end{aligned}$$

The second to the last step follows from the Intensional Choice. Now it only remains to show that taking the canonical quotient $[\square]_{\mathsf{r}}$ of the right-hand side agrees with taking the quotient of the left-hand side by the equivalence relation $[\square]_{\mathsf{r}}\sim[\square]_{\mathsf{r}}$, which is an easy exercise. ■

Chapter 4

Equilogical Spaces and Related Categories

4.1 Equilogical Spaces

Equilogical spaces were defined by Dana Scott in 1996. The motivation was to have a category with good closure properties, say complete, cocomplete and cartesian closed, that contained many interesting and well known subcategories, such as topological spaces, domains, and PER models. At the same time the category was supposed to be "easy to describe to the mathematician in the street". Equilogical spaces and equivariant maps between them can be explained as follows:

> An *equilogical space* is a topological space with an equivalence relation.
>
> An *equivariant map* is a continuous map that preserves the equivalence relations.

The original definition requires the spaces also to be T_0, but this is inessential. In this chapter we limit attention just to *countably based* equilogical spaces, which are those equilogical spaces whose underlying topological space is countably based and T_0. Hence, by equilogical space we always mean a countably based one.

The category of equilogical spaces Equ is a realizability model since it is equivalent to $\mathsf{Mod}(\mathbb{P})$. In Subsection 4.1.2 we define *effective* equilogical spaces $\mathsf{Equ}_{\mathsf{eff}}$, which are equivalent to $\mathsf{Mod}(\mathbb{P}, \mathbb{P}_\sharp)$. Even though Equ and $\mathsf{Mod}(\mathbb{P})$ are equivalent and we have developed a general theory of modest sets, it is illuminating to explore equilogical spaces directly, rather than just apply the theory to $\mathsf{Mod}(\mathbb{P})$. The advantage is that in Equ we avoid the details about the underlying PCA $\mathbb{P}$, and think in terms of topological spaces and continuous maps instead.

An interesting variation is to take equivalence relations on 0-*dimensional* countably based T_0-spaces. These equilogical spaces are called 0-*equilogical* spaces. It turns out that the 0-equilogical spaces form a cartesian closed category $\mathsf{0Equ}$. Moreover, $\mathsf{0Equ}$ is equivalent to the category $\mathsf{Mod}(\mathbb{B})$. Thus, with 0-equilogical we can circumvent the somewhat unpleasant technicalities of the second Kleene algebra and replace them with arguments about 0-dimensional spaces. This is further explored in Section 4.2.

4.1.1 Equilogical Spaces

Definition 4.1.1 An *equilogical space* $X = (|X|, \equiv_X)$ is a countably based T_0-space $|X|$ together with an equivalence relation $\equiv_X$ on X. If X and Y are equilogical spaces, a continuous map $f\colon |X| \to |Y|$ is *equivariant* when, for all $x, y \in |X|$,

$$x \equiv_X y \Longrightarrow fx \equiv_Y fy\ .$$

Two equivariant maps $f, g\colon |X| \to |Y|$ are *equivalent*, written $f \equiv_{X \to Y} g$, when for all $x, y \in |X|$

$$x \equiv_X y \Longrightarrow fx \equiv_Y gy\ .$$

A morphism $[f]\colon X \to Y$ between equilogical spaces X and Y is an equivalence class of equivariant maps. We say that an equivariant map f *represents* the morphism $[f]$. Composition of morphisms is defined by $[g] \circ [f] = [g \circ f]$. The identity morphism on X is represented by the identity on $|X|$. The category of equilogical spaces and morphisms between them is denoted by Equ.

There are several equivalent versions of the category of equilogical spaces. The category PEqu of partial equivalence relations on algebraic lattices is one of them.

Definition 4.1.2 An object $A = (|A|, \approx_A)$ of the category PEqu, also called an *equilogical space*, is a countably based algebraic lattice $|A|$ with a partial equivalence relation $\approx_A$ on $|A|$. If A and B are equilogical spaces, a continuous map $f\colon |A| \to |B|$ is *equivariant* when for all $x, y \in |A|$

$$x \approx_A y \Longrightarrow fx \approx_B fy\ .$$

Two equivariant maps $f, g\colon |A| \to |B|$ are *equivalent*, written $f \approx_{A \to B} g$, when for all $x, y \in |A|$

$$x \approx_A y \Longrightarrow fx \approx_B gy\ .$$

A morphism $[f]\colon A \to B$ between equilogical spaces A and B is an equivalence class of equivariant maps.

If $A = (|A|, \approx_A)$ is an equilogical space, we denote the domain of $\approx_A$ by $\|A\|$, i.e., $\|A\| = \big\{x \in |A| \bigm| x \approx_A x\big\}$.

Theorem 4.1.3 *The categories* Equ*,* PEqu*,* $\mathsf{PER}(\mathbb{P})$*,* $\mathsf{Mod}(\mathbb{P})$*, and* $\mathsf{Mod}(\mathbb{V})$ *are equivalent.*

Proof. We specify the equivalences functors between the categories, and leave the proof that they really are equivalences as exercise.

First we establish an equivalence between Equ and $\mathsf{PER}(\mathbb{P})$. Given an object $P = (\mathbb{P}, \approx_P) \in \mathsf{PER}(\mathbb{P})$, the corresponding equilogical space is the restriction $RP = (\|P\|, \approx_P)$, where $\|P\|$ is the domain of $\approx_P$ equipped with the subspace topology inherited from $\mathbb{P}$. A morphism $[f]\colon P \to Q$ is mapped to the restriction $R[f] = [f{\restriction}_{\|P\|}]$. This defines a functor $R\colon \mathsf{PER}(\mathbb{P}) \to \mathsf{Equ}$. In the other direction, suppose X is an equilogical space. By the Embedding Theorem there exists an embedding $e_X\colon X \hookrightarrow \mathbb{P}$. Define IX to be the partial equivalence relation $(\mathbb{P}, \approx_X)$ determined by

$$e_X x \approx_X e_X y \iff x \equiv_X y,$$

where $x, y \in |X|$. Suppose X and Y are objects in Equ and $[f]\colon X \to Y$ is a morphism between them. Think of $|X|$ and $|Y|$ as subspaces of $\mathbb{P}$. By the Extension Theorem there exists a continuous extension $F\colon \mathbb{P} \to \mathbb{P}$ of the map $f\colon |X| \to |Y|$. Let $I[f]$ be the morphism $[F]\colon IX \to IY$. This defines a functor $I\colon \mathsf{Equ} \to \mathsf{PER}(\mathbb{P})$. It is easy to check that functors R and I form an equivalence of categories.

The categories PEqu and $\mathsf{PER}(\mathbb{P})$ are equivalent because every $A \in \mathsf{PEqu}$ is isomorphic to some object in $\mathsf{PER}(\mathbb{P})$, since by the Embedding Theorem every countably based algebraic lattice $|A|$ can be embedded in $\mathbb{P}$.

We already know from Section 1.2.1 that $\mathsf{PER}(\mathbb{P})$ and $\mathsf{Mod}(\mathbb{P})$ are equivalent. Lastly, the categories $\mathsf{Mod}(\mathbb{P})$ and $\mathsf{Mod}(\mathbb{V})$ are equivalent by Theorem 1.4.11, because the PCAs $\mathbb{P}$ and $\mathbb{V}$ are equivalent, which we proved in Subsection 1.4.5. ∎

For another characterization of equilogical spaces, as *dense* partial equivalence relations on Scott domains, see Theorem 4.1.21.

Proposition 4.1.4 *The category of equilogical spaces* Equ *has countable limits and colimits.*

Proof. We need to show that Equ has countable products, equalizers, countable coproducts, and coequalizers. The product of a countable family $(X_i)_{i\in I}$ of equilogical spaces is the equilogical space $X = \prod_{i\in I} X_i$ whose underlying topological space is the topological product

$$|X| = \prod_{i\in I} |X_i|$$

and the equivalence relation is the product of equivalence relations

$$(x_i)_{i\in I} \equiv_X (y_i)_{i\in I} \iff \forall\, i \in I \,.\, x_i \equiv_{X_i} y_i \;.$$

The equalizer of morphisms $[f], [g]\colon U \to V$ is an morphism $[e]\colon E \to U$ where

$$|E| = \left\{ x \in |U| \;\middle|\; fx \equiv_V gx \right\} \;,$$

$\equiv_E$ is the restriction of $\equiv_U$ to $|E|$, and $e\colon |E| \to |U|$ is the inclusion map.

The coproduct of a countable family $(Y_i)_{i\in I}$ is the equilogical space $Y = \coprod_{i\in I} Y_i$ whose underlying topological space is the topological coproduct

$$|Y| = \coprod_{i\in I} |Y_i|$$

and the equivalence relation is the coproduct of equivalence relations

$$\langle i, x\rangle \equiv_Y \langle j, y\rangle \iff i = j \text{ and } x \equiv_{Y_i} y \;.$$

The coequalizer of morphisms $[f], [g]\colon U \to V$ is a morphism $[q]\colon V \to Q$ where $|Q| = |V|$, and $\equiv_Q$ is the least equivalence relation satisfying

$$\exists\, u \in |U| \,.\, (x \equiv_V fu \text{ and } y \equiv_V gu) \Longrightarrow x \equiv_Q y \;.$$

The map $q\colon |V| \to |Q|$ is the identity map. Note that this does not mean that $[q]$ is the identity morphism! ∎

The category of equilogical spaces is not only cartesian closed, but also locally cartesian closed. We know this already because any category of modest sets is locally cartesian closed. In Subsection 4.1.4 we will need an explicit description of dependent products in PEqu, so we give a proof here.

Theorem 4.1.5 *The category of equilogical spaces is cartesian closed and locally cartesian closed.*

Proof. This proposition is most easily proved in PEqu. By Proposition 4.1.4 the category PEqu has finite products. For $A, B \in \mathsf{PEqu}$ the exponential B^A is the equilogical space $B^A = ([|A| \to |B|], \approx_{A\to B})$ where $[|A| \to |B|]$ is the algebraic lattice of continuous maps from $|A|$ to $|B|$, and $\approx_{A\to B}$ is defined by

$$f \approx_{A\to B} g \iff \forall\, x, y \in |A| \,.\, (x \approx_A y \Longrightarrow fx \approx_B gy) \ .$$

The evaluation map $[e]\colon B^A \times A \to B$ is represented by the evaluation map $e\colon [|A| \to |B|] \times |A| \to |B|$ for the underlying lattices. The verification that B^A really is an exponential is left as exercise.

Let $I \in \mathsf{PEqu}$ be an equilogical space. As in any slice category, the terminal object in PEqu/I is the identity map $1_I\colon I \to I$. The slice PEqu/I has binary products because they are the pullbacks in PEqu. More explicitly, the product of objects $[a]\colon A \to I$ and $[b]\colon B \to I$ is the morphism $[a \otimes b]\colon A \otimes B \to I$, where the underlying lattice of $A \otimes B$ is $|A \otimes B| = |A| \times |B|$, the partial equivalence relation is defined by

$$\langle x, y\rangle \approx_{A\otimes B} \langle x', y'\rangle \iff x \equiv_A x' \wedge y \equiv_B y' \wedge ax \equiv_I by' \ ,$$

and the morphism $[a \otimes b]$ is represented by the map $a \otimes b = a \circ \mathsf{fst} = b \circ \mathsf{snd}$.

It remains to be shown that PEqu/I has exponentials. Let $[b]\colon B \to I$ and $[c]\colon C \to I$ be two objects in PEqu/I. The exponential $(E, [e]) = (C, [c])^{(B,[b])}$ is defined as follows. The underlying lattice of E is $|E| = [|B| \to |C|] \times |I|$, and the partial equivalence relation $\approx_E$ is defined by

$$\begin{aligned}
&\langle f, i\rangle \approx_E \langle g, j\rangle \iff \\
&\qquad i \approx_I j \text{ and} \\
&\qquad \forall\, x, y \in |I| \,.\, (x \equiv_I y \wedge bx \equiv_I i \Longrightarrow fx \equiv_B gy \wedge c(fx) \equiv_I i) \ .
\end{aligned}$$

The morphism $[e]\colon E \to I$ is represented by the canonical projection $e = \mathsf{snd}\colon [|B| \to |C|] \times |I| \to |I|$. In the following we simplify notation by writing a in place of $(A, [a]\colon A \to I)$, and similarly for other objects. To complete the proof, we define natural isomorphisms

$$\begin{aligned}
\phi_{a,b,c}&\colon \mathsf{Hom}(a \otimes b, c) \to \mathsf{Hom}(a, c^b) \ , \\
\psi_{a,b,c}&\colon \mathsf{Hom}(a, c^b) \to \mathsf{Hom}(a \otimes b, c) \ .
\end{aligned}$$

Isomorphism ϕ is defined by

$$\phi_{a,b,c}[f] = [\lambda x \in |A| \,.\, \langle \lambda y \in |A| \,.\, f(x, y), ax\rangle] \ ,$$

and its inverse ψ is

$$\psi_{a,b,c}[g] = [\lambda \langle x, y\rangle \in |A| \times |B| \,.\, ((\mathsf{fst}\,(gx)y)] \ .$$

The following calculation shows that $\psi \circ \phi = 1$:

$$((\psi \circ \phi)f)\langle x, y\rangle = (\psi(\lambda w \in |A| \,.\, \langle \lambda z \in |B| \,.\, f(w,z), aw\rangle))\langle x, y\rangle$$
$$= (\mathsf{fst}\,\langle \lambda z \in |B| \,.\, f(x,z), ax\rangle)y = f(x,y) \ .$$

Similarly, $\phi \circ \psi = 1$:

$$((\phi \circ \psi)g)x = (\phi(\lambda\langle w, z\rangle \in |A| \times |B| \,.\, (\mathsf{fst}\,(gw))z))x$$
$$= \langle \lambda y \in |A| \,.\, (\mathsf{fst}\,(gx))y, ax\rangle = (\mathsf{fst}\,(gx), ax)$$
$$\equiv (\mathsf{fst}\,(gx), \mathsf{snd}\,(gx)) = gx \ .$$

The verification that ψ and ϕ are well defined and natural is left as exercise. ∎

Every countably based T_0-space X can be viewed as an equilogical space $IX = (X, =_X)$ where $=_X$ is the identity relation on X. A continuous map $f\colon X \to Y$ determines a morphism $If = [f]\colon IX \to IY$. This defines a functor $I\colon \omega\mathsf{Top}_0 \to \mathsf{Equ}$ which is obviously full and faithful.

Theorem 4.1.6 *The inclusion* $I\colon \omega\mathsf{Top}_0 \to \mathsf{Equ}$ *is full and faithful, preserves limits, coproducts, and all exponentials that exist in* $\omega\mathsf{Top}_0$.

Proof. Only the claim that I preserves exponentials is not obvious. Suppose $X, Y \in \omega\mathsf{Top}_0$ and the exponential Y^X exists in $\omega\mathsf{Top}_0$. It is well known that the underlying set of Y^X is the set of continuous maps from X to Y, and that the evaluation map $e\colon Y^X \times X \to Y$ is the actual evaluation $e(f, x) = fx$.

We verify directly that $I(Y^X)$ is the exponential of IX and IX in Equ, with the evaluation map $[e]\colon I(Y^X) \times IX \to IY$. Let A be an equilogical space and $[f]\colon A \times IX \to IY$ a morphism. Because $f\colon |A| \times X \to Y$ is a continuous map there exists a continuous map $\widetilde{f}\colon |A| \to Y^X$ such that for all $a \in |A|$ and $x \in X$

$$e(\widetilde{f}a, x) = (\widetilde{f}a)x = f(a, x) \ .$$

Taking into account equivariance of f, we see that the map $\widetilde{f}$ is equivariant because $a \equiv_A b$ implies that for all $x \in X$

$$(\widetilde{f}a)x = f(a,x) = f(b,x) = (\widetilde{f}b)x \ ,$$

This means that the functions $\widetilde{f}a$ and $\widetilde{f}b$ are equal, as required. To see that the morphism $[\widetilde{f}]$ is independent of the choice of the representative f, suppose $g \equiv_{A \times X \to Y} f$ and let $\widetilde{g}\colon |A| \to Y^X$ be the curried form of g. If $a \equiv_A b$ then $(\widetilde{g}a)x = g(a,x) \equiv f(b,x) = (\widetilde{f}b)x$ for all $x \in X$, hence $\widetilde{g} \equiv_{A \to I(Y^X)} \widetilde{f}$. We have now shown that for every morphism $[f]\colon A \times IX \to IY$ there exists a morphism $[\widetilde{f}]\colon A \to I(Y^X)$ such that

$$[e] \circ ([\widetilde{f}] \times [1_X]) = [f] \ .$$

Suppose $[g]\colon A \to I(Y^X)$ is also a morphism such that $[e] \circ ([g] \times [1_X]) = [f]$. Then $[g]$ coincides with $[\widetilde{f}]$ because $a \equiv_A b$ implies that for all $x \in X$

$$(ga)x = (e \circ (g \times 1_X))\langle a, x\rangle = f(a,x) = f(b,x) = (\widetilde{f}b)x \ ,$$

therefore $[g] = [\widetilde{f}]$, which proves uniqueness of $[\widetilde{f}]$. ∎

Note that in the above proof we never used the uniqueness of $\widetilde{f}$, which suggests that *weak* exponentials in $\omega\mathsf{Top}_0$ might be related to exponentials in Equ. A *weak* exponential of spaces $X, Y \in \omega\mathsf{Top}_0$ is a space $W \in \omega\mathsf{Top}_0$ with a continuous evaluation map $e\colon W \times X \to Y$ such that for every $A \in \omega\mathsf{Top}_0$ and for every continuous map $f\colon A \times X \to Y$ there exists a (not necessarily unique!) continuous map $\widetilde{f}\colon A \to W$ such that $e(\widetilde{f}a, x) = f(a, x)$ for all $a \in A$ and $x \in X$.

Proposition 4.1.7 *If A and B are equilogical spaces and $V \in \omega\mathsf{Top}_0$ is a weak exponential of $|A|$ and $|B|$ with an evaluation map $e\colon V \times |A| \to |B|$, then the exponential B^A in Equ is an equilogical space $W = (|W|, \equiv_W)$ whose underlying space is the subspace $|W| \subseteq V$, defined by*

$$|W| = \{f \in V \mid f \equiv_W f\} \ ,$$

and where $\equiv_W$ is the relation on V, defined by

$$f \equiv_W g \iff \forall\, a, a' \in |A| \,.\, \big(a \equiv_A a' \Longrightarrow e(f, a) \equiv_B e(g, a')\big) \ .$$

The evaluation morphism is $[e{\restriction}_{|W|}]\colon W \times A \to B$.

Proof. The relation $\equiv_W$ is clearly symmetric and transitive on V, but it is not necessarily reflexive. That is why we need to restrict the weak exponential V to an appropriate subspace $|W|$. Let $[f]\colon C \times A \to B$ be a morphism in Equ. Because $f\colon |C| \times |A| \to |B|$ is a continuous map, there exists $\widetilde{f}\colon |C| \to V$ such that $e(\widetilde{f}c, a) = f(c, a)$ for all $c \in |C|$ and $a \in |A|$. It is easy to check that $\widetilde{f}$ maps into $|W|$. The map $\widetilde{f}$ is equivariant because $c \equiv_C c'$ and $a \equiv_A a'$ implies

$$e(\widetilde{f}c, a) = f(c, a) \equiv_B f(c', a') = e(\widetilde{f}c', a') \ .$$

From this it follows that $c \equiv_C c'$ implies $\widetilde{f}c \equiv_W \widetilde{f}c'$, hence $\widetilde{f}$ is equivariant. The morphism $[\widetilde{f}]\colon C \to W$ does not depend on the choice of the representative f or the choice of $\widetilde{f}$. Indeed, suppose $g \equiv_{C \times A \to B} f$ and $\widetilde{g}$ is a weak curried form of g. If $c \equiv_C c'$ and $a \equiv_A a'$ then

$$e(\widetilde{g}c, a) = g(c, a) \equiv_B f(c', a') = e(\widetilde{f}c', a') \ ,$$

which means that $\widetilde{g}c \equiv_W \widetilde{f}c'$ and so $\widetilde{g} \equiv_{C \to W} \widetilde{f}$. The identity $[e] \circ ([\widetilde{f}] \times 1_A) = [f]$ is obvious because we even have $e \circ (\widetilde{f} \times 1_A) = f$. Uniqueness of $[\widetilde{f}]$ is easily proved. Suppose $[h]\colon C \to W$ is a morphism such that $[e] \circ ([h] \times 1_A) = [f]$. If $c \equiv_C c'$ and $a \equiv_A a'$ then

$$e(hc, a) \equiv_B f(c, a) \equiv_B f(c', a') = e(\widetilde{f}c', a') \ .$$

Therefore $c \equiv_C c'$ implies $hc \equiv_W \widetilde{f}c'$, which means that $h \equiv_{C \to W} \widetilde{f}$. ∎

Let $[\Box]\colon \mathsf{Equ} \to \mathsf{Top}$ be the "quotient" functor which maps an equilogical space A to the topological quotient $[A] = |A|/{\equiv_A}$, and a morphism $[f]\colon A \to B$ to the unique continuous map $[f]\colon QA \to QB$ such that $[f][x] = [fx]$ for all $x \in |A|$. Here $[x]$ is the equivalence class of x. Note that $[A]$ does not have to be countably based or a T_0-space.

An equilogical space which is isomorphic to a topological space is called a *topological object.* By Theorem 1.3.4, topological objects are exactly the projective modest sets in $\mathsf{Mod}(\mathbb{P})$.

Proposition 4.1.8 *An equilogical space A is a topological object if, and only if, there exists an equivariant retraction $r\colon |A| \to |A|$.*

Proof. Suppose A is isomorphic to a topological space X. Let $[f]: A \to X$ be an isomorphism and let $[g]: X \to A$ be its inverse. Then $r = g \circ f$ is the required retraction because $f \circ g = 1_X$.

Conversely, if $r: |A| \to |A|$ is an equivariant retraction, then A is isomorphic to the image of r. The image of r is a countably based T_0-space because it is a subspace of a countably based T_0-space $|A|$. ∎

Corollary 4.1.9 *An equilogical space A is a topological object if, and only if, the quotient $[A]$ is a countably based T_0-space and the canonical projection $q: |A| \to [A]$ represents an isomorphism $[q]: A \to [A]$.*

Proof. Suppose A is a topological object. By Proposition 4.1.8 there exists an equivariant retraction $r: |A| \to |A|$. Let $R \subseteq |A|$ be the image of r and let $s: [A] \to R$ be defined by $s[x] = rx$. It is clear that $q{\restriction}_R \circ s = 1_{[A]}$ and $s \circ q{\restriction}_R = 1_R$, so only continuity of s remains to be demonstrated. Suppose $U \subseteq R$ is an open subset of R. Then $q^*(s^*(U)) = r^*(U)$, which is an open subset of $|A|$ because r is continuous. It follows that $s^*(U)$ is open since q is an open map. ∎

4.1.2 Effective Equilogical Spaces

Countably based equilogical spaces correspond to the modest sets $\mathsf{Mod}(\mathbb{P})$. Since the computability predicate in $\mathsf{Mod}(\mathbb{P})$ is trivial Equ is not a suitable category for studying computability. We refine the notion of equilogical spaces so that the resulting category of *effective equilogical spaces* correspond to the modest sets $\mathsf{Mod}(\mathbb{P}, \mathbb{P}_\sharp)$. We do this by first defining *effective topological spaces*, and then constructing the effective equilogical spaces as equivalence relations on effective topological spaces.

Effective Topological Spaces

Recall that an enumeration operator $f: \mathbb{P} \to \mathbb{P}$ is computable when its graph Γf is an r.e. set. By the Embedding Theorem, every countably based T_0-space X can be embedded in $\mathbb{P}$, and every continuous map $g: X \to Y$ can be extended to an enumeration operator $\bar{g}: \mathbb{P} \to \mathbb{P}$, so that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, tail] & Y \arrow[d, tail] \\ \mathbb{P} \arrow[r, "\bar{g}"] & \mathbb{P} \end{tikzcd}$$

The embeddings $X \rightarrowtail \mathbb{P}$ and $Y \rightarrowtail \mathbb{P}$ are determined by a choice of subbases for X and Y. Once such subbases are chosen, we can define a *computable continuous map* $g: X \to Y$ to be a continuous map for which there exists a computable enumeration operator $\bar{g}: \mathbb{P} \to \mathbb{P}$ which makes the above diagram commute. This idea gives the following definition of effective topological spaces.

Definition 4.1.10 An *effective topological space* is a pair (X, S_X) where X is a countably based T_0-space and $S_X: \mathbb{N} \to \mathcal{O}(X)$ is an enumeration of a countable subbasis for X. A *computable continuous map* $f: (X, S_X) \to (Y, S_Y)$ is a continuous map $f: X \to Y$ for which there exists an r.e. relation $F \subseteq \mathbb{P}_0 \times \mathbb{N}$ such that:

1. F is monotone: $x \subseteq y$ and $F(x,m)$ implies $F(y,m)$.
2. F approximates f: if $F(\{n_1,\ldots,n_k\},m)$ then $S_X(n_1) \cap \cdots \cap S_X(n_k) \subseteq f^*(S_Y(m))$.
3. F converges to f: if $ft \in S_Y(m)$ then there exists $\{n_1,\ldots,n_k\} \in \mathbb{P}_0$ such that $t \in S_X(n_1) \cap \cdots \cap S_X(n_k)$ and $F(\{n_1,\ldots,n_k\},m)$.

The relation F is called an *r.e. realizer* for f. We also say that F *tracks* f. The category of effective topological spaces and computable continuous maps is denoted by $\mathsf{Top}_{\mathsf{eff}}$.

Note that in the above definition the empty set is allowed as a subbasic open set. We often simplify notation and denote an effective topological space (X, S_X) by X. The category $\mathsf{Top}_{\mathsf{eff}}$ is well-defined. The identity morphism $1_X : (X, S_X) \to (X, S_X)$ is the identity function $1_X : X \to X$, which has an r.e. realizer I_X, defined by

$$I_X(\{n_1,\ldots,n_k\},m) \iff m \in \{n_1,\ldots,n_k\} \ .$$

The composition of computable maps $f : X \to Y$ and $g : Y \to Z$ is again a computable map $g \circ f : X \to Z$ because it has an r.e. realizer H defined by

$$H(x,n) \iff \exists\, y \in \mathbb{P}_0 \,.\, \Big(G(y,n) \wedge \bigwedge_{m \in y} F(x,m)\Big) \ ,$$

where F and G are r.e. realizers for f and g, respectively.

The monotonicity condition in Definition 4.1.10 is redundant, for if F is an r.e. relation that satisfies the second and the third condition, then we can recover monotonicity by defining a new relation F' by

$$F'(x,n) \iff \bigvee_{y \subseteq x} F(y,n) \ ,$$

It is easy to see that F' satisfies all three conditions and realizes the same function as f.

The singleton space $1 = \{\star\}$ is an effective space with the subbase $S_1 n = \{\star\}$. We can use it to define computability of points as follows.

Definition 4.1.11 A point $t \in X$ of an effective space (X, S_X) is *computable* when the continuous map $1 \to X$, defined by $\star \mapsto t$, is computable.

This definition amounts to saying that a point $t \in X$ is computable when the set of indices of its subbasic open neighborhoods $\{n \in \mathbb{N} \mid t \in S_X n\}$ is r.e.

The algebraic lattice $\mathbb{P}$ is an effective space with the subbasis $S_{\mathbb{P}}$ given by

$$S_{\mathbb{P}}(n) = \uparrow\{n\} \ ,$$

where $\mathsf{finset} : \mathbb{N} \to \mathbb{P}_0$ is a standard enumeration of finite elements of $\mathbb{P}$. We also pick a particular subbase for the algebraic lattice $\mathbb{P}^{\mathbb{P}}$ of enumeration operators, namely

$$S_{\mathbb{P}^{\mathbb{P}}}(\langle m,n\rangle) = \uparrow \mathsf{step}_{\mathsf{finset}\, m, \{n\}} \ .$$

The coding functions $\langle \Box, \Box \rangle$ and finset are described in Subsection 1.1.3, and the step function $\mathsf{step}_{x,y}$ is defined by

$$\mathsf{step}_{x,y}(z) = \begin{cases} y & \text{if } x \subseteq z \ , \\ \emptyset & \text{otherwise} \ . \end{cases} \qquad (x, y \in \mathbb{P}_0)$$

It is easily checked that with this choice of subbases for $\mathbb{P}$ and $\mathbb{P}^{\mathbb{P}}$, the computable continuous maps $\mathbb{P} \to \mathbb{P}$ are exactly the r.e. enumeration operators, the evaluation map $e \colon \mathbb{P}^{\mathbb{P}} \times \mathbb{P} \to \mathbb{P}$ is computable, and so are the pairing function $\langle \Box, \Box \rangle \colon \mathbb{P} \times \mathbb{P} \to \mathbb{P}$, the graph function $\Gamma \colon \mathbb{P}^{\mathbb{P}} \to \mathbb{P}$, and the retraction $\Lambda \colon \mathbb{P} \to \mathbb{P}^{\mathbb{P}}$.

Next, we prove effective versions of the Embedding and Extension Theorems.

Theorem 4.1.12 (Effective Embedding Theorem) *Every effective topological space can be effectively embedded into $\mathbb{P}$.*

Proof. Let (X, S_X) be an effective topological space. We show that the embedding $e \colon X \to \mathbb{P}$, defined in the proof of the Embedding Theorem 1.1.2 by

$$et = \left\{ n \in \mathbb{N} \mid t \in S_X(n) \right\} \ ,$$

is a computable map. It has an r.e. realizer E defined by

$$E(\{n_0, \ldots, n_k\}, m) \iff m \in \{n_0, \ldots, n_k\} \ .$$

This is obviously an r.e. relation which is monotone in the first argument. Clearly,

$$E(\{n_0, \ldots, n_k\}, m) \Longrightarrow S_X(n_0) \cap \cdots \cap S_x(n_k) \subseteq e^*(\uparrow m) = S_X(m) \ .$$

Suppose $et \in \uparrow \{m\}$. Then $t \in S_X(m)$, and $E(\{m\}, m)$. Therefore, e is a computable map because it satisfies all three conditions from Definition 4.1.10. ∎

Theorem 4.1.13 (Effective Extension Theorem) *Let X and Y be effective topological spaces and $f \colon X \to Y$ a computable map between them. Then there exists a computable map $\overline{f} \colon \mathbb{P} \to \mathbb{P}$ such that the following diagram commutes:*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "e_X"'] & Y \arrow[d, "e_Y"] \\ \mathbb{P} \arrow[r, "\overline{f}"'] & \mathbb{P} \end{tikzcd}$$

The maps e_X and e_Y are the computable embeddings defined in the Theorem 4.1.12.

Proof. Let F be an r.e. realizer for f. We define the map $\overline{f}\colon \mathbb{P} \to \mathbb{P}$ by defining its graph to be F, i.e.,

$$m \in \overline{f}(\{n_0, \ldots, n_k\}) \iff F(\{n_0, \ldots, n_k\}, m) .$$

All we have to show is that this choice of $\overline{f}$ makes the diagram commute. For any $t \in X$,

$$\begin{aligned}
&\overline{f}(e_X t)\\
&\quad = \{m \in \mathbb{N} \mid \exists\, n_0, \ldots, n_k \in \mathbb{N}.\, (t \in S_X(n_0) \cap \cdots \cap S_X(n_k) \wedge F(\{n_0, \ldots, n_k\}, m))\}\\
&\qquad = \{m \in \mathbb{N} \mid ft \in S_Y(m)\} = e_Y(ft) .
\end{aligned}$$

The second set in the above sequence of equalities is contained in the third set because f satisfies the third condition of Definition 4.1.10. The reverse inclusion follows from the second condition for f. ∎

Theorem 4.1.14 *The category of effective topological spaces has finite limits, finite coproducts, and weak exponentials.*

Proof. We need to show that $\mathsf{Top}_{\mathsf{eff}}$ has a terminal object, binary products, equalizers, an initial object, binary coproducts, and weak exponentials. The proof is the same as for the category of topological spaces, except that we just have to find appropriate subbases so that the continuous maps occurring in these constructions have r.e. realizers.

The terminal object is the one-point space $1 = (\{\star\}, S_1)$ with the subbasis $S_1(n) = \{\star\}$.

The product of (X, S_X) and (Y, S_Y) is the effective space $(X \times Y, S_{X \times Y})$ with the subbasis

$$S_{X \times Y}(\langle m, n\rangle) = S_X(m) \times S_Y(n) .$$

It is easily checked that the projection maps $\mathsf{fst}\colon X \times Y \to X$ and $\mathsf{snd}\colon X \times Y \to Y$ are computable. Given computable maps $f\colon Z \to X$ and $g\colon Z \to Y$, the map $\langle f, g\rangle\colon Z \to X \times Y$ is computable because it has an r.e. realizer H defined by

$$H(x, \langle m, n\rangle) \iff F(x, m) \wedge G(x, n) .$$

where F and G are r.e. realizers for f and g, respectively.

If (Y, S_Y) is an effective space and $X \subseteq Y$ is a subspace of Y, then the induced *effective subspace* (X, S_X) has the subbasis

$$S_X(n) = S_Y(n) \cap X .$$

The equalizer of computable maps $f, g\colon Y \to Z$ is the effective subspace $E = \{t \in Y \mid ft = gt\}$ with the inclusion map $e\colon X \hookrightarrow Y$ which is computable because it has an r.e. realizer I defined by

$$E(x, m) \iff m \in x .$$

Given any computable map $k\colon X \to Y$ with an r.e. realizer K such that $f \circ h = g \circ h$, the unique map $k'\colon X \to E$ for which $k = i \circ k'$ is computable because it is realized by K.

The initial object is the empty space $\mathsf{0} = (\emptyset, S_{\mathsf{0}})$ with the subbasis $S_{\mathsf{0}}(n) = \emptyset$.

The coproduct of (X, S_X) and (Y, S_Y) is the effective space $(X + Y, S_{X+Y})$ with the subbasis

$$\begin{aligned}
S_{X+Y}(2n) &= S_X(n) ,\\
S_{X+Y}(2n+1) &= S_Y(n) .
\end{aligned}$$

It is easily checked that the inclusion maps $\iota_0 \colon X \to X+Y$ and $\iota_1 \colon Y \to X+Y$ are computable. Given computable maps $f \colon X \to Z$ and $g \colon Y \to Z$, the map $f+g \colon X+Y \to Z$ is computable because it has an r.e. realizer H defined by

$$H(x,m) \iff F(\left\{n \mid 2n \in x\right\}, m) \wedge G(\left\{n \mid 2n+1 \in x\right\}, m) \ ,$$

where F and G are r.e. realizers for f and g, respectively.

Finally, a weak exponential (W, S_W) of effective spaces (X, S_X) and (Y, S_Y) is the effective subspace $W \subseteq \mathbb{P}^{\mathbb{P}}$ given by

$$W = \left\{f \in \mathbb{P}^{\mathbb{P}} \mid \forall\, t \in X \,.\, (f(e_X t) \in e_Y(Y))\right\} \ ,$$

where $e_X : X \hookrightarrow \mathbb{P}$ and $e_Y : Y \hookrightarrow \mathbb{P}$ are the embeddings determined by S_X and S_Y, respectively, and $e_Y(Y) \subseteq \mathbb{P}$ is the image of Y in $\mathbb{P}$. In other words, W is the space of those enumeration operators which restrict to continuous maps between the (images of) spaces X and Y. Since the evaluation map $e \colon \mathbb{P}^{\mathbb{P}} \times \mathbb{P} \to \mathbb{P}$ is computable, its restriction to $W \times X \to Y$ is computable as well, via the same r.e. realizer. Similarly, if $f \colon Z \times X \to Y$ is an computable map, it has a computable weak curried form $\tilde{f} \colon Z \to W$ because currying is λ-definable. ∎

Effective Equilogical Spaces

With a notion of effective topological spaces at hand, we can define the effective equilogical spaces just like the ordinary ones, except that we replace topological spaces and continuous maps by their effective versions.

Definition 4.1.15 An *effective equilogical space* $A = (|A|, \equiv_A, S_A)$ is an effective topological space $(|A|, S_A)$ together with an equivalence relation $\equiv_A$ on $|A|$. Two computable equivariant maps $f, g \colon |A| \to |B|$ are equivalent when they map related elements to related elements. A morphism $[f] \colon A \to B$ between effective equilogical spaces is an equivalence class of computable equivariant maps. The category of effective equilogical spaces and morphisms between them is denoted by $\mathsf{Equ}_{\mathsf{eff}}$.

We check that we got the definition right by proving that $\mathsf{Equ}_{\mathsf{eff}}$ is equivalent to $\mathsf{Mod}(\mathbb{P}, \mathbb{P}_\sharp)$, as we originally intended it to be.

Proposition 4.1.16 *The categories* $\mathsf{Mod}(\mathbb{P}, \mathbb{P}_\sharp)$*,* $\mathsf{PER}(\mathbb{P}, \mathbb{P}_\sharp)$*, and* $\mathsf{Equ}_{\mathsf{eff}}$ *are equivalent.*

Proof. We show equivalence of $\mathsf{PER}(\mathbb{P}, \mathbb{P}_\sharp)$ and $\mathsf{Equ}_{\mathsf{eff}}$. First, we define an equivalence functor $R \colon \mathsf{PER}(\mathbb{P}, \mathbb{P}_\sharp) \to \mathsf{Equ}_{\mathsf{eff}}$ as follows. Given an effective partial equivalence relation $P = (\mathbb{P}, \approx_P) \in \mathsf{PER}(\mathbb{P}, \mathbb{P}_\sharp)$, let $RP = (\|P\|, S_P, \approx_P)$ be the effective equilogical space whose underlying effective topological space is the domain $\|P\|$ of $\approx_P$, considered as an effective subspace of $\mathbb{P}$. A morphism $[f] \colon P \to Q$ is mapped by R to the morphism $R[f] = [f{\restriction}_{\|P\|}]$, where $f{\restriction}_{\|P\|}$ is the restriction of f to $\|P\|$. The map $f{\restriction}_{\|P\}}$ is obviously equivariant. Its r.e. realizer is the graph of the enumeration operator f.

Next we define a functor $I \colon \mathsf{Equ}_{\mathsf{eff}} \to \mathsf{PER}(\mathbb{P}, \mathbb{P}_\sharp)$. Given an effective equilogical space $A = (|A|, S_A, \equiv_A)$, let $e_A \colon |A| \to \mathbb{P}$ be the embedding from the Effective Embedding Theorem 4.1.12, and let IA be the partial equivalence relation on $\mathbb{P}$ defined by

$$x \approx_{IA} y \iff \exists\, t, y \in |A| \,.\, (x = e_A t \wedge y = e_A u \wedge t \equiv_A u) \ .$$

A computable map $[f]\colon A \to B$ is mapped to the morphism $I[f] = [\overline{f}]\colon IA \to IB$, where $\overline{f}\colon \mathbb{P} \to \mathbb{P}$ is an effective extension of f, which exists by the Effective Extension Theorem 4.1.13. It is obvious that such an extension is equivariant, and that the morphism $[\overline{f}]$ does not depend on the choice of $\overline{f}$. We leave the easy proof that I and R constitute an equivalence of categories as an exercise. ∎

The basic constructions in the category $\mathsf{Equ}_{\mathsf{eff}}$, such as products, coproducts, limits, colimits, and exponentials, are the same as those in Equ. The reason for this is that all the morphisms needed in these constructions in Equ, such as the projection maps, are built up using λ-calculus and are thus automatically computable. It is useful to describe exponentials in $\mathsf{Equ}_{\mathsf{eff}}$ directly in terms of effective weak exponentials in $\mathsf{Top}_{\mathsf{eff}}$.

Proposition 4.1.17 *If A and B are effective equilogical spaces and V is an effective weak exponential of $|A|$ and $|B|$ with an evaluation map $e\colon V \times |A| \to |B|$, then the exponential B^A in $\mathsf{Equ}_{\mathsf{eff}}$ is an effective equilogical space $W = (|W|, S_{|W|}, \equiv_W)$ whose underlying space is the effective subspace $|W| \subseteq V$,*

$$|W| = \left\{ f \in V \mid f \equiv_W f \right\} ,$$

where $\equiv_W$ is the partial equivalence relation on V, defined by

$$f \equiv_W g \iff \forall\, a, a' \in |A| \,.\, \left(a \equiv_A a' \Longrightarrow e(f,a) \equiv_B e(g,a')\right) .$$

The evaluation morphism is $[e{\restriction}_{|W|}]\colon W \times A \to B$.

Proof. The proof is analogous to the proof of Proposition 4.1.7, just replace the topological spaces with their effective versions. ∎

We conclude this section with the definition of a *'sharp' operator*, $\#\colon \mathsf{Equ}_{\mathsf{eff}} \to \mathsf{Equ}_{\mathsf{eff}}$. The basic idea is that the elements of $\#A$ are the computable elements of $\#A$.

Definition 4.1.18 The sharp operator $\#\colon \mathsf{PER}(\mathbb{P}, \mathbb{P}_\sharp) \to \mathsf{PER}(\mathbb{P}, \mathbb{P}_\sharp)$ is a functor defined as follows. If $P \in \mathsf{Mod}(\mathbb{P}, \mathbb{P}_\sharp)$ then $\#P$ is the equivalence relation $\approx_P \cap\, (\mathbb{P}_\sharp \times \mathbb{P}_\sharp)$, that is,

$$x \approx_{\#P} y \iff x \approx_P y \text{ and } x, y \in \mathbb{P}_\sharp .$$

If $[f]\colon P \to R$ is a morphism represented by a computable enumeration operator $f\colon \mathbb{P} \to \mathbb{P}$, then $\#[f] = [f]$ is the morphism represented by the same operator.

The action of $\#$ on morphisms is well-defined because a computable enumeration operator applied to an r.e. set yields an r.e set. The same functor can be defined equivalently on $\mathsf{Equ}_{\mathsf{eff}}$ as follows. If $A \in \mathsf{Equ}_{\mathsf{eff}}$ then the underlying space $|\#A|$ is

$$|\#A| = \left\{ t \in |A| \;\middle|\; \left\{ n \in \mathbb{N} \mid t \in S_n \right\} \in \mathbb{P}_\sharp \right\} \subseteq |A| ,$$

the subbasis $S_{\#A}$ is

$$S_{\#A}(n) = S_A(n) \cap |\#A| ,$$

and the equivalence relation $\equiv_{\#A}$ is the restriction of $\equiv_A$ to $|\#A|$. If $[f]\colon A \to B$ is a morphism in $\mathsf{Equ}_{\mathsf{eff}}$, then $\#[f]$ is the restriction $\#[f] = [f{\restriction}_{|\#A|}]$. This is well defined because $f{\restriction}_{|\#A|}$ has the same r.e. realizer as f.

Note that $\#A$ is a subobject of A, and that $\#$ is idempotent, which means $\# \circ \# = \#$. In other words, $\#$ is a comonad on $\mathsf{Equ}_{\mathsf{eff}}$ whose counit $\epsilon\colon \# \Longrightarrow 1$ at A is the inclusion $\#A \hookrightarrow A$, and comultiplication $\mu\colon \# \Longrightarrow \#^2$ at A is the identity map $\mu_A = 1_A$.

4.1.3 Effectively Presented Domains as a Subcategory of $\mathsf{Equ}_{\mathsf{eff}}$

An *effectively presented domain* (D, b) is a countably based continuous domain D with an enumeration of a countable basis $b_0, b_1, b_2, \dots$ such that the relation $b_n \ll b_m$ is r.e. in $\langle n, m\rangle \in \mathbb{N} \times \mathbb{N}$. A continuous function $f\colon D \to E$ between effectively presented domains (D, b) and (E, c) is *computable* if the relation $c_m \ll f b_n$ is r.e. in $\langle n, m\rangle \in \mathbb{N} \times \mathbb{N}$. Note that the requirement that the way below relation $\ll$ be r.e. on the basis elements is equivalent to the requirement that the identity function on the domain be computable.

Let $\mathsf{CDom}_{\mathsf{eff}}$ be the category of effectively presented domains and computable functions between them. Effectively presented domains have been used successfully both in denotational semantics of programming languages and as computational models of structures from mainstream mathematics [Eda97, Eda95, Eda96, Esc97, EH98, ES99b, ES99a]. We show that computability in $\mathsf{CDom}_{\mathsf{eff}}$ agrees with that of $\mathsf{Equ}_{\mathsf{eff}}$ because $\mathsf{CDom}_{\mathsf{eff}}$ is a full subcategory of $\mathsf{Top}_{\mathsf{eff}}$.

Lemma 4.1.19 *Let $f\colon D \to E$ be a continuous function between continuous domains D and E. Suppose $b_0, b_1, \dots$ is a basis for D, and $c_0, c_1, \dots$ is a basis for E. If $c_m \ll f t$ for some $t \in D$ and $m \in \mathbb{N}$ then there exists $n \in \mathbb{N}$ such that $b_n \ll t$ and $c_m \ll f b_n$.*

Proof. Proof by contradiction: suppose that $c_m \not\ll f b_n$ for all $b_n \ll t$. Then the directed set $\{f b_n \mid b_n \ll t\}$ is a subset of the closed set $E \setminus \{u \in E \mid c_m \ll u\}$, therefore also its supremum $f t$ is in the set, which means that $c_m \not\ll f t$. ∎

Theorem 4.1.20 *The category $\mathsf{CDom}_{\mathsf{eff}}$ embeds fully and faithfully into the category $\mathsf{Top}_{\mathsf{eff}}$.*

Proof. An effectively presented domain (D, b) can be viewed as an effective topological space (D, S_D) where S_D is the topological basis for the Scott topology on D, defined by

$$S_D(n) = \{t \in D \mid b_n \ll t\} \ .$$

Suppose $f\colon (D, b) \to (E, c)$ is a computable map, in the sense of effectively presented domains. We need to show that it has an r.e. realizer F, in the sense of effective topological spaces. Define F by

$$F(\{n_0, \dots, n_k\}, m) \iff c_m \ll f b_{n_0} \land \dots \land c_m \ll f b_{n_k} \ .$$

We show that F approximates f and converges to f. Note that F is not monotone in the first argument, but that is not a problem, as was explained in the remark following Definition 4.1.10. The second condition is satisfied because of the following chain of implications:

$$\begin{aligned} F(\{n_0, \dots, n_k\}, m) &\Longrightarrow c_m \ll f b_{n_0} \land \dots \land c_m \ll f b_{n_k} \\ &\Longrightarrow f b_{n_0} \in S_E(m) \land \dots \land f b_{n_k} \in S_E(m) \\ &\Longrightarrow b_{n_0} \in f^*(S_E(m)) \land \dots \land b_{n_k} \in f^*(S_E(m)) \\ &\Longrightarrow S_D(n_0) \subseteq f^*(S_E(m)) \land \dots \land S_D(n_k) \subseteq f^*(S_E(m)) \\ &\Longrightarrow S_D(n_0) \cap \dots \cap S_D(n_k) \subseteq f^*(S_E(m)) \ . \end{aligned}$$

Suppose $f t \in S_E(m)$. Then $c_m \ll f t$ and by Lemma 4.1.19 there exists $n \in \mathbb{N}$ such that $b_n \ll t$ and $c_m \ll f b_n$. This means that $t \in S_D(n)$ and $F(\{n\}, m)$, therefore the third condition is satisfied as well. Therefore, F is an r.e. realizer for f.

Conversely, suppose $f\colon D \to E$ has an r.e. realizer, in the sense of effective topological spaces. We need to show that the relation $c_m \ll fb_n$ is r.e. in $\langle m,n\rangle \in \mathbb{N}\times\mathbb{N}$. The statement $c_m \ll fb_n$ is equivalent to the following statements:

$$\begin{aligned} c_m \ll fb_n &\iff fb_n \in S_E(m) \\ &\iff \exists\, n_0,\dots,n_k \in \mathbb{N}\,.\,(b_n \in S_D(n_0)\cap\dots\cap S_D(n_k) \wedge F(\{n_0,\dots,n_k\},m)) \\ &\iff \exists\, n_0,\dots,n_k \in \mathbb{N}\,.\,(b_{n_0} \ll b_n \wedge\dots\wedge b_{n_k} \ll b_n \wedge F(\{n_0,\dots,n_k\},m))\ . \end{aligned}$$

The last line is an r.e. relation in $\langle m,n\rangle \in \mathbb{N}\times\mathbb{N}$, therefore f is a computable map, in the sense of effectively presented domains. ∎

4.1.4 Domains with Totality as a Subcategory of Equilogical Spaces

In this section we study the relationship between equilogical spaces and totality for domains, as developed and studied by Normann, Berger, and others [Ber93, Sch96, Ber97a, Nor98a, Plo98]. Berger [Ber00] can serve as motivation and overview of the topic of totality. The main result of this section is a "goodness of fit" theorem which show that, in a precise sense, equilogical spaces are a generalization of dense and codense totalities on domains. We prove that dense and codense totalities on domains form a cartesian closed subcategory of Equ. An important consequence of this result is that the Kleene-Kreisel countable functionals of finite type [Kle59] arise in Equ by repeated exponentiation of the natural numbers object $\mathbb{N}$. We then generalize these results to continuous functionals of dependent types. As another consequence we obtain a generalization of Markov's principle for equilogical spaces.

For the purposes of this section it is convenient to take equilogical spaces as partial equivalence relations on algebraic lattices. Thus, we work with the category PEqu, which was defined in Definition 4.1.2.

Domains with Totality

A *Scott domain* $(D,\leq)$ is a countably based, algebraic consistently-complete, directed-complete, partially ordered set with a least element.[1] We may view domains as topological spaces with their Scott topologies. Let $\omega\mathsf{Dom}$ be the category of Scott domains and continuous functions. We refer to Scott domains simply as "domains". Domains can also be considered as topologically closed non-empty subsets of countably based algebraic lattices. Thus $\omega\mathsf{ALat}$, the category of countably based algebraic lattices and continuous maps, is a full subcategory of $\omega\mathsf{Dom}$. Additionally $\omega\mathsf{Dom}$ is a cartesian closed category, and $\omega\mathsf{ALat}$ is a full cartesian closed subcategory of $\omega\mathsf{Dom}$. A domain becomes an algebraic lattice if a "top" element is added to the poset. This construction produces a functor which, however, is not a reflection and it does not preserve the ccc-structure.

The following definitions are taken from Berger [Ber93]. A subset $M \subseteq D$ is *dense* if it is dense in the topological sense, *i.e.*, the closure of M is D. We write $x \uparrow y$ when elements $x, y \in D$ are bounded, and $x \not\uparrow y$ when they are unbounded. A finite subset $\{x_0,\dots,x_k\} \subseteq D$ is *separable* when there exist open subsets $U_0,\dots,U_k \subseteq D$ such that $x_0 \in U_0,\dots,x_k \in U_k$ and $U_0\cap\dots\cap U_k = \emptyset$. We say that $U_0,\dots,U_k$ *separate* $x_0,\dots,x_k$. It is easily seen that a finite set is separable if, and

[1] For background material on domain theory we suggest [SHLG94] or [AC98].

only if, it is unbounded. A family of open sets U is *separating* when it separates every separable finite set, *i.e.*, for every separable $\{x_0, \ldots, x_k\} \subseteq D$ there exist members of U that separate it.

The *Boolean domain* $\mathbb{B}_\perp$ is the flat domain for the Boolean values 1 and 0. A *partial continuous predicate* (pcp) on a domain D is a continuous function $p \colon D \to \mathbb{B}_\perp$. The function-space domain $D \to \mathbb{B}_\perp$ is denoted by $\mathrm{pcp}(D)$. With each pcp p we associate two disjoint open sets by inverse images:

$$p^+ = p^{-1}(\{1\}) \qquad \text{and} \qquad p^- = p^{-1}(\{0\}) \ .$$

A subset $P \subseteq \mathrm{pcp}(D)$ is *separating* when the corresponding family $\{p^+ \mid p \in P\}$ is separating.

Given a set $M \subseteq D$ let

$$\mathcal{E}_D(M) = \{p \in \mathrm{pcp}(D) \mid \forall x \in M . p(x) \neq \perp\} \ .$$

A set M is *codense* in D when the family $\mathcal{E}_D(M)$ is separating. An element $x \in D$ is said to be codense when the singleton $\{x\}$ is codense in D. Every element of a codense set is codense, but not every set of codense elements is codense. If $M \subseteq D$ is a codense set then the consistency relation $\uparrow$ is an equivalence relation on M. Thus, a codense set $M \subseteq D$ can be viewed as a domain D together with a partial equivalence relation $\uparrow_M$, which is just the relation $\uparrow$ restricted to M.

A *totality* on a domain, in the sense of Berger [Ber93], is a dense and codense subset of a domain. Note that in the original paper by Berger [Ber93] codense sets are called *total.* Here we are using the newer terminology of Berger [Ber97a]. We use the following notation for totalities on domains. A totality on a domain is a pair $D = (\|D\|, |D|)$ where $|D|$ is a domain and $\|D\| \subseteq |D|$ is a totality on $|D|$. The consistency relation restricted to $\|D\|$ is denoted by $\uparrow_D$.

Given totalities D and E, it is easily seen that the set $\|D\| \times \|E\|$ is again a totality on the domain $|D| \times |E|$. Similarly, by the Density Theorem in Berger [Ber93] the set

$$\|D \to E\| = \{f \in D \to E \mid f_*(\|D\|) \subseteq \|E\|\}$$

is a totality on the function-space domain $D \to E$. This idea of totality generalizes the simple-minded connection between total and partial functions using flat domains. If A is any set, let $A_\perp$ be the flat domain obtained by adding a bottom element. Then A itself is a totality on $A_\perp$, and the total set-theoretic functions $A \to B$ correspond to (equivalence classes) of functions in $\|A \to B\|$ considered as elements of $A_\perp \to B_\perp$.

Let $\mathsf{PER}(\omega\mathsf{Dom})$ be the category formed just like PEqu except that domains are used instead of algebraic lattices, i.e., an object of $\mathsf{PER}(\omega\mathsf{Dom})$ is a pair $(D, \approx_D)$ where D is a domain and $\approx_D$ is a partial equivalence relation on D. The category $\mathsf{PER}(\omega\mathsf{Dom})$ is cartesian closed, and for $D, E \in \mathsf{PER}(\omega\mathsf{Dom})$ we choose the canonical product and exponential $D \times E$ and $D \to E$ whose underlying domains are the standard product and exponential in $\omega\mathsf{Dom}$, and the partial equivalence relations are defined by

$$\begin{aligned} \langle x_1, y_1\rangle \approx_{D\times E} \langle x_2, y_2\rangle &\iff x_1 \approx_D x_2 \wedge y_1 \approx_E y_2 \ , \\ f \approx_{D\to E} g &\iff \forall\, x, y \in D\, .\, (x \approx_D y \Longrightarrow fx \approx_E gy) \ . \end{aligned}$$

We say that a partial equivalence relation $\approx_D$ on a domain D is *dense* when its support $\mathsf{dom}(\approx_D) = \{x \in D \mid x \approx_D x\}$ is a dense subset of D. Because every algebraic lattice is a domain, PEqu is a full subcategory of $\mathsf{PER}(\omega\mathsf{Dom})$. The top-adding functor

$$T \colon \mathsf{PER}(\omega\mathsf{Dom}) \to \mathsf{PEqu}$$

maps an object $(D, \approx_D) \in \mathsf{PER}(\omega\mathsf{Dom})$ to the object

$$TD = (D \cup \{\top\}, \approx_D)$$

where $D \cup \{\top\}$ is the algebraic lattice obtained from the underlying domain of D by attaching a compact top element. The functor T maps a morphism $[f]\colon D \to E$ to the morphism $T[f]$ represented by the map

$$(Tf)x = \begin{cases} fx & \text{if } x \neq \top\,, \\ \top & \text{if } x = \top\,. \end{cases}$$

The top-adding functor is a product-preserving reflection, hence PEqu is an exponential ideal and a sub-ccc of $\mathsf{PER}(\omega\mathsf{Dom})$.

In the category $\omega\mathsf{Dom}$ it is *not* the case that every continuous map $f\colon D' \to E$ defined on an arbitrary non-empty subset $D' \subseteq D$ has a continuous extension to the whole domain D. Because of this fact the category $\mathsf{PER}(\omega\mathsf{Dom})$ has certain undesirable properties. However, it is true that every continuous map defined on a *dense* subset has a continuous extension; this is an easy consequence of the Extension Theorem and the fact that a domain becomes an algebraic lattice when a top element is added to it. These observations suggest that we should consider only the dense partial equivalence relations on domains.

Let $\mathsf{DPER}(\omega\mathsf{Dom})$ be the full subcategory of $\mathsf{PER}(\omega\mathsf{Dom})$ whose partial equivalence relations are either *dense* or *empty*. We are including the empty equivalence relation here because the only map from an empty subset always has a continuous extension. The objects whose partial equivalence relations are empty are exactly the initial objects of $\mathsf{DPER}(\omega\mathsf{Dom})$. We have the following theorem.

Theorem 4.1.21 $\mathsf{DPER}(\omega\mathsf{Dom})$ *and* PEqu *are equivalent.*

Proof. In one direction, the equivalence is established by the top-adding functor

$$T\colon \mathsf{DPER}(\omega\mathsf{Dom}) \to \mathsf{PEqu}\,.$$

In the other direction, the equivalence functor $K\colon \mathsf{PEqu} \to \mathsf{DPER}(\omega\mathsf{Dom})$ is defined as follows. For an initial object $A = (|A|, \emptyset)$, define $KA = A$. Otherwise, let K map an object $A \in \mathsf{PEqu}$ to the object KA whose underlying domain is the set $|KA| = \overline{\mathsf{dom}(\approx_A)}$, which is the topological closure of $\mathsf{dom}(\approx_A)$ in $|A|$, equipped with the subspace topology. The partial equivalence relation for KA is just $\approx_A$ restricted to $|KA|$. The functor K maps a morphism $[f]\colon A \to B$ to the morphism represented by the restriction $f\restriction_{|KA|}$. Here we assume that the morphism from an initial object $A = (|A|, \emptyset)$ is represented by the constant map $f\colon x \mapsto \bot$. If A is initial, $K[f]$ is obviously well defined. When A is not initial, $K[f]$ is well defined because continuity of f implies that

$$f_*(|KA|) = f_*(\overline{\mathsf{dom}(\approx_A)}) \subseteq \overline{f_*(\mathsf{dom}(\approx_A))} \subseteq \overline{\mathsf{dom}(\approx_B)} = |KB|\,.$$

It is easily checked that K and T establish an equivalence between PEqu and $\mathsf{DPER}(\omega\mathsf{Dom})$. ∎

We would like to represent domains with totality as equilogical spaces. If $D = (\|D\|, |D|)$ is a domain with totality, let $(|D|, \uparrow_D)$ be the object of $\mathsf{PER}(\omega\mathsf{Dom})$ whose underlying domain is $|D|$ and the partial equivalence relation is $\uparrow_D$, the consistency relation restricted to $\|D\|$. This identifies domains with totality as objects of $\mathsf{DPER}(\omega\mathsf{Dom})$. The following result shows that the morphisms of $\mathsf{DPER}(\omega\mathsf{Dom})$ are the right ones, because the cartesian closed structure of $\mathsf{DPER}(\omega\mathsf{Dom})$ agrees with the formation of products and function-space objects with totality.

Theorem 4.1.22 *Let D and E be domains with totality. Then in* $\mathsf{DPER}(\omega\mathsf{Dom})$

$$\begin{aligned}(|D|, \uparrow_D) \times (|E|, \uparrow_E) &= (|D| \times |E|, \uparrow_{D\times E}) \ , \\ (|D|, \uparrow_D) \to (|E|, \uparrow_E) &= (|D| \to |E|, \uparrow_{D\to E}) \ .\end{aligned}$$

Proof. Here it is understood that the product $(D, \approx_D) \times (E, \approx_E)$ and the exponential $(D, \approx_D) \to (E, \approx_E)$ are the canonical ones for $\mathsf{PER}(\omega\mathsf{Dom})$. They are objects in $\mathsf{DPER}(\omega\mathsf{Dom})$ by the Density Theorem in Berger [Ber93]. The first equality follows from the observation that $\langle x_1, y_1 \rangle \uparrow \langle x_2, y_2 \rangle$ if, and only if, $x_1 \uparrow x_2$ and $y_1 \uparrow y_2$. Let $X = (D, \approx_D) \to (E, \approx_E)$ and $Y = (D \to E, \sim_{\langle M,N \rangle})$. Objects X and Y have the same underlying domains, so we only have to show that the two partial equivalence relations coincide. The partial equivalence relation on X is

$$f \uparrow_X g \iff f, g \in \langle M, N \rangle \text{ and } \forall x, y \in M. \big(x \uparrow y \Longrightarrow fx \uparrow gy\big).$$

Suppose $f \uparrow_X g$. Then $f, g \in \langle M, N \rangle$ and it remains to be shown that $f \uparrow g$. For every $x \in M$, since $x \uparrow x$ and $f \uparrow_X g$, $fx \uparrow gx$, thus by Lemma 7 in Berger [Ber93] f and g are inseparable, which is equivalent to them being bounded. Conversely, suppose $f, g \in \langle M, N \rangle$ and $f \uparrow g$. For every $x, y \in M$ such that $x \uparrow y$, it follows that $fx \uparrow gy$ because $fx \leq (f \vee g)(x \vee y)$ and $gy \leq (f \vee g)(x \vee y)$. This means that $f \uparrow_X g$. ■

The Kleene-Kreisel Countable Functionals

The category PEqu is a full sub-ccc of $\mathsf{PER}(\omega\mathsf{Dom})$. Since $\mathsf{DPER}(\omega\mathsf{Dom})$ is a full subcategory of $\mathsf{PER}(\omega\mathsf{Dom})$ and is equivalent to PEqu, it is a full sub-ccc of $\mathsf{PER}(\omega\mathsf{Dom})$ as well. Theorem 4.1.22 states that, for any totalities D and E , the exponential $(|D|, \uparrow_D) \to (|E|, \uparrow_E)$ coincides with the object $(|D| \to |E|, \uparrow_{\|D\to E\|})$. We may use this to show that in PEqu the countable functionals of finite types arise as iterated function spaces of the natural numbers object. For simplicity we only concentrate on pure finite types ι, $\iota \to \iota$, $(\iota \to \iota) \to \iota$, … and skip the details of how to extend this to the full hierarchy of finite types generated by ι, o, $\times$, and $\to$.

The natural numbers object in $\mathsf{DPER}(\omega\mathsf{Dom})$ is the object

$$N_0 = (\mathbb{N}_\perp, \uparrow_{N_0})$$

whose underlying domain is the flat domain of natural numbers $\mathbb{N}_\perp = \mathbb{N} \cup \{\perp\}$ and the partial equivalence relation $\uparrow_{N_0}$ is the restriction of identity to $\mathbb{N}$. Define the hierarchy $N_1, N_2, \ldots$ inductively by

$$N_{j+1} = N_j \to N_0$$

where the arrow is formed in $\mathsf{DPER}(\omega\mathsf{Dom})$. By Theorem 4.1.22, this hierarchy is contained in $\mathsf{DPER}(\omega\mathsf{Dom})$ and corresponds exactly to Ershov's and Berger's construction of countable functionals of pure finite types. It is well known that the equivalence classes of N_j correspond naturally to the original Kleene-Kreisel countable functionals of pure type j, see Berger [Ber93] or Ershov [Ers77].

In PEqu the natural numbers object is

$$M_0 = (\mathbb{N}_{\perp,\top}, \uparrow_{M_0}),$$

where $\mathbb{N}_{\perp,\top} = \mathbb{N} \cup \{\perp, \top\}$ is the algebraic lattice of flat natural numbers with bottom and top, and $\uparrow_{M_0}$ is the restriction of identity to $\mathbb{N}$. The iterated function spaces $M_1, M_2, \ldots$ are defined inductively by

$$M_j = M_{j-1} \to M_0.$$

The hierarchies $N_0, N_1, \ldots$ and $M_0, M_1, \ldots$ correspond to each other in view of the equivalence between $\mathsf{DPER}(\omega\mathsf{Dom})$ and PEqu, because they are both built from the natural numbers object by iterated use of exponentiation, hence the equivalence classes of M_j correspond naturally to the Kleene-Kreisel countable functionals of pure type j.

Totality on Domains for Dependent Types

We now generalize the results about the relationship between domains with totality and equilogical spaces to dependent type hierarchies. As we have shown, dense and codense totalities on Scott domains form a sub-ccc of equilogical spaces. Berger [Ber97a] extended the theory of totality for domains to dependent sums and products. We show that his constructions agree with the locally cartesian closed structure of equilogical spaces.

First we give a brief overview of the rather technical theorems and proofs that follow. We do not provide any proofs or references for the claims made in this overview, because they are repeated in more detail in the rest of the section. Berger [Ber97a, Ber97b] contains material on totalities for parameterizations on domains.

Let $F = (|F|, \|F\|)$ be a dense, codense and consistent totality on $D = (|D|, \|D\|)$, i.e., $(|F|, |D|)$ is a consistent parameterization on the domain $|D|$, $\|D\| \subseteq |D|$ is a dense and codense totality on $|D|$, and $(\|D\|, \|F\|)$ is a dense and codense dependent totality for $|F|$. We can explain the main point of the proof that the dependent types in domains with totality agree with dependent types in equilogical spaces by looking at how the dependent products are constructed in both setting. In the domain-theoretic setting a total element of the dependent product $P = \Pi(D, F)$ is a continuous map $f = \langle f_1, f_2\rangle \colon |D| \to |\Sigma(D, F)|$ that maps total elements to total elements and satisfies for all $x \in \|D\|$

$$f_1 x = x\ .$$

In PEqu a total element of the dependent product $Q = \prod_D F$ is a continuous map

$$g = \langle g_1, g_2\rangle \colon |D|^\top \to |\Sigma(D, F)|^\top$$

that preserves the partial equivalence relations and satisfies for all $x \in \|D\|$

$$g_1 x \uparrow_D x\ .$$

Here $\uparrow_D$ is the consistency relation on domain $|D|$, restricted to the totality $\|D\|$. In order to prove that P and Q are isomorphic we need to be able to translate an element $f \in \|P\|$ to one in $\|Q\|$, and vice versa. It is easy enough to translate $f \in \|P\|$ since we can just use f itself again. This is so because $f_1 x = x$ implies $f_1 x \uparrow_D x$. However, given a $g \in \|Q\|$, it is not obvious how to get a corresponding function in $\|P\|$. We need a way of continuously *transporting* 'level' $\|F(g_1 x)\|$ to 'level' $\|Fx\|$. In other words, we need a continuous map t such that whenever $x, y \in \|D\|$, $x \uparrow y$, and $u \in \|Fy\|$ then $t(y,x)u \in \|Fx\|$ and $\langle x, t(y,x)u\rangle \uparrow \langle y, u\rangle$ in $|\Sigma(D,F)|$. Given such a map t, the element of $\|P\|$ corresponding to $g \in \|Q\|$ is the map

$$x \mapsto \langle x, t(g_1 x, x)(g_2 x)\rangle.$$

The theory of totality for parameterizations on domains provides exactly what we need. Every consistent parameterization F has a *transporter* t, which has the desired properties. In addition, we must also require that the parameterization F be *natural*, which guarantees that $t(y,x)$ maps $\|Fy\|$ to $\|Fx\|$ whenever x and y are total and consistent. Berger [Ber97a] used the naturality conditions for dependent totalities to show that the consistency relation coincides with extensional equality. As equality of functions in equilogical spaces is defined extensionally, it is not surprising that naturality is needed in order to show the correspondence between the equilogical and domain-theoretic settings.

We explicitly describe the locally cartesian closed structure of PEqu. Let the *support* of an equilogical space A be the set

$$\|A\| = \left\{x \in |A| \;\middle|\; x \approx_A x\right\}.$$

Let $r\colon J \to I$ be a morphism in PEqu. The *pullback along* r^* is the functor

$$r^*\colon \mathsf{PEqu}/I \to \mathsf{PEqu}/J$$

that maps an object $a\colon A \to I$ over I to an object $r^*\colon r^*A \to J$ over J, as in the pullback diagram

$$\begin{tikzcd} r^*A \arrow[r] \arrow[d, "r^*a"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "a"] \\ J \arrow[r, "r"'] & I \end{tikzcd}$$

The pullback functor r^* has left and right adjoints. The left adjoint is the *dependent sum along* r

$$\textstyle\sum_r\colon \mathsf{PEqu}/J \to \mathsf{PEqu}/I$$

that maps an object $b\colon B \to J$ over J to the the object $\sum_r b = r \circ b\colon B \to\ I$ over I. The right adjoint to the pullback functor r^* is the *dependent product along* r

$$\textstyle\prod_r\colon \mathsf{PEqu}/J \to \mathsf{PEqu}/I,$$

defined as follows. Let $b\colon B \to J$ be an object in the slice over J. Let $\sim$ be a partial equivalence relation on the algebraic lattice $|I| \times (|J| \to |B|)$ defined by

$$\langle i, f\rangle \sim \langle i', f'\rangle$$
$$\text{if and only if}$$
$$i \approx_I i' \wedge \forall j, j' \in |J|\,.\,\big(j \approx_J j' \wedge r(j) \approx_I i \Longrightarrow f(j) \approx_B f'(j') \wedge b(f(j)) \approx_J j\big)$$

The dependent product $\prod_r b$ is the object $(|\prod_r b|, \sim)$, where

$$|\prod_r b| = |I| \times (|J| \to |B|) \ . \tag{4.1}$$

The map $\prod_r b \colon \prod_r b \to I$ is the obvious projection $\langle i, f\rangle \mapsto i$.

We define the 'top' functor $\Box^\top \colon \omega\mathsf{Dom} \to \omega\mathsf{ALat}$ by setting $D^\top$ to be the domain D with a new compact top element added to it. Given a map $f \colon D \to E$, let $f^\top \colon D^\top \to E^\top$ be defined by

$$f^\top x = \begin{cases} fx & \text{if } x \neq \top_D \\ \top_E & \text{if } x = \top_D \ . \end{cases}$$

It is is easily checked that $f^\top$ is a continuous map. We are going to use the following two lemmas and corollary later on.

Lemma 4.1.23 *Let C, D, and E be Scott domains and $f \colon C \to (D \to E^\top)$ a continuous map. Then the map $f' \colon C \to (D^\top \to E^\top)$, defined by*

$$f'xy = \begin{cases} fxy & \textit{if } y \neq \top_D \\ \top_E & \textit{if } y = \top_D \end{cases}$$

is also continuous.

Proof. We prove the equivalent claim that if $f \colon C \times D \to E^\top$ is continuous then $f' \colon C \times D^\top \to E^\top$, defined by

$$f'(x,y) = \begin{cases} f(x,y) & \text{if } y \neq \top_D \\ \top_E & \text{if } y = \top_D \end{cases}$$

is also continuous. First observe that if $V \subseteq C \times D$ is an open subset then $V \cup (C \times \{\top_D\})$ is an open subset of $C \times D^\top$. Hence, if $U \subseteq E^\top$ is a non-empty open set then $\top_E \in U$ and so the inverse image of U is

$$f'^*(U) = f^*(U \setminus \{\top_E\}) \cup (C \times \{\top_D\}),$$

which is an open set. ∎

Corollary 4.1.24 *Let D, and E be Scott domains and $f \colon D \to E^\top$ a continuous map. Then the map $f' \colon D^\top \to E^\top$ defined by*

$$f'y = \begin{cases} fy & \textit{if } y \neq \top_D \\ \top_E & \textit{if } y = \top_D \end{cases}$$

is also continuous.

Proof. Apply Lemma 4.1.23 with $C = \{\bot\}$. ∎

Lemma 4.1.25 *Suppose D and E are Scott domains, $S \subseteq D$ is an open subset, and $f \colon D \setminus S \to E^\top$ is a continuous map from the Scott domain $D \setminus S$ to the algebraic lattice $E^\top$. Then the map $f' \colon D \to E^\top$ defined by*

$$f'x = \begin{cases} fx & \textit{if } x \notin S \\ \top_E & \textit{if } x \in S \end{cases}$$

is also continuous.

Proof. Suppose $U \subseteq E^{\top}$ is a non-empty open subset. Its inverse image is

$$f'^{*}(U) = f^{*}(U) \cup S.$$

Because $f^{*}(U)$ is an open subset of $D \setminus S$ there exists an open subset $V \subseteq D$ such that $f^{*}(U) = V \cap (D \setminus S)$. Now it is clear that $f'^{*}(U)$ is open in D because it is equal to $V \cup S$. ∎

Recall that to each dense and codense totality D we assigned an equilogical space

$$\mathrm{Q}D = (|D|^{\top}, \uparrow_D) \tag{4.2}$$

where $\uparrow_D$ is the consistency relation restricted to the totality $\|D\|$, i.e., $x \uparrow_D y$ if, and only if, $x, y \in \|D\| \wedge x \uparrow y$.

Dependent Totalities for Domains

The following is a summary of totality for dependent types, as presented in Berger [Ber97a]. A *parameterization* on a domain $|D|$ is a co-continuous functor $F \colon |D| \to \omega\mathsf{Dom}^{\mathrm{ep}}$ from $|D|$, viewed as a category, to the category $\omega\mathsf{Dom}^{\mathrm{ep}}$ of Scott domains and *good* embeddings. Recall from [Ber97a] that an embedding-projection pair is good when the projection preserves arbitrary suprema. Whenever $x, y \in |D|$, $x \leq y$, there is an embedding $F(x \leq y)^{+} \colon Fx \to Fy$ and a projection $F(x \leq y)^{-} \colon Fy \to Fx$. We abbreviate these as follows, for $u \in Fx$ and $v \in Fy$:

$$u^{[y]} = F(x \leq y)^{+}(u) \;,$$
$$v_{[x]} = F(x \leq y)^{-}(v) \;.$$

A parameterization F on $|D|$ is *consistent* when it has a *transporter*. A transporter is a continuous map t such that for every $x, y \in |D|$, $t(x, y)$ is a map from Fx to Fy, satisfying:

(1) if $x \leq y$ then $F(x \leq y)^{+} \leq t(x, y)$ and $F(x \leq y)^{-} \leq t(y, x)$,

(2) $t(x, y)$ is strict,

(3) $t(y, z) \circ t(x, y) \leq t(x, z)$.

Let D be a totality. A *dependent totality* on D is a pair $F = (|F|, \|F\|)$ where $|F| \colon |D| \to \omega\mathsf{Dom}^{\mathrm{ep}}$ is a parameterization and $(\|D\|, \|F\|)$ is a totality for the parameterization $(|D|, |F|)$. Just like for totalities on domains, there are notions of *dense* and *codense* dependent totalities. See Berger [Ber97a] for definitions of these and also for definitions of *dependent sum* $\Sigma(D, F)$ and *dependent product* $\Pi(D, F)$. From now on we only consider dense and codense dependent totalities on consistent parameterizations.

A dependent totality F on D is *natural* if $\|D\|$ is upward closed in $|D|$, $\|Fx\|$ is upward closed in $|Fx|$ for all $x \in \|D\|$, and whenever $x \leq y \in \|D\|$ then

$$\forall\, v \in |Fy| \,.\, \big(v \in \|Fy\| \iff v_{[x]} \in \|Fx\|\big) \;.$$

Note that the above condition implies

$$\forall\, u \in |Fx| \,.\, \Big(u \in \|Fx\| \iff u^{[y]} \in \|Fy\|\Big) \;.$$

Lemma 4.1.26 *Let F be a natural dependent totality on D. Since F is consistent, it has a transporter t. Let $x, y \in \|D\|$, $x \uparrow y$, and $u \in \|Fy\|$. Then $t(y, x)u \in \|Fx\|$ and $\langle y, u\rangle \uparrow \langle x, t(y, x)u\rangle$ in $|\Sigma(D, F)|$.*

Proof. By naturality of F we have $(u^{[x\vee y]})_{[x]} \in \|Fx\|$, and since

$$(u^{[x\vee y]})_{[x]} \leq t(x \vee y, x)(t(y, x \vee y)u) \leq t(y,x)u$$

also $t(y,x)u \in \|Fx\|$. Furthermore, $\langle y, u\rangle \uparrow \langle x, t(y,x)u\rangle$ in $|\Sigma(D,F)|$ because $x \uparrow y$ and $u^{[x\vee y]} \uparrow (t(y,x)u)^{[x\vee y]}$, which follows from the common upper bound

$$\begin{aligned} u^{[x\vee y]} &\leq t(y, x\vee y)u, \\ (t(y,x)u)^{[x\vee y]} &\leq (t(x, x\vee y) \circ t(y,x))u \leq t(y, x\vee y)u \ . \end{aligned}$$

This completes the proof. ∎

Let F be a dependent totality on D and let G be a dependent totality on $\Sigma(D,F)$. Define a *parametrized* dependent totality $\widetilde{G}$, i.e., a co-continuous functor from D to the category of parameterizations [Ber97a], by

$$\widetilde{G}x = \lambda u \in Fx \,.\, G(x,u) \ .$$

More precisely, for each $x \in D$, $\widetilde{G}x$ is a dependent totality on Fx, defined by the curried form of G as above. In [Ber97a], which provides more details, $\widetilde{G}$ is called the *large Currying* of G. Given such a $\widetilde{G}$, there are parametrized versions of dependent sum $\Sigma(F,G)$ and dependent product $\Pi(F,G)$, which are dependent totalities on D, defined for $x \in D$ by

$$\begin{aligned} \Pi(F,G)x &= \Pi(Fx, \widetilde{G}x) \ , \\ \Sigma(F,G)x &= \Sigma(Fx, \widetilde{G}x) \ . \end{aligned}$$

To each natural dependent totality F on D we assign an equilogical space

$$\mathrm{q}(D,F) \colon \mathrm{Q}(D,F) \to \mathrm{Q}D$$

in the slice over $\mathrm{Q}D$ by defining

$$\mathrm{Q}(D,F) = \mathrm{Q}(\Sigma(D,F)) \tag{4.3}$$

$$\mathrm{q}(D,F) = \pi_1^{\top} \ , \tag{4.4}$$

where π_1 is the first projection $\pi_1 \colon |\Sigma(D,F)| \to |D|$, $\pi_1 \colon \langle x, u\rangle \mapsto x$.

Comparison of Dependent Types

We show that dependent sums and products on totalities coincide with those on equilogical spaces.

Theorem 4.1.27 (Main Theorem) *Let F be a dependent totality on D, and let G be a dependent totality on $\Sigma(D,F)$. The construction of dependent sum $\Sigma(F,G)$ and dependent product $\Pi(F,G)$ agrees with the construction of dependent sum and dependent product in* $\mathsf{PER}(\omega\mathsf{ALat})$, *i.e.,*

$$\begin{aligned} \mathrm{Q}(D, \Sigma(F,G)) &\cong \textstyle\sum_{\mathrm{q}(D,F)} \mathrm{q}(\Sigma(D,F), G) \ , \\ \mathrm{Q}(D, \Pi(F,G)) &\cong \textstyle\prod_{\mathrm{q}(D,F)} \mathrm{q}(\Sigma(D,F), G) \end{aligned}$$

in the slice over $\mathrm{Q}D$.

The rest of this subsection constitutes a proof of the Main Theorem, but before we embark on it, let us explain its significance. We have defined a translation Q from domain-theoretic dependent totalities to equilogical spaces. The Main Theorem says that this translation commutes with the construction of dependent sums and products. Thus, Q preserves the implicit local cartesian closed structure of totalities $\Sigma(F, G)$ and $\Pi(F, G)$. It may seem odd that we did not define a *functor* Q that would embed the dependent totalities into $\mathsf{PER}(\omega\mathsf{ALat})$ and preserve the locally cartesian closed structure. This can be done easily enough, by defining the morphisms $(D, F) \to (E, G)$ to be (equivalence classes of) equivalence-preserving continuous maps $\mathrm{Q}(D, F) \to \mathrm{Q}(E, G)$, i.e., essentially as the morphisms in $\mathsf{PER}(\omega\mathsf{ALat})$. Note that this is different from the definition of morphisms between parameterizations, as defined in Berger [Ber97a], where the motivation was to build the hierarchies in the first place, rather than to study an interpretation of dependent type theory. Thus, a notion of morphism suitable for the interpretation of dependent type theory was never explicitly given, although it is fairly obvious what it should be. In this manner we trivially obtain a full and faithful functor Q. The crux of the matter is that with such a choice of morphisms, the domain-theoretic constructions $\Sigma(F, G)$ and $\Pi(F, G)$ indeed yield the category-theoretic dependent sums and products. This is the main purpose of our work—to show that the domain theoretic constructions of dependent functionals, which has at times been judged arcane and ad hoc, is essentially the same as the dependent functionals arising in the realizability topos $\mathsf{RT}(\mathcal{P}\omega)$, which is much smoother and better understood from the category-theoretic point of view. The benefits of this correspondence go both ways. On the one hand, the domain-theoretic construction, which was conceived through a sharp conceptual analysis of the underlying domain-theoretic notions, is more easily understood and accepted by a category theorist. On the other hand, we can transfer the domain-theoretic results about the dependent functionals to Equ and $\mathsf{RT}(\mathcal{P}\omega)$, e.g., the Continuous Choice Principle from Sect 4.1.4. It is not clear how to obtain the Continuous Choice Principle directly in the realizability setting.

Lastly, we note that the Main Theorem is formulated for dependent sums and products with *parameters*, i.e., for parameterizations of parameterizations on domains; a parameter-free formulation states only that $Q(\Pi(D, F)) \cong \prod q(D, F)$. We need the theorem with parameters in order to establish the full correspondence between the lccc structures. We now proceed with the proof of the Main Theorem.

Dependent Sums. Dependent sums are easily dealt with because all we have to do is unravel all the definitions. For this purpose, let

$$
\begin{aligned}
X &= \mathrm{Q}(D, \Sigma(F, G)) \;, \\
Y &= \textstyle\sum_{\mathrm{q}(D,F)} \mathrm{q}(\Sigma(D, F), G) \;.
\end{aligned}
$$

In order to simplify the presentation we assume that ordered pairs and tuples satisfy the identities $\langle x, y, z\rangle = \langle\langle x, y\rangle, z\rangle = \langle x, \langle y, z\rangle\rangle$. This does affect the correctness of the proof, since it just amounts to leaving out the appropriate canonical isomorphisms. In particular, this assumption implies the equality $|\Sigma(\Sigma(D, F), G)| = |\Sigma(D, \Sigma(F, G))|$. From this it follows that the underlying lattices $|X|$ and $|Y|$ agree because

$$
\begin{aligned}
|Y| = |\mathrm{Q}(\Sigma(D, F), G)| &= |\Sigma(\Sigma(D, F), G)|^{\top} \\
&= |\Sigma(D, \Sigma(F, G))|^{\top} = |\mathrm{Q}(D, \Sigma(F, G))| = |X| \;.
\end{aligned}
$$

It remains to show that the partial equivalence relations on X and Y agree as well. It is easily checked that $\|X\| = \|Y\|$. For every $\langle x, u, v\rangle, \langle x', u', v'\rangle \in \|X\|$,

$$\begin{gathered}
\langle x, u, v\rangle \approx_X \langle x', u', v'\rangle \iff \\
x \uparrow x' \wedge \langle u, v\rangle^{[x \vee x']} \uparrow \langle u', v'\rangle^{[x \vee x']} \iff \\
x \uparrow x' \wedge \left(u^{[x \vee x']} \uparrow u'^{[x \vee x']} \wedge v^{[\langle x,u\rangle \vee \langle x',u'\rangle]} \uparrow v'^{[\langle x,u\rangle \vee \langle x',u'\rangle]}\right) .
\end{gathered}$$

Similarly, if $\langle x, u, v\rangle, \langle x', u', v'\rangle \in \|Y\|$ then we have

$$\begin{gathered}
\langle x, u, v\rangle \approx_Y \langle x', u', v'\rangle \iff \\
\langle x, u\rangle \uparrow \langle x', u'\rangle \wedge v^{[\langle x,u\rangle \vee \langle x',u'\rangle]} \uparrow v'^{[\langle x,u\rangle \vee \langle x',u'\rangle]} \iff \\
\left(x \uparrow x' \wedge u^{[x \vee x']} \uparrow u'^{[x \vee x']}\right) \wedge v^{[\langle x,u\rangle \vee \langle x',u'\rangle]} \uparrow v'^{[\langle x,u\rangle \vee \langle x',u'\rangle]} .
\end{gathered}$$

Dependent Products. Dependent products are more complicated. Let

$$\begin{aligned}
U &= \mathrm{Q}(D, \Pi(F, G)) , \\
V &= \textstyle\prod_{\mathrm{q}(D,F)} \mathrm{q}(\Sigma(D, F), G) .
\end{aligned}$$

Let us explicitly describe U and V. The underlying lattice of U is

$$|U| = |\Sigma(D, \Pi(F, G))|^{\top} . \tag{4.5}$$

The partial equivalence relation on U relates $\langle x, f\rangle \in |U|$ and $\langle y, g\rangle \in |U|$ if, and only if,

$$\begin{aligned}
& x \uparrow_D y \wedge \\
& \forall\, u \in \|Fx\| \,.\, fu \in \|G(x, u)\| \wedge \\
& \forall\, v \in \|Fy\| \,.\, gv \in \|G(y, v)\| \wedge \\
& \forall\, w \in |F(x \vee y)| \,.\, \left((f(w_{[x]}))^{[\langle x \vee y, w\rangle]} \uparrow (g(w_{[y]}))^{[\langle x \vee y, w\rangle]}\right) .
\end{aligned}$$

By (4.1), the underlying lattice of V is

$$|V| = |D|^{\top} \times \left(|\Sigma(D, F)|^{\top} \to |\Sigma(\Sigma(D, F), G)|^{\top}\right) . \tag{4.6}$$

Elements $\langle x, y\rangle \in |V|$ and $\langle y, g\rangle \in |V|$ are related if, and only if, the following holds: $x \uparrow_D y$, and for all $z, z' \in |D|$ such that $z \uparrow_D x$ and $z' \uparrow_D x$, and for all $w \in |Fz|$, $w' \in |Fz'|$ such that $w^{[z \vee z']} \uparrow_{F(z \vee z')} w'^{[z \vee z']}$,

$$\begin{aligned}
& f\langle z, w\rangle \uparrow_{\Sigma(\Sigma(D,F),G)} g(z', w') \wedge \\
& \pi_1(f\langle z, w\rangle) \uparrow_{\Sigma(D,F)} \langle z, w\rangle \wedge \\
& \pi_1(g\langle z', w'\rangle) \uparrow_{\Sigma(D,F)} \langle z', w'\rangle .
\end{aligned}$$

We define maps $\phi\colon |U| \to |V|$ and $\theta\colon |V| \to |U|$, and verify that they represent isomorphisms between U and V. Let t be a transporter for the parameterization F. Define the map $\phi\colon |U| \to |V|$ by

$$\begin{aligned}
\phi\top &= \top , \\
\phi(x, f) &= \langle x, \phi_2(x, f)\rangle ,
\end{aligned}$$

where $\phi_2(x, f)\colon |\Sigma(D, F)|^\top \to |\Sigma(\Sigma(D, F), G)|^\top$ is

$$\begin{aligned}\phi_2(x, f)\top &= \top\ ,\\ \phi_2(x, f)(y, u) &= \langle x, t(y, x)u, f(t(y, x)u)\rangle\ .\end{aligned}$$

Let s be a transporter for the parameterization G on $\Sigma(D, F)$. Define the map $\theta\colon |V| \to |U|$ by

$$\begin{aligned}\theta(\top, g) &= \top\\ \theta(x, g) &= \texttt{if}\ \ \exists\, u \in |Fx|\,.\, g(x, u) = \top\\ &\ \texttt{then}\ \ \top\\ &\ \texttt{else}\ \ \langle x, \lambda u \in |Fx|\,.\, s(g_1(x, u), \langle x, u\rangle)(g_2(x, u))\rangle\end{aligned}$$

where $g = \langle g_1, g_2\rangle\colon |\Sigma(D, F)| \to |\Sigma(\Sigma(D, F), G)|$.

It is easy and tedious to verify that ϕ and θ have the intended types. Continuity of ϕ follows directly from Corollary 4.1.24 and Lemma 4.1.23. Continuity of θ follows from Lemmas 4.1.23 and 4.1.25. We can apply Lemma 4.1.25 because the set

$$\big\{\langle x, g\rangle \bigm| \exists\, u \in |Fx|\,.\, g(x, u) = \top\big\} \subseteq |D| \times \Big(|\Sigma(D, F)|^\top \to |\Sigma(\Sigma(D, F), G)|^\top\Big)$$

is open, as it is a projection of the open set

$$\big\{\langle x, u, g\rangle \bigm| g(x, u) = \top\big\} \subseteq |\Sigma(D, F)| \times \Big(|\Sigma(D, F)|^\top \to |\Sigma(\Sigma(D, F), G)|^\top\Big)\ .$$

Next we verify that ϕ and θ represent morphisms and that they are inverses of each other. Since we only work with total elements from now on, we do not have to worry about the cases when $\top$ appears as an argument or a result of an application.

(1) ϕ represents a morphism $U \to V$ in the slice over $\mathsf{Q}D$. Let $\langle x, f\rangle, \langle x', f'\rangle \in \|U\|$ and suppose $\langle x, f\rangle \uparrow \langle x', f'\rangle$. This means that $x \uparrow x'$ and $f^{[x\vee x']} \uparrow f'^{[x\vee x']}$, i.e., for every $w \in |F(x \vee x')|$

$$(f(w_{[x]}))^{[\langle x\vee x', w\rangle]} \uparrow (f'(w_{[x']}))^{[\langle x\vee x', w\rangle]}\ .$$

We prove that $\phi(x, f) \approx_V \phi(x', f')$. Clearly, $x \uparrow_D x'$ since $x \uparrow x'$ and $x, x' \in \|D\|$. Let

$$\begin{aligned}g &= \pi_2(\phi(x, f)) = \lambda\langle y, u\rangle \in |\Sigma(D, F)|\,.\,\langle x, t(y, x)u, f(t(y, x)u)\rangle\\ g' &= \pi_2(\phi(x', f')) = \lambda\langle y, u\rangle \in |\Sigma(D, F)|\,.\,\langle x', t(y, x')u, f'(t(y, x')u)\rangle\ .\end{aligned}$$

Let $y, y' \in \|D\|$ such that $y \uparrow y'$ and $y \uparrow x$. Let $u \in \|Fy\|$ and $u' \in \|Fy'\|$ such that $u^{[y\vee y']} \uparrow u'^{[y\vee y']}$. We need to show the following:

(a) $\langle y, u\rangle \uparrow \langle x, t(y, x)u\rangle$

(b) $g(y, u) \in \|\Sigma(\Sigma(D, F), G)\|$

(c) $g'(y', u') \in \|\Sigma(\Sigma(D, F), G)\|$

(d) $(g(y, u))^{[\langle y,u\rangle\vee\langle y',u'\rangle]} \uparrow (g'(y', u'))^{[\langle y,u\rangle\vee\langle y',u'\rangle]}$.

Proof of (a): by assumption $y \uparrow x$, and $u^{[x \vee y]} \uparrow t(y,x)(u)^{[x \vee y]}$ holds because of the common upper bound:

$$\begin{aligned} u^{[x\vee y]} &\leq t(y, x \vee y)u \\ (t(y,x)u)^{[x\vee y]} &\leq (t(x, x \vee y) \circ t(y,x))u \leq t(y, x \vee y)u \ . \end{aligned}$$

Proof of (b): by assumption $x \in \|D\|$, and also $t(y,x)u \in \|Fx\|$ because $x, y \in \|D\|$, $x \uparrow y$ and $u \in \|Fy\|$. Finally, $f(t(y,x)u) \in \|G(x, t(y,x)u)\|$ because $f \in \|\Pi(Fx, \widetilde{G}x)\|$. The proof of (c) is analogous to the proof (b).
Proof of (d): by assumption $x \uparrow x'$, and $(t(y,x)u)^{[x\vee x']} \uparrow (t(y',x')u')^{[x\vee x']}$ holds because

$$\begin{aligned} (t(y,x)u)^{[x\vee x']} &\leq t(y, x\vee x')u \leq t(y\vee y', x \vee x')(u^{[y\vee y']}) \\ (t(y',x')u')^{[x\vee x']} &\leq t(y', x\vee x')u' \leq t(y\vee y', x \vee x')(u'^{[y\vee y']}) \end{aligned}$$

and $u^{[y\vee y']} \uparrow u'^{[y\vee y']}$. Let $z = t(y,x)u$ and $z' = t(y',x')u'$, and let $w = z^{[x\vee x']} \vee z'^{[x\vee x']}$. We claim that

$$(fz)^{[\langle x\vee x', w\rangle]} = (fz)^{[\langle x,z\rangle \vee \langle x',z'\rangle]} \uparrow (f'z')^{[\langle x,z\rangle \vee \langle x',z'\rangle]} = (f'z')^{[\langle x\vee x', w\rangle]} \ .$$

From $z \leq w_{[x]}$ it follows that $fz \leq f(w_{[x]})$, hence

$$(fz)^{[\langle x\vee x', w\rangle]} \leq (f(w_{[x]}))^{[\langle x\vee x', w\rangle]} \ ,$$

and similarly,

$$(f'z')^{[\langle x\vee x', w\rangle]} \leq (f'(w_{[x']}))^{[\langle x\vee x', w\rangle]} \ .$$

The claim holds because $f(w_{[x]})^{[\langle x\vee x', w\rangle]} \uparrow f'(w_{[x']})^{[\langle x\vee x', w\rangle]}$.

(2) θ represents a morphism $V \to U$ in the slice over $\mathrm{Q}D$. Suppose $\langle x, g\rangle \approx_V \langle x', g'\rangle$ and let

$$\begin{aligned} f &= \pi_2(\theta(x,g)) = \lambda u \in |Fx| \,.\, s(g_1(x,u), \langle x,u\rangle)(g_2(x,u)) \ , \\ f' &= \pi_2(\theta(x',g')) = \lambda u' \in |Fx'| \,.\, s(g_1'(x',u'), \langle x',u'\rangle)(g_2'(x',u')) \ . \end{aligned}$$

We need to show that

(a) $fu \in \|G(x,u)\|$ for every $u \in \|Fx\|$

(b) $f'u' \in \|G(x',u')\|$ for every $u' \in \|Fx'\|$

(c) for every $w \in |F(x\vee x')|$,

$$(f(w_{[x]}))^{[\langle x\vee x', w\rangle]} \uparrow (f'(w_{[x']}))^{[\langle x\vee x', w\rangle]} \ .$$

Proof of (a): $\langle x,u\rangle \in \|\Sigma(D,F)\|$, $g(x,u) \in \|\Sigma(\Sigma(D,F),G)\|$ and $\langle x,u\rangle \uparrow g_1(x,y)$ imply

$$fu = s(g_1(x,u), \langle x,u\rangle)(g_2(x,u)) \in \|G(x,u)\|.$$

The proof of (b) is similar.

Proof of (c): since $x \vee x' \in \|D\|$, $\|F(x \vee x')\|$ is a dense subset of $|F(x \vee x')|$, so it is sufficient to verify the claim for $w \in \|F(x \vee x')\|$. For such a w it follows from naturality of F that $w_{[x]} \in \|F(x)\|$, and similarly $w_{[x']} \in \|F(x')\|$. From $\langle x, w_{[x]}\rangle \uparrow \langle x', w_{[x']}\rangle$ we may conclude that $g(x, w_{[x]}) \uparrow g'(x', w_{[x']})$, hence

$$s(g_1(x, w_{[x]}), \langle x \vee x', w\rangle)(g_2(x, w_{[x]})) \uparrow s(g_1'(x', w_{[x']}), \langle x \vee x', w\rangle)(g_2'(x', w_{[x']})) \ .$$

Finally, observe that

$$\begin{aligned}
(f(w_{[x]}))^{[\langle x \vee x', w\rangle]} &= (s(g_1(x, w_{[x]}), \langle x, w_{[x]}\rangle)(g_2(x, w_{[x]})))^{[\langle x \vee x', w\rangle]} \\
&\leq s(g_1(x, w_{[x]}), \langle x \vee x', w\rangle)(g_2(x, w_{[x]})) \ , \\
(f'(w_{[x']}))^{[\langle x \vee x', w\rangle]} &= (s(g_1'(x', w_{[x']}), \langle x', w_{[x']}\rangle)(g_2'(x', w_{[x']})))^{[\langle x \vee x', w\rangle]} \\
&\leq s(g_1'(x', w_{[x']}), \langle x \vee x', w\rangle)(g_2'(x', w_{[x']})) \ .
\end{aligned}$$

(3) $\theta \circ \phi \approx_{U \to U} 1_U$. Let $\langle x, f\rangle \in \|U\|$. We need to show that $\theta(\phi(x, f)) \uparrow \langle x, f\rangle$. The first component is obvious since $\pi_1(\theta(\phi(x, f))) = x$. As for the second component, for any $v \in \|Fx\|$,

$$\begin{aligned}
(\pi_2(\theta(\phi(x, f))))v &= s(\langle x, t(x, x)v\rangle, \langle x, v\rangle)(f(t(x, x)v)) \\
&\geq s(\langle x, v\rangle, \langle x, v\rangle)(fv) \\
&\geq fv \ ,
\end{aligned}$$

hence $\pi_2(\theta(\phi(x, f))) \uparrow f$.

(4) $\phi \circ \theta \approx_{V \to V} 1_V$. Let $\langle x, g\rangle \in \|V\|$. We need to show that $\phi(\theta(x, g)) \approx_V \langle x, g\rangle$. Again, the first component is obvious since $\pi_1(\phi(\theta(x, g))) = x$. For the second component, given any $\langle y, u\rangle \in \|\Sigma(D, F, \|)$ such that $x \uparrow y$, what has to be shown is

$$\langle x, t(y, x)u, s(g_1(x, t(y, x)u), \langle x, t(y, x)u\rangle)(g_2(x, t(y, x)u))\rangle \uparrow g(y, u) \ .$$

First, we have

$$\langle x, t(y, x)u\rangle \uparrow \langle y, u\rangle \text{ and } \langle y, u\rangle \uparrow g_1(y, u),$$

and since these are elements of a codense totality, we may conclude by transitivity that $\langle x, t(y, x)u\rangle \uparrow g_1(y, u)$. Let $z = g_1(y, u)$ and $w = \langle x, t(y, x)u\rangle$. The relation

$$(g_2(y, u))^{[z \vee w]} \uparrow (s(g_1 w, w)(g_2 w))^{[z \vee w]}$$

holds because

$$\begin{aligned}
(g_2(y, u))^{[z \vee w]} &\leq s(z, z \vee w)(g_2(y, u)) \\
s(g_1 w, w)(g_2 w)^{[z \vee w]} &\leq s(g_1 w, z \vee w)(g_2 w) \ ,
\end{aligned}$$

and $(y, u) \uparrow w$ together with monotonicity of the function $s(g_1 \Box, z \vee w)(g_2 \Box)$ imply that

$$s(z, z \vee w)(g_2(y, u)) \uparrow s(g_1 w, z \vee w)(g_2 w) \ .$$

This concludes the proof of the Main Theorem.

Let $\mathcal{B}$ be the full subcategory of Equ on objects $\mathrm{Q}D$ where D is a natural totality, i.e., $\|D\|$ is a dense, codense, and upward closed subset of $|D|$. It is the case that $\mathcal{B}$ is a cartesian closed subcategory of Equ. However, note that the Main Theorem does not imply that $\mathcal{B}$ is a locally cartesian closed subcategory of Equ. We only showed that $\mathcal{B}$ is closed under those dependent sums and products that correspond to parameterizations on domains. In order to resolve the question whether $\mathcal{B}$ is locally cartesian closed it would be useful to have a good characterization of $\mathcal{B}$ in terms of the categorical structure of Equ.

Continuous Choice Principle

As an application of the Main Theorem, we translate Berger's Continuous Choice Principle for dependent totalities [Ber97a] into a Choice Principle expressed in the internal logic of Equ.

Let (D, F) be a dependent totality. By [Ber97a, Proposition 3.5.2] there is a continuous functional

$$\mathsf{choose} \in |\Pi(x : D, (Fx \to \mathbb{B}_\perp) \to Fx)|$$

such that for all $x \in \|D\|$ and $p \in \|Fx \to \mathbb{B}\|$, if $p^*(\mathsf{true}) \neq \emptyset$, then $(\mathsf{choose}\, x)p \in p^*(\mathsf{true}) \cap \|Fx\|$. By looking at the proof of [Ber97a, Proposition 3.5.2], we see that choose is *not* a *total* functional of type $\|\Pi(x : D, (Fx \to \mathbb{B}_\perp) \to Fx)\|$ because choose applied to the constant function $\lambda x.\,\mathsf{false}$ yields $\perp$, which is not total. This means that choose does not represent a morphism in Equ. Nevertheless we can use it to construct a realizer for the following Choice Principle, stated in the internal logic of Equ:

$$\forall\, p \in (\textstyle\sum_{x : D} Fx) \to \mathsf{2}\,.\, \Big(\big(\forall\, x \in D\,.\, \neg\neg\exists\, y \in Fx\,.\,(p(x,y) = 1)\,\big) \Longrightarrow \big(\exists\, h \in \textstyle\prod_{x : D} Fx\,.\, \forall\, x \in D\,.\, p(x, hx) = 1\big)\Big) \tag{4.7}$$

We omit the proof. Suffice it to say that (4.7) is realized using choose in much the same way as in the proof of [Ber97a, Corollary 3.5.3].

If we specialize (4.7) by setting $D = \mathsf{1}$ and $F = \mathbb{N}$, we obtain

$$\forall\, p \in \mathbb{N} \to \mathsf{2}\,.\, \big((\neg\neg\exists\, y \in \mathbb{N}\,.\, py = \mathsf{true}) \Longrightarrow \exists\, z \in \mathbb{N}\,.\, pz = \mathsf{true}\big)$$

This is a form of *Markov's Principle*. Thus, (4.7) is a generalization of Markov's Principle. This view is in accordance with the construction of the choose functional in [Ber97a], which works by *searching* for a witness.

We conclude this section with a comment on the significance of the density and codensity theorems [Ber97a] for the results presented here. We defined a translation from dependent totalities to equilogical spaces, and showed that it preserves dependent sums and products. The density theorems for dependent totalities ensure that the translation is well defined in the first place. Thus, density plays a fundamental role, which is further supported by Theorem 4.1.21, which states that the category of equilogical spaces is equivalent to the category of *dense* partial equivalence relations on Scott domains. The effect of codensity is that the translation of domain-theoretic totalities into equilogical spaces gives a rather special kind of *totally disconnected equilogical spaces*. An equilogical space X is totally disconnected when the curried form of the evaluation map $X \to \mathsf{2}^{\mathsf{2}^X}$ is monic, or equivalently, when the topological quotient $\|X\|/{\approx_X}$ is a totally disconnected space.[2]

[2] A topological space is totally disconnected when every two distinct points can be separated by a clopen set.

There are totally disconnected equilogical spaces that do not arise as dense and codense totalities. The subcategory of totally disconnected equilogical spaces is a locally cartesian closed subcategory of Equ. Perhaps the notion of total disconnectedness, or some refinement of it, can be used to prove the Choice Principle (4.7) directly in Equ.

4.1.5 Equilogical Spaces as a Subcategory of Domain Representations

A *domain representation* is a partial continuous surjection from a Scott domain to a topological space. Alternatively, a domain representation can be viewed as a partial equivalence relation on a Scott domain. Domain representations were used by Blanck [Bla97a, Bla97b, Bla99] to study computability on topological spaces. There is an evident similarity between domain representations and equilogical spaces, as the latter ones can be viewed as partial equivalence relations on countably based algebraic lattices. In this section we investigate the connection between domain representations and equilogical spaces.

Let $\mathsf{PER}(\omega\mathsf{Dom})$ be the category of partial equivalence relations and equivalence classes of equivariant maps between them. It is defined just like the category $\mathsf{PER}(\omega\mathsf{ALat})$ except that countably based algebraic lattices are replaced with Scott domains. First we show that the domain representations are equivalent to a category of modest sets.

Theorem 4.1.28 *The categories* $\mathsf{PER}(\omega\mathsf{Dom})$ *and* $\mathsf{Mod}(\mathbb{U})$ *are equivalent.*

Proof. For convenience, we work with $\mathsf{PER}(\mathbb{U})$ instead of $\mathsf{Mod}(\mathbb{U})$. Let us define the equivalence functors

$$\mathsf{PER}(\omega\mathsf{Dom}) \underset{I}{\overset{F}{\rightleftarrows}} \mathsf{PER}(\mathbb{U}) \ .$$

The functor I is just the inclusion of $\mathsf{PER}(\mathbb{U})$ into $\mathsf{PER}(\omega\mathsf{Dom})$. The functor F maps a per $(D, \approx_D)$ to the per $(\mathbb{U}, \approx_{FD})$, where $\approx_{FD}$ is defined by

$$u \approx_{FD} v \iff \exists\, x, y \in D\,.\,(u = i_D x \land v = i_D y \land x \approx_D y) \ .$$

A morphism $[f]\colon (D, \approx_D) \to (E, \approx_E)$ is mapped to the morphism

$$F[f] = [Ff] = [i_E \circ f \circ p_D]\colon (\mathbb{U}, \approx_{FD}) \to (\mathbb{U}, \approx_{FE}) \ .$$

The map Ff preserves pers, for whenever $x \approx_D y$ then $fx \approx_E fy$, therefore $i_E(fx) \approx_{FE} i_E(fy)$, hence

$$(Ff)(i_D x) = (i_E \circ f \circ p_D \circ i_D)x = (i_E \circ f)x \approx_{FE} (i_E \circ f)y = (Ff)(i_D y) \ .$$

A similar argument shows that the definition of $F[f]$ does not depend on the choice of the representative f. It remains to be shown that $F \circ I$ and $I \circ F$ are both naturally isomorphic to the identity functors.

By construction, for every object $(\mathbb{U}, \approx)$ in $\mathsf{PER}(\mathbb{U})$, $F(I(\mathbb{U}, \approx)) = F(\mathbb{U}, \approx) = (\mathbb{U}, \approx)$, since we defined I to be the inclusion and we assumed that $(i_{\mathbb{U}}, p_{\mathbb{U}})$ is the identity.

Take any object $(D, \approx_D)$ in $\mathsf{PER}(\omega\mathsf{Dom})$. We prove that $I(F(D, \approx_D)) \cong (D, \approx_D)$. The embedding i_D and the projection p_D serve as the representatives of isomorphisms. The embedding i_D preserves the pers by construction. The projection p_D also preserves pers, because whenever $u \approx_{FD} v$ then, for some $x, y \in D$, $u = i_D x$, $v = i_D y$ and $x \approx_D y$, therefore

$$p_D u = p_D(i_D x) = x \approx_D y = p_D(i_D y) = p_D v \ .$$

Lastly, we need to check that $[p_D]$ and $[i_D]$ are inverses of each other. Since $p_D \circ i_D = 1_D$, it follows immediately that $[p_D] \circ [i_D] = [1_D]$. For the other composition, assume $u \approx_{FD} u$. Then $u = i_D x$ and $x \approx_D x$ for some $x \in D$, therefore

$$i_D(p_D u) = (i_D \circ p_D \circ i_D)x = i_D x = u \approx_{FD} u\ .$$

■

Let $\mathsf{PER}(\mathsf{Dom}_{\mathsf{eff}})$ be the category of partial equivalence relations on effective domains and equivalence relations of computable equivariant maps. The following theorem is an effective version of Theorem 4.1.28. It identifies partial equivalence relations on effective domains as the category $\mathsf{Mod}(\mathbb{U}, \mathbb{U}_\sharp)$.

Theorem 4.1.29 *The categories* $\mathsf{PER}(\mathsf{Dom}_{\mathsf{eff}})$ *and* $\mathsf{Mod}(\mathbb{U}, \mathbb{U}_\sharp)$ *are equivalent.*

Proof. The proof is the same as the proof of 4.1.28, except that we have to pay attention to computability of maps. If D is an effective domain, then there exists a computable embedding-projection pair $i_D \colon D \to \mathbb{U}$ and $p_D \colon \mathbb{U} \to D$. Therefore the equivalence functor $F \colon \mathsf{PER}(\omega\mathsf{Dom}) \to \mathsf{PER}(\mathbb{U}, \mathbb{U}_\sharp)$, which is defined by $F[f] = [i_E \circ f \circ p_D]$, really maps computable maps to computable maps. ■

By Theorem 4.1.21, equilogical spaces are equivalent to *dense* partial equivalence relations on Scott domains. It makes sense to ask whether $\mathsf{PER}(\omega\mathsf{Dom})$ and equilogical spaces are actually equivalent. It is easier to compare $\mathsf{PER}(\omega\mathsf{Dom})$ with $\mathsf{PER}(\omega\mathsf{ALat})$ than the equivalent category Equ.

Theorem 4.1.30 *The categories* $\mathsf{PER}(\omega\mathsf{Dom})$ *and* $\mathsf{PER}(\omega\mathsf{ALat})$ *are* not *equivalent.*

Proof. Since $\mathsf{PER}(\omega\mathsf{Dom})$ is equivalent to $\mathsf{Mod}(\mathbb{U})$ and $\mathsf{PER}(\omega\mathsf{ALat})$ is equivalent to $\mathsf{Mod}(\mathbb{P})$, it is sufficient to show that $\mathsf{Mod}(\mathbb{U})$ and $\mathsf{Mod}(\mathbb{P})$ are not equivalent.

If two categories are equivalent, then the equivalence functors preserve and reflect projective objects,[3] cf. Definition 1.3.1. Furthermore, the equivalence functors preserve the global points of an object.[4] So, if two categories are equivalent, then they must have the same number of isomorphism classes of projective objects with two global points. But we are now going to show that $\mathsf{Mod}(\mathbb{P})$ has two such classes, whereas $\mathsf{Mod}(\mathbb{U})$ has three.

By Theorem 1.3.4, an object in a category of modest sets is projective if, and only if, it is isomorphic to a canonically separated one, cf. Definition 1.3.2. In other words, it is sufficient to consider modest sets for which every element is realized by exactly one realizer. To say that a modest set in $\mathsf{Mod}(\mathbb{P})$ or $\mathsf{Mod}(\mathbb{U})$ has two global points is to say that it has two elements. This description makes it clear that a canonically separated modest set with two global points amounts to a two-element subset of the underlying PCA.

How many non-isomorphic types of two-element subsets $A = \{a_0, a_1\} \subseteq \mathbb{P}$ are there? It is easy to see that there are just two. If a_0 and a_1 are comparable, then A is isomorphic to the set $\{\emptyset, \mathbb{N}\}$, and if a_0 and a_1 are incomparable then A is isomorphic to the set $\{\{0\}, \{1\}\}$. The isomorphism are obtained by the Extension Theorem 1.1.3. It is obvious that $\{\emptyset, \mathbb{N}\}$ and $\{\{0\}, \{1\}\}$ are not isomorphic, since only the constant maps from the former into the latter one are realized.

[3] This means that an object is projective if, and only if, its image under the functor is.

[4] The global points of an object X are the morphism $\mathsf{Hom}(1, X)$ from the terminal object into X.

However, in $\mathsf{Mod}(\mathbb{U})$ there are three possibilities. First, note that $\mathbb{U}$ is not a lattice, because a retract of a lattice is a lattice, whereas every domain appears as a retract of $\mathbb{U}$. Therefore, we can find two inconsistent elements $x_0, x_1 \in \mathbb{U}$. Clearly, we can also find two incomparable consistent elements y_0, y_1, and finally, we can find two comparable elements z_0, z_1. These are the three types of two-element subsets of $\mathbb{U}$. Indeed, suppose $B = \{b_0, b_1\} \subseteq \mathbb{U}$ is a two-element subset. If b_0 and b_1 are inconsistent, the desired isomorphism $B \to \{x_0, x_1\}$ is the join of two step functions

$$\mathsf{step}_{b'_0,x_0} \vee \mathsf{step}_{b'_1,x_1} \ ,$$

where b'_0 and b'_1 are incomparable compact elements below b_0 and b_1, respectively. If b_0 and b_1 are incomparable and consistent then use $\mathsf{step}_{b'_0,y_0} \vee \mathsf{step}_{b'_1,y_1}$ to establish an isomorphism $B \to \{y_0, y_1\}$. If b_0 and b_1 are comparable then there exist compact elements b'_0 and b'_1 such that $b'_0 \leq b_0$, $b'_1 \leq b_1$, and $b'_1 \not\leq b_0$. The function $\mathsf{step}_{b_0{}',z_0} \vee \mathsf{step}_{b_1{}',z_1}$ to is an isomorphism from B to $\{z_0, z_1\}$. The sets $\{x_0, x_1\}$, $\{y_0, y_1\}$, and $\{z_0, z_1\}$ are pairwise non-isomorphic because only constant maps from $\{z_0, z_1\}$ into the other two are realized, and only constant maps from $\{y_0, y_1\}$ to $\{z_0, z_1\}$ are realized. ∎

In Subsection 1.4.3 we constructed an applicative inclusion $(\eta \vdash \zeta)\colon \mathsf{Mod}(\mathbb{P}, \mathbb{P}_\sharp) \to \mathsf{Mod}(\mathbb{U}, \mathbb{U}_\sharp)$. Since both η and ζ are discrete they induce a pair of functors:

$$\widehat{\eta}\colon \mathsf{Mod}(\mathbb{U}, \mathbb{U}_\sharp) \to \mathsf{Mod}(\mathbb{P}, \mathbb{P}_\sharp) \ , \qquad \widehat{\eta}\colon \mathsf{Mod}(\mathbb{P}, \mathbb{P}_\sharp) \to \mathsf{Mod}(\mathbb{U}, \mathbb{U}_\sharp) \ .$$

We summarize their properties in the following proposition.

Proposition 4.1.31 *There are adjoint functors $\widehat{\eta} \dashv \widehat{\zeta}$,*

$$\begin{tikzcd} \mathsf{Mod}(\mathbb{P}, \mathbb{P}_\sharp) \arrow[r, shift right, "\widehat{\zeta}"'] & \mathsf{Mod}(\mathbb{U}, \mathbb{U}_\sharp) \arrow[l, shift right, "\widehat{\eta}"'] \end{tikzcd}$$

where:

(1) $\widehat{\eta}$ and $\widehat{\zeta}$ are induced by an applicative inclusion $\eta \dashv \zeta$, as defined in Subsection 1.4.3.

(2) $\widehat{\eta} \circ \widehat{\zeta}$ is naturally isomorphic to $1_{\mathsf{Mod}(\mathbb{P},\mathbb{P}_\sharp)}$.

(3) $\widehat{\zeta}$ is full and faithful, and $\widehat{\eta}$ is full but not faithful.

(4) $\widehat{\zeta}$ preserves regular epis, and $\widehat{\eta}$ preserves finite limits.

(5) $\widehat{\eta}$ preserves the natural numbers object, whereas $\widehat{\zeta}$ does not.

(6) $\widehat{\zeta}$ preserves exponentials, whereas $\widehat{\eta}$ does not.

Proof. (1) By construction. (2) By Theorem 1.4.12(2). (3) $\widehat{\zeta}$ is full and faithful by Proposition 1.4.13(1), $\widehat{\eta}$ is faithful by Proposition 1.4.5(1). (4) By Theorem 1.4.12(1). (5) By Proposition 1.4.5(3). (6) By Proposition 1.4.13(1). ∎

The applicative morphisms η and ζ also induce functors on the non-effective versions of modest sets,

$$\widehat{\eta}\colon \mathsf{Mod}(\mathbb{U},\mathbb{U}_\sharp) \to \mathsf{Mod}(\mathbb{P},\mathbb{P}_\sharp)\ , \qquad \widehat{\eta}\colon \mathsf{Mod}(\mathbb{P},\mathbb{P}_\sharp) \to \mathsf{Mod}(\mathbb{U},\mathbb{U}_\sharp)\ ,$$

which have all of the properties stated in Proposition 4.1.31. Curiously, $\widehat{\eta}$ has a further left adjoint, which does not exist in the effective version of modest sets.

Theorem 4.1.32 *The functor* $\widehat{\eta}\colon \mathsf{Mod}(\mathbb{U}) \to \mathsf{Mod}(\mathbb{P})$ *has a left adjoint* $K\colon \mathsf{Mod}(\mathbb{P}) \to \mathsf{Mod}(\mathbb{U})$.

Proof. The functor K has already been constructed in the proof of Theorem 4.1.21. Let us repeat the construction in terms of $\mathsf{Mod}(\mathbb{U})$ and $\mathsf{Mod}(\mathbb{P})$. The empty modest set $\emptyset$ over $\mathbb{P}$ is mapped to the empty modest set $K\emptyset = \emptyset$ over $\mathbb{U}$. For a non-empty modest set S over $\mathbb{P}$, let $D_S = \overline{\bigcup_{x \in S} \mathsf{E}_S x}$. The subspace $D_S \subseteq \mathbb{P}$ is the topological closure of a non-empty subset of $\mathbb{P}$, hence it is a Scott domain. There exists an embedding-projection pair[5] $i_S\colon D_S \to \mathbb{U}$, $p_S\colon \mathbb{U} \to D_S$. Define KS to be the set $|S|$ with the existence predicate defined by

$$\mathsf{E}_{SK} x = \left\{ i_S a \;\middle|\; a \in \mathsf{E}_S x \right\}\ .$$

A morphism $f\colon S \to T$ is mapped to the morphism $Kf = f\colon |KS| \to |KT|$, which is tracked by $(i_T \circ g \circ p_S\colon \mathbb{U} \to \mathbb{U}$, where $g\colon \mathbb{P} \to \mathbb{P}$ tracks f.

Let $S \in \mathsf{Mod}(\mathbb{P})$ and $T \in \mathsf{Mod}(\mathbb{U})$. Suppose $f\colon |S| \to |\widehat{\eta}T|$ is tracked by $p\colon \mathbb{P} \to \mathbb{P}$. Then $f\colon |KS| \to |T|$ is tracked by $\eta^{-1} \circ p \circ p_S$. Conversely, if $h\colon |KS| \to |T|$ is tracked by $u\colon \mathbb{U} \to \mathbb{U}$, then $h\colon |S| \to |\widehat{\eta}T|$ is tracked by any continuous extension of $\eta \circ u \circ i_S\colon D_S \to \mathbb{P}$. Therefore, K is the left adjoint of $\widehat{\eta}$. ■

4.2 Equilogical Spaces and Type Two Effectivity

Type Two Effectivity (TTE) is the framework for the study of computability developed and used by Weihrauch and coworkers [Wei00, BW99, Wei95, Wei87, Wei85, KW85]. The basic notion in TTE is a *representation* of a topological space, which is a partial continuous surjection $\delta_X : \mathbb{N}^{\mathbb{N}} \rightharpoonup X$. This is just a modest set in $\mathsf{Mod}(\mathbb{B})$, as explained in Subsection 1.2.2. The Baire space $\mathbb{N}^{\mathbb{N}}$ can be replaced by the Cantor space $S^{\mathbb{N}}$ where S is a finite alphabet of symbols. The Cantor space is a PCA, in a similar fashion as the Baire space. The two PCAs are equivalent as they are retracts of each other. We prefer to work with representations on the Baire space. The notions of computability and realized maps in TTE agrees with the definitions of computability and morphisms in $\mathsf{Mod}(\mathbb{B},\mathbb{B}_\sharp)$.

We compare equilogical spaces and TTE in three ways. First, in Subsection 4.2.1 we show that $\mathsf{Mod}(\mathbb{B},\mathbb{B}_\sharp)$ is a full coreflective subcategory of Equ. This also gives us a very topological description of $\mathsf{Mod}(\mathbb{B})$ as the category of 0-equilogical spaces. Second, in Subsection 4.2.4 we show that Equ and $\mathsf{Mod}(\mathbb{B})$ share a common cartesian closed subcategory that contains $\omega\mathsf{Top}_0$. With this result we obtain a transfer principle between equilogical spaces and TTE at the level of cartesian closed structure. We can also explain why some domain theoretic models are so successful, even though they seem to be completely inappropriate from the point of view of the internal logic of Equ. Third, in Section 4.3 we apply some tools from topos theory to further compare Equ and $\mathsf{Mod}(\mathbb{B})$, this time by embedding them both into toposes of sheaves. As a result we obtain a transfer principle between equilogical spaces and TTE.

[5] This is where the construction of K fails in the effective case, because an arbitrary closed subspace of $\mathbb{P}$ need not be an effective domain.

4.2.1 $\mathsf{Mod}(\mathbb{B}, \mathbb{B}_\sharp)$ as a Subcategory of Effective Equilogical Spaces

In Subsection 1.4.2 we defined an applicative retraction

$$(\iota \dashv \delta) \colon (\mathbb{P}, \mathbb{P}_\sharp) \xrightarrow{\mathsf{PCA}} (\mathbb{B}, \mathbb{B}_\sharp) ,$$

which induces the following relation between equilogical spaces and TTE.

Theorem 4.2.1 *There are adjoint functors $\widehat{\iota} \dashv \widehat{\delta}$*

$$\begin{tikzcd} \mathsf{Mod}(\mathbb{P}, \mathbb{P}_\sharp) \arrow[r, shift right, "\widehat{\delta}"'] & \mathsf{Mod}(\mathbb{B}, \mathbb{B}_\sharp) \arrow[l, shift right, "\widehat{\iota}"'] \end{tikzcd}$$

where:

(1) The functors $\widehat{\iota}$ and $\widehat{\delta}$ are induced by an applicative retraction $\iota \vdash \delta \colon (\mathbb{P}, \mathbb{P}_\sharp) \xrightarrow{\mathsf{PCA}} (\mathbb{B}, \mathbb{B}_\sharp)$, as defined in Subsection 1.4.2.

(2) The unit of the adjunction $1_{\mathsf{Mod}(\mathbb{B},\mathbb{B}_\sharp)} \Longrightarrow \widehat{\delta} \circ \widehat{\iota}$ is a natural isomorphism.

(3) $\widehat{\iota}$ is full and faithful, preserves finite limits and reflects isomorphisms.

(4) $\widehat{\delta}$ preserves finite colimits.

(5) $\widehat{\iota}$ and $\widehat{\delta}$ preserve the natural numbers object.

Proof. (1) By construction. (2) By Theorem 1.4.12(3). (3) By Proposition 1.4.5(1), $\widehat{\iota}$ is faithful; by (2) it is full; by Theorem 1.4.12(1), it preserves finite limits; by Proposition 1.4.13(2), it reflects isomorphisms. (4) By Proposition 1.4.13(2). (5) By Corollary 1.4.9 and Proposition 1.4.5. ∎

The applicative inclusion $\iota \vdash \delta$ also induces an adjunction between $\mathsf{Mod}(\mathbb{P})$ and $\mathsf{Mod}(\mathbb{B})$.

Proposition 4.2.2 *There are adjoint functors $\widehat{\iota} \vdash \widehat{\delta}$*

$$\begin{tikzcd} \mathsf{Mod}(\mathbb{P}) \arrow[r, shift right, "\widehat{\delta}"'] & \mathsf{Mod}(\mathbb{B}) \arrow[l, shift right, "\widehat{\iota}"'] \end{tikzcd}$$

which have the same properties as the functors in Theorem 4.2.1. In addition, $\widehat{\delta}$ preserves countable coproducts.

Proof. The proof is the same as the proof of Theorem 4.2.1. That $\widehat{\delta}$ preserves countable coproducts will be evident from the description of $\widehat{\delta}$ as a functor from $\mathsf{0Equ}$ to Equ. ∎

We have a topological description of $\mathsf{Mod}(\mathbb{P})$, namely the category of equilogical spaces. The category $\mathsf{Mod}(\mathbb{B})$ can be described in a similar way. A 0-*dimensional space* is a topological space whose family of clopen sets forms a base for its topology. We denote by $\mathsf{0Dim}$ the category of countably based 0-dimensional T_0-spaces and continuous maps. These spaces are in fact Hausdorff.

Definition 4.2.3 A 0-*equilogical space* is an equilogical space whose underlying topological space is 0-dimensional. The category $\mathsf{0Equ}$ is the full subcategory of Equ on 0-equilogical spaces.

In other words, $\mathsf{0Equ}$ is formed just like Equ, except that we use $\mathsf{0Dim}$ instead of $\omega\mathsf{Top}_0$ for the underlying spaces.

Theorem 4.2.4 *The categories* $\mathsf{0Equ}$ *and* $\mathsf{Mod}(\mathbb{B})$ *are equivalent.*

Proof. This is essentially the same proof as the proof of Theorem 4.1.3 that Equ and $\mathsf{PER}(\mathbb{P})$ are equivalent, except that we use the Embedding and Extension Theorems for $\mathbb{B}$ instead of $\mathbb{P}$.

By Embedding Theorem 1.1.5 for $\mathbb{B}$, a countably based T_0-space is 0-dimensional if, and only if, it embeds in $\mathbb{B}$. Thus every 0-equilogical space is isomorphic to one whose underlying topological space is a subspace of $\mathbb{B}$. This make it clear that equivalence relations on 0-dimensional countably based T_0-spaces correspond to partial equivalence relations on $\mathbb{B}$. Morphisms work out, too, since by the Extension Theorem for $\mathbb{B}$ 1.1.6 every partial continuous map on $\mathbb{B}$ can be extended to a realized one. ∎

The adjunction $\widehat{\iota} \dashv \widehat{\delta}$ from Proposition 4.2.2 can be described topologically as an adjunction $I \dashv D$,

$$\mathsf{Equ} \underset{D}{\overset{I}{\rightleftarrows}} \mathsf{0Equ}\ .$$

The functor I, which corresponds to $\widehat{\iota}$, is just the inclusion of the subcategory $\mathsf{0Equ}$ into Equ. The functor D, which corresponds to $\widehat{\delta}$, is defined as follows. For every countably based T_0-space X there exists an admissible representation $\delta_X : \mathbb{B} \rightharpoonup X$. The subspace $X_0 = \mathsf{dom}(\delta) \subseteq \mathbb{B}$ is a countably based 0-dimensional Hausdorff space. Now if $(X, \equiv_X)$ is an equilogical space, let $(DX, \equiv_{DX})$ be the 0-equilogical space $DX = X_0$ with the equivalence relation

$$a \equiv_{DX} b \iff \delta_X a \equiv_X \delta_X b\ .$$

If $[f]\colon (X, \equiv_X) \to (Y, \equiv_Y)$ is a morphism in Equ, then $D[f]$ is the morphism represented by a continuous $g\colon X_0 \to Y_0$ that tracks $f\colon X \to Y$, as shown in the following commutative diagram:

$$\begin{tikzcd} X_0 \arrow[r, "g"] \arrow[d, "\delta_X"'] & Y_0 \arrow[d, "\delta_Y"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

Such a map g exists because δ_X and δ_Y were chosen to be admissible representations.

Proposition 4.2.5 *(a) The functor* $D\colon \mathsf{EPQ}_0 \to \mathsf{0Equ}$ *preserves cartesian closed structure. (b) If* $(X, \equiv_X), (Y, \equiv_Y) \in \mathsf{0Equ}$ *and in* $\omega\mathsf{Top}_0$ *there exists a* 0*-dimensional weak exponential of* X *and* Y*, then* I *preserves the exponential* $(Y, \equiv_Y)^{(X, \equiv_X)}$*. (c)* I *preserves the exponentials* $\mathbb{N}^{\mathbb{N}}$ *and* $2^{\mathbb{N}}$*, and the real numbers object.*

Proof. (a) Immediate from commutativity of (4.2.4). In particular, this means that D preserves exponentials of topological objects. (b) If $W \in \mathsf{0Dim}$ is a weak exponential of X and Y in $\omega\mathsf{Top}_0$, then it is also a weak exponential of X and Y in $\mathsf{0Dim}$. Now the construction of Y^X from W in Equ coincides with the one in $\mathsf{0Equ}$. (c) The Baire space $\mathbb{N}^{\mathbb{N}}$ and the Cantor space $2^{\mathbb{N}}$ both satisfy the condition from (b), the real numbers object is a regular quotient of $2^{\mathbb{N}}$, and I preserves coequalizers because it is a left adjoint. ∎

In Section 4.3 we will be able to show that I does not preserve the higher types $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$ and $\mathbb{R}^{\mathbb{R}}$.

4.2.2 Sequential Spaces

Let X be a topological space, $V \subseteq X$, and $\langle x_n \rangle_{n \in \mathbb{N}}$ a sequence in X. We say that $\langle x_n \rangle_{n \in \mathbb{N}}$ is *eventually in* V when there exists n_0 such that $x_n \in V$ for all $n \geq n_0$. A sequence $\langle x_n \rangle_{n \in \mathbb{N}}$ *converges* to $x_\infty \in X$, written $\langle x_n \rangle_{n \in \mathbb{N}} \to x_\infty$, when $\langle x_n \rangle_{n \in \mathbb{N}}$ is eventually in every neighborhood $x_\infty \in U \in \mathcal{O}(X)$ of x_∞. The point x_∞ is *a limit* of the sequence $\langle x_n \rangle_{n \in \mathbb{N}}$. Note that a limit of a sequence need not be unique. In fact, a space is T_1 if, and only if, every sequence has at most one limit.

Let $\overline{\omega}$ be the one-point compactification of the discrete space $\omega = \{0, 1, 2, \ldots\}$, $\overline{\omega} = \omega \cup \{\infty\}$. The space $\overline{\omega}$ is homeomorphic to the subset $\{1/2^n \mid n \in \omega\} \cup \{0\}$ of the real line. Convergent sequences with their limits are in bijective correspondence with continuous maps $\overline{\omega} \to X$. A sequence $\langle x_n \rangle_{n \in \mathbb{N}}$ that converges to x_∞ corresponds to the continuous map $x\colon \overline{\omega} \to X$, defined by

$$\begin{aligned} x &\colon n \mapsto x_n \qquad (n \in \omega) \\ x &\colon \infty \mapsto x_\infty \,. \end{aligned}$$

Definition 4.2.6 A subset $S \subseteq X$ is *sequentially open* when, for any sequence $\langle x_n \rangle_{n \in \mathbb{N}}$ converging to a limit $x_\infty \in S$, $\langle x_n \rangle_{n \in \mathbb{N}}$ is eventually in S. A function $f\colon X \to Y$ is *sequentially continuous* when it preserves converging sequences, i.e., $\langle x_n \rangle_{n \in \mathbb{N}} \to x_\infty$ implies $\langle f x_n \rangle_{n \in \mathbb{N}} \to f x_\infty$.

Proposition 4.2.7 *The family of sequentially open subsets of a space* X *forms a topology, called the* sequential topology *on* X*. The space* X *equipped with the sequential topology is denoted by* $\sigma(X)$*. The sequential topology is finer that the original topology, and* $\sigma(\sigma(X)) = \sigma(X)$*.*

Proof. Suppose $\{V_i \mid i \in I\}$ is a family of sequentially open sets, $V = \bigcup_{i \in I} V_i$ and $\langle x_n \rangle_{n \in \mathbb{N}} \to x_\infty \in V$. There exists $i \in I$ such that $x_\infty \in V_i$. Because V_i is sequentially open, $\langle x_n \rangle_{n \in \mathbb{N}}$ is eventually in V_i, therefore eventually in V as well. This means that V is sequentially open. Suppose V_1 and V_2 are sequentially open and $\langle x_n \rangle_{n \in \mathbb{N}} \to_{\mathcal{S}} x_\infty \in V_1 \cap V_2$. There exists $n_1 \in \omega$ such that $x_n \in V_1$ for all $n \geq n_1$, and there exists $n_2 \in \omega$ such that $x_n \in V_2$ for all $n \geq n_2$. Then for all $n \geq \max(n_1, n_2)$, $x_n \in V_1 \cap V_2$. We showed that sequentially open sets form a topology.

It is trivially true that every open set is sequentially open, which means that $\mathcal{O}(X) \subseteq \mathcal{O}(\sigma(X))$.

For the last part we only need to prove that the topology on $\sigma(\sigma(X))$ is finer than the topology of $\sigma(X)$. It is sufficient to show that $\langle x_n \rangle_{n \in \mathbb{N}} \to_X x_\infty$ implies $\langle x_n \rangle_{n \in \mathbb{N}} \to_{\sigma(X)} x_\infty$, but this follows immediately from the definition of sequentially open sets. ∎

Proposition 4.2.8 *(a) A map $f\colon X \to Y$ is sequentially continuous if, and only if, it is continuous as a map $f\colon \sigma(X) \to \sigma(Y)$. (b) Every continuous map $f\colon X \to Y$ between spaces X and Y is sequentially continuous.*

Proof. (a) Suppose f maps convergent sequences to convergent sequences. Let $V \subseteq Y$ be a sequentially open set. Suppose $\langle x_n\rangle_{n\in\mathbb{N}} \to x_\infty \in f^*V$. Because f is sequentially continuous, $\langle fx_n\rangle_{n\in\mathbb{N}} \to fx_\infty \in V$, and so $\langle fx_n\rangle_{n\in\mathbb{N}}$ is eventually in V, which means that $\langle x_n\rangle_{n\in\mathbb{N}}$ is eventually in f^*V, therefore f^*V is sequentially open. Conversely, suppose $f\colon \sigma(X) \to \sigma(Y)$ is continuous, $\langle x_n\rangle_{n\in\mathbb{N}} \to x_\infty$ in X, and $fx_\infty \in V \in \mathcal{O}(Y)$. Since V is open it is also sequentially open, hence f^*V is sequentially open in X. Because $x_\infty \in f^*V$, $\langle x_n\rangle_{n\in\mathbb{N}}$ is eventually in f^*V, therefore $\langle fx_n\rangle_{n\in\mathbb{N}}$ is eventually in V.

(b) Suppose $f\colon X \to Y$ is continuous, $\langle x_n\rangle_{n\in\mathbb{N}} \to x_\infty$ in X, and $fx_\infty \in V \in \mathcal{O}(Y)$. Then $x_\infty \in f^*V$ and since f^*V is open, $\langle x_n\rangle_{n\in\mathbb{N}}$ is eventually in f^*V, therefore $\langle fx_n\rangle_{n\in\mathbb{N}}$ is eventually in V. ∎

Definition 4.2.9 A topological space X is a *sequential space* when every sequentially open set $V \subseteq X$ is open in X, that is $X = \sigma(X)$. The category of sequential spaces and continuous maps between them is denoted by Seq.

Proposition 4.2.10 *Every first-countable space is sequential.*

Proof. Recall that a first countable space is a space X such that every point $x \in X$ has a countable neighborhood base $U_0^x, U_1^x, \dots$ We may further assume that the neighborhoods U_i^x, $i \in \mathbb{N}$, are nested, that is $U_0^x \supseteq U_1^x \supseteq \cdots$ Suppose that $V \subseteq X$ is not open. There exists a point $x \in V$ such that $U_i^x \not\subseteq V$ for all $i \in \mathbb{N}$. Pick a point $x_i \in U_i^x \setminus V$ for every $i \in \mathbb{N}$. Then $\langle x_n\rangle_{n\in\mathbb{N}} \to x$ but $\langle x_n\rangle_{n\in\mathbb{N}}$ is not eventually in V, which means that V is not sequentially open. ∎

Proposition 4.2.11 *A topological quotient of a sequential space is sequential.*

Proof. Let X be a sequential space and $q\colon X \to Y$ a quotient map. Suppose $U \subseteq Y$ is sequentially open. It suffices to show that q^*U is also sequentially open. Suppose $\langle x_n\rangle_{n\in\mathbb{N}} \to x_\infty \in q^*U$. Then $\langle qx_n\rangle_{n\in\mathbb{N}} \to qx_\infty \in U$, therefore the sequence $\langle qx_n\rangle_{n\in\mathbb{N}}$ is eventually in U, but that means the original sequence $\langle x_n\rangle_{n\in\mathbb{N}}$ is eventually in q^*U. ∎

Proposition 4.2.12 *Let $\langle x_n\rangle_{n\in\mathbb{N}}$ be a sequence in a topological space X and $x_\infty \in X$. Then $\langle x_n\rangle_{n\in\mathbb{N}} \to_X x_\infty$ if, and only if, $\langle x_n\rangle_{n\in\mathbb{N}} \to_{\sigma(X)} x_\infty$.*

Proof. Since $\sigma(\overline{\omega}) = \overline{\omega}$, a continuous map $x\colon \overline{\omega} \to X$ is also continuous as a map $x\colon \overline{\omega} \to \sigma(X)$ by Proposition 4.2.8(a). A continuous map $x\colon \overline{\omega} \to \sigma(X)$ is continuous as a map $x\colon \overline{\omega} \to X$ because $\mathcal{O}(X) \subseteq \mathcal{O}(\sigma(X))$. ∎

Proposition 4.2.13 *Let X be a topological space. Sequential topology on X is the finest topology on X that has the same converging sequences as the topology of X.*

Proof. Let ρ be a topology on X that has the same converging sequences as $\mathcal{O}(X)$. Suppose V is sequentially open in ρ. If $\langle x_n \rangle_{n \in \mathbb{N}} \to_{\mathcal{O}(X)} x_\infty$ then by assumption $\langle x_n \rangle_{n \in \mathbb{N}} \to_\rho x_\infty$, hence $\langle x_n \rangle_{n \in \mathbb{N}}$ is eventually in V. Therefore, V is sequentially open in $\mathcal{O}(X)$. ∎

Theorem 4.2.14 *The category* Seq *is cartesian closed.*

Proof. This is well known and follows from the fact that Seq is a reflective subcategory of the cartesian-closed category Lim of *limit spaces* [Kur52], and the reflection preserves products [Fra65, MS00]. We describe binary products and exponentials but omit the proof that they form a cartesian closed category.

Suppose X and Y are sequential spaces. The *sequential product* $X \times_\sigma Y$ is the space $\sigma(X \times Y)$, with the usual canonical projections $\mathsf{fst} : X \times_\sigma Y \to X$ and $\mathsf{snd} : X \times_\sigma Y \to Y$. The projections are continuous because the product topology $X \times Y$ is smaller than the sequential topology $\sigma(X \times Y)$. The reason for taking $\sigma(X \times Y)$ rather than $X \times Y$ is that the topological product of sequential spaces need not be a sequential space. Given sequentially continuous maps $f : Z \to X$ and $g : Z \to Y$ from a sequential space Z , the unique map $\langle f, g \rangle : Z \to X \times_\sigma Y$ that satisfies, for all $z \in Z$, $\langle f, g \rangle z = \langle fz, gz \rangle$, is sequentially continuous because Z is a sequential space.

For sequential spaces X and Y, their exponential Y^X in Seq is the set $\mathcal{C}(X, Y)$ of sequentially continuous maps from X to Y, with the usual evaluation map. Let us find out what the sequential structure on Y^X ought to be. Since the evaluation map is supposed to be sequentially continuous, it must be the case that whenever $\langle f_n \rangle_{n \in \mathbb{N}} \to f_\infty$ and $\langle x_n \rangle_{n \in \mathbb{N}} \to x_\infty$ then $\langle f_n x_n \rangle_{n \in \omega} \to f_\infty x_\infty$. On the other hand, we want as few convergent sequences in $\mathcal{C}(X, Y)$ as possible, in order for the transpose of a map to be continuous. Therefore, we specify that $\langle f_n \rangle_{n \in \mathbb{N}} \to f_\infty$ in $\mathcal{C}(X, Y)$ if, and only if, whenever $\langle x_n \rangle_{n \in \mathbb{N}} \to x_\infty$ in X then $\langle f_n x_n \rangle_{n \in \omega} \to f_\infty x_\infty$ in Y. Then we take the topology on $\mathcal{C}(X, Y)$ to consist of the sequentially open sets with respect to this notion of convergence. ∎

4.2.3 Admissible Representations

A representation $\delta_S : \mathbb{B} \rightharpoonup S$ of a set S induces a quotient topology on S, defined by

$$U \subseteq S \text{ open} \iff \delta_S^* U \text{ open in } \mathsf{dom}(\delta_S) \ .$$

We denote by $[S]$ the topological space S with the quotient topology induced by δ_S. It is easy to check that every realized function $f : (S, \delta_S) \to (T, \delta_T)$ becomes a continuous map $f : [S] \to [T]$ this way.

In TTE we are typically interested in representations of topological spaces, rather than arbitrary sets. For this reason it is important to represent a topological space X with a representation (X, δ_X) which has a reasonable relation to the topology of X. An obvious requirement is that the original topology $\mathcal{O}(X)$ should coincide with the quotient topology $[X]$. However, as is well known by the school of TTE, this requirement is too weak because the same space may have many non-isomorphic representations. Even worse, not every continuous map $f : [X] \to [Y]$ need be realized. Thus, we are lead to further restricting the allowable representations of topological spaces.

Definition 4.2.15 An *admissible representation* for a topological space X is a partial continuous surjection $\delta : \mathbb{B} \rightharpoonup X$ such that every partial continuous map $f : \mathbb{B} \rightharpoonup X$ can be factored through δ: there exists $g : \mathbb{B} \rightharpoonup \mathbb{B}$ such that $f\alpha = \delta(g\alpha)$ for all $\alpha \in \mathsf{dom}(f)$.

Note that we do not require explicitly that the quotient topology $[X]$ of an admissible representation coincide with the original topology $\mathcal{O}(X)$.

Proposition 4.2.16 *(a) Suppose $\delta\colon \mathbb{B} \rightharpoonup X$ is a quotient map. Then X is a sequential space. (b) If X is a sequential space and $\delta\colon \mathbb{B} \rightharpoonup X$ is an admissible representation then δ is a quotient map.*

Proof. (a) Every subspace of the Baire space is a sequential space and X is a topological quotient of such a subspace, therefore sequential by Proposition 4.2.11.

(b) Suppose that for some set $U \subseteq X$ the set $\delta^* U$ is open in $\mathsf{dom}(\delta)$. It is sufficient to show that U is sequentially open. Suppose $\langle x_n \rangle_{n \in \mathbb{N}} \to x_\infty \in U$. The continuous map $x\colon \overline{\omega} \to X$ that represents the sequence and its limit factors through δ via a map $\chi\colon \overline{\omega} \to \mathbb{B}$ so that $x = \delta \circ \chi$, because $\overline{\omega}$ is a subspace of $\mathbb{B}$. Now $\chi_\infty \in \delta^* U$ and since $\langle \chi_n \rangle_{n \in \mathbb{N}} \to \chi_\infty$, $\langle \chi_n \rangle_{n \in \mathbb{N}}$ is eventually in $\delta^* U$, therefore $\langle \delta \circ \chi_n \rangle_{n \in \mathbb{N}} = \langle x_n \rangle_{n \in \mathbb{N}}$ is eventually in U. Thus U is sequentially open. ∎

Suppose $\mathcal{C}$ is some class of topological spaces that we would like to study in TTE. We represent each $X \in \mathcal{C}$ by a representation $IX = (X, \delta_X)$. In order for this to make sense, $I\colon \mathcal{C} \to \mathsf{Mod}(\mathbb{B})$ should be a functor. The definition of admissible representations is tailored in such as way that I becomes a faithful functor when each IX is an admissible representation, because then every continuous map $f\colon X \to Y$ is tracked in $\mathsf{Mod}(\mathbb{B})$. The functor I is full, provided that every representation $\delta_X : \mathbb{B} \rightharpoonup X$ is a quotient map. This explains why it is advantageous to consider admissible representations that are quotient maps. The first part of Proposition 4.2.16 tells us that any class of topological spaces $\mathcal{C}$ that is embedded in $\mathsf{Mod}(\mathbb{B})$ in this way is a subcategory of Seq. The second part tells us that the representation functor $I\colon \mathcal{C} \to \mathsf{Mod}(\mathbb{B})$ is automatically full, as long as $\mathcal{C}$ is a subcategory of Seq.

A *pseudobase* of a space X is a family $\mathcal{B}$ of subsets of X such that whenever $\langle x_n \rangle_{n \in \mathbb{N}} \to_{\mathcal{O}(X)} x_\infty$ and $x_\infty \in U \in \mathcal{O}(X)$ then there exists $B \in \mathcal{B}$ such that $x_\infty \in B \subseteq U$ and $\langle x_n \rangle_{n \in \mathbb{N}}$ is eventually in B.

Lemma 4.2.17 *Suppose $\mathcal{B} = \{B_i \mid i \in \mathbb{N}\}$ is a countable pseudobase for a countably based T_0-space Y. Let X be a first-countable space and $f\colon X \to Y$ a continuous map. For every $x \in X$ and every neighborhood V of fx there exists a neighborhood U of x and $i \in \mathbb{N}$ such that $fx \in f_*(U) \subseteq B_i \subseteq V$.*

Proof. Note that the elements of the pseudobase do not have to be open sets, so this is not just a trivial consequence of continuity of f. We prove the lemma by contradiction. Suppose there were $x \in X$ and a neighborhood V of fx such that for every neighborhood U of x and for every $i \in \mathbb{N}$, if $B_i \subseteq V$ then $f_*(U) \not\subseteq B_i$. Let $U_0 \supseteq U_1 \supseteq \cdots$ be a descending countable neighborhood system for x. Let $p\colon \mathbb{N} \to \mathbb{N}$ be a surjective map that attains each value infinitely often, that is for all $k, j \in \mathbb{N}$ there exists $i \geq k$ such that $pi = j$. For every $i \in \mathbb{N}$, if $B_{pi} \subseteq V$ then $f_*(U_i) \not\subseteq B_{pi}$. Therefore, for every $i \in \mathbb{N}$ there exists $x_i \in U_i$ such that if $B_{pi} \subseteq V$ then $fx_i \notin B_{pi}$. The sequence $\langle x_n \rangle_{n \in \mathbb{N}}$ converges to x, hence $\langle fx_n \rangle_{n \in \mathbb{N}}$ converges to fx. Because $\mathcal{B}$ is a pseudobase there exists $j \in \mathbb{N}$ such that $B_j \subseteq V$ and $\langle fx_n \rangle_{n \in \mathbb{N}}$ is eventually in B_j, say from the k-th term onwards. There exists $i \geq k$ such that $pi = j$. Now we get $fx_i \in B_{pi} \subseteq V$, which is a contradiction. ∎

Theorem 4.2.18 (Schröder [Sch00]) *A T_0-space has an admissible representation if, and only if, it has a countable pseudobase.*

Proof. We follow the argument given by Schröder. Suppose $\delta \colon \mathbb{B} \rightharpoonup X$ is an admissible representation. Let $\mathcal{B} = \{B_a \mid a \in \mathbb{N}^*\}$ be the family of sets $B_a = \delta_*(a{::}\mathbb{B})$. We show that $\mathcal{B}$ is a pseudobase for X. Suppose $\langle x_n \rangle_{n\in\mathbb{N}} \to_{\mathcal{O}(X)} x_\infty$ and $x_\infty \in U \in \mathcal{O}(X)$. We can view the sequence $\langle x_n \rangle_{n\in\mathbb{N}}$ and its limit x_∞ as a continuous map $x \colon \overline{\omega} \to X$. Since $\overline{\omega}$ embeds in $\mathbb{B}$ and δ is an admissible representation there exists a continuous map $y \colon \overline{\omega} \to \mathbb{B}$ such that $x_n = \delta y_n$ for all $n \in \overline{\omega}$. Because δ is continuous there exists a basic open neighborhood $a{::}\mathbb{B}$ such that $y_\infty \in a{::}\mathbb{B}$ and $x_\infty = \delta(y_\infty) \in B_a = \delta_*(a{::}\mathbb{B}) \subseteq U$. This shows that $\mathcal{B}$ is a countable pseudobase.

Conversely, suppose $\mathcal{B} = \{B_k \mid k \in \mathbb{N}\}$ is a countable pseudobase for a space X. Define a relation $D \subseteq X \times \mathbb{B}$ by

$$D(\alpha, x) \iff \forall n \in \mathbb{N} \,.\, (x \in B_{\alpha n}) \land \forall U \in \mathcal{O}(X) \,.\, (x \in U \Longrightarrow \exists n \in \mathbb{N} \,.\, B_{\alpha n} \subseteq U)$$

Suppose $D(\alpha, x)$ and $D(\alpha, y)$. If $x \in U \in \mathcal{O}(X)$ then there exists $n \in \mathbb{N}$ such that $B_{\alpha n} \subseteq U$, therefore $y \in U$. Hence, y belongs to all the open sets that x belongs to, and vice versa by a symmetric argument. Because X is a T_0-space x and y are the same point. The relation D is single-valued and is the graph of a partial map $\delta \colon \mathbb{B} \rightharpoonup X$, defined by

$$\delta\alpha = x \iff D(\alpha, x).$$

It is easy to see that δ is surjective because $\mathcal{B}$ is a pseudobase. To see that it is continuous, suppose $\delta\alpha \in U \in \mathcal{O}(X)$. There exists $n \in \mathbb{N}$ such that $B_{\alpha n} \subseteq U$. If for a $\beta \in \mathsf{dom}(\delta)$ there exists $m \in \mathbb{N}$ such that $\beta m = \alpha n$ then $\delta\beta \in B_{\alpha n} \subseteq U$. The set $S = \{\beta \in \mathbb{B} \mid \exists m \in \mathbb{N} \,.\, \beta m = \alpha n\}$ is an open subset of $\mathbb{B}$. It follows that $\delta\alpha \in \delta_*(S) \subseteq U$. Therefore, δ is continuous.

We show that δ is an admissible representation for X. Let $f \colon \mathbb{B} \rightharpoonup X$ be a continuous partial map. Suppose $f\alpha \in U \in \mathcal{O}(X)$. We claim that there exist $n \in \mathbb{N}$ and $a \in \mathbb{N}^*$ such that $a \sqsubseteq \alpha$ and $f_*(a{::}\mathbb{B}) \subseteq B_n \subseteq U$. This follows from Lemma 4.2.17 because $\mathsf{dom}(f)$ is a first-countable space. Define a partial map $g \colon \mathbb{B} \to \mathbb{B}$ for $\alpha \in \mathsf{dom}(f)$ by

$$\begin{aligned} &(g\alpha)n = 0 && \text{if } \forall k \in \mathbb{N} \,.\, f_*([\alpha 0, \dots, \alpha n]{::}\mathbb{B}) \not\subseteq B_k \\ &(g\alpha)n = 1 + \min\{k \in \mathbb{N} \mid f_*([\alpha 0, \dots, \alpha n]{::}\mathbb{B}) \subseteq B_k\} && \text{otherwise} \end{aligned}$$

The map g is continuous because the value of $(g\alpha)n$ depends only on finitely many values of the arguments, namely n and $\alpha 0, \dots, \alpha n$. Now define a map $h \colon \mathbb{B} \to \mathbb{B}$ by

$$(h\alpha)n = (g\alpha)(\min\{m \geq n \mid (g\alpha)m \neq 0\}) - 1.$$

The function h is well defined because for every n there do exist $m \geq n$ and $k \in \mathbb{N}$ such that $f_*([\alpha 0, \dots, \alpha m]{::}\mathbb{B}) \subseteq B_k$, which follows from the earlier claim. For every $\alpha \in \mathsf{dom}(f)$, the set $\{(h\alpha)n \mid n \in \mathbb{N}\}$ is the same as the set

$$\{k \in \mathbb{N} \mid \exists n \in \mathbb{N} \,.\, f_*([\alpha 0, \dots, \alpha n]{::}\mathbb{B}) \subseteq B_k\}.$$

We show that $f\alpha = \delta(h\alpha)$ for all $\alpha \in \mathsf{dom}(f)$ by showing that $f\alpha$ and $\delta(h\alpha)$ belong to the same open sets. Suppose $f\alpha \in U \in \mathcal{O}(X)$. By the earlier claim, there exist $k, n \in \mathbb{N}$ such that $f_*([\alpha 0, \dots, \alpha k]{::}\mathbb{B}) \subseteq B_n \subseteq U$. Thus, there is $m \in \mathbb{N}$ such that $h(\alpha)m = k$, hence $\delta(h\alpha) \in B_k \subseteq U$. Conversely, suppose $\delta(h\alpha) \in U \in \mathcal{O}(X)$. There exists $k \in \mathbb{N}$ such that $\delta(h\alpha) \in B_k \subseteq U$, therefore for some $m \in \mathbb{N}$, $(h\alpha)m = k$. This means that for some $n \in \mathbb{N}$, $f\alpha \in f_*([\alpha 0, \dots, \alpha n]{::}) \subseteq B_k \subseteq U$.
∎

Corollary 4.2.19 *Every countably based T_0-space has an admissible representation.*

Proof. A countable base is a countable pseudobase. ∎

4.2.4 A Common Subcategory of Equilogical Spaces and $\mathsf{Mod}(\mathbb{B})$

In this subsection we combine results by Menni and Simpson [MS00] about a common subcategory PQ_0 of Equ and Seq, and by Schröder [Sch00] on admissible representations to prove that Equ, $\mathsf{Mod}(\mathbb{B})$, and Seq share a common cartesian closed subcategory PQ_0 that contains $\omega\mathsf{Top}_0$ as a full subcategory.

First we review the relevant parts of Menni and Simpson [MS00]. In their paper, the ambient category is $\mathsf{EQU}(\omega\mathsf{Top})$, which is the category of equilogical spaces built from arbitrary countably based spaces, as opposed to just countably based T_0-spaces. However, all the results needed here apply to Equ. One just has to check that the T_0-condition does not get in the way, and that it is preserved by the relevant constructions. The best way to do this is to observe that Equ is an exponential ideal of $\mathsf{EQU}(\omega\mathsf{Top})$.

Definition 4.2.20 Let $X \in \omega\mathsf{Top}_0$ and $q \colon X \to Y$ be a continuous map. Then q is said to be *ω-projecting* when for every $Z \in \omega\mathsf{Top}_0$ and every continuous map $f \colon Z \to Y$ there exists a lifting $g \colon Z \to X$ such that $f = q \circ g$.

An equilogical space $(X, \equiv_X)$ is *ω-projecting* when the canonical quotient map $X \to X/{\equiv_X}$ is ω-projecting. The full subcategory of Equ on the ω-projecting equilogical spaces is denoted by EPQ_0. Let PQ_0 be the category of those T_0-spaces Y for which there exists an ω-projecting map $q \colon X \to Y$.

Proposition 4.2.21 *If Y is a sequential space and $q \colon X \to Y$ is an ω-projecting map then q is a quotient map.*

Proof. Suppose $U \subseteq Y$ and $q^*(U)$ is an open subset of X. Because X and Y are sequential spaces it suffices to show that U is sequentially open. Suppose $\langle x_n \rangle_{n\in\mathbb{N}} \to x_\infty \in U$. Because q is ω-projecting we can lift the continuous map $x \colon \overline{\omega} \to Y$ to a continuous map $\overline{x} \colon \overline{\omega} \to X$ such that $x = q \circ \overline{x}$. Because $q^*(U)$ is sequentially open and $\langle \overline{x}_n \rangle_{n\in\mathbb{N}} \to \overline{x}_\infty \in q^*(U)$, the sequence $\langle \overline{x}_n \rangle_{n\in\mathbb{N}}$ is eventually in $q^*(U)$, therefore the sequence $\langle x_n \rangle_{n\in\mathbb{N}}$ is eventually in U. This proves that U is sequentially open. ∎

Notice the similarity between the definitions of ω-projecting maps and admissible representations. The name PQ_0 stands for "ω-projecting quotient", and EPQ_0 stands for "equilogical ω-projecting quotient".

Theorem 4.2.22 (Menni & Simpson [MS00]) *The category PQ_0 is a cartesian closed subcategory of Seq, EPQ_0 is a cartesian closed subcategory of Equ, and the categories PQ_0 and EPQ_0 are equivalent.*

Proof. See [MS00]. The equivalence functor between EPQ_0 and PQ_0 maps an ω-projecting equilogical space $(X, \equiv_X)$ to the quotient $X/{\equiv_X}$. In fact, Menni and Simpson prove that in a precise sense EPQ_0 is the largest common cartesian closed subcategory of Equ and Seq. ∎

We also need the following result.

Theorem 4.2.23 (Schröder [Sch00]) *Let (X, δ_X) and (Y, δ_Y) be admissible representations for sequential T_0-spaces X and Y. Then the product $(X, \delta_X) \times (Y, \delta_Y)$ formed in $\mathsf{Mod}(\mathbb{B})$ is an admissible representation for the product $X \times Y$ formed in Seq, and similarly the exponential $(Y, \delta_Y)^{(X, \delta_X)}$ formed in $\mathsf{Mod}(\mathbb{B})$ is an admissible representation for the exponential Y^X formed in Seq.*

Proof. See [Sch00]. ∎

Let AdmSeq be the full subcategory of Seq on those sequential T_0-spaces that have an admissible representation. Then we can define a functor $I \colon \mathsf{AdmSeq} \to \mathsf{Mod}(\mathbb{B})$, where $X \in \mathsf{AdmSeq}$ is mapped to its admissible representation $IX = (X, \delta_X)$, and a continuous maps $f \colon X \to Y$ is mapped to itself, $If = f$. Theorem 4.2.23 states that I is a full and faithful cartesian closed functor, as summarized in the following corollary.

Corollary 4.2.24 *The category AdmSeq is a cartesian closed subcategory of $\mathsf{Mod}(\mathbb{B})$.*

Proof. The functor maps a sequential space X to its admissible representation (X, δ_X). It is cartesian closed by Theorem 4.2.23. ∎

Theorem 4.2.25 PQ_0 *and* AdmSeq *are the same category.*

Proof. It was independently observed by Schröder that PQ_0 is a full subcategory of AdmSeq, which is the easier of the two inclusions. The proof goes as follows. Suppose $q \colon X \to Y$ is an ω-projecting quotient map. We need to show that Y is a sequential space with an admissible representation. It is sequential by Propositions 4.2.10 and 4.2.11. By Corollary 4.2.19 there exists an admissible representation $\delta_X : \mathbb{B} \rightharpoonup X$. Let $\delta_Y = q \circ \delta_X$. Suppose $f \colon \mathbb{B} \rightharpoonup Y$ is a continuous partial map. Because q is ω-projecting f lifts though X, and because δ_X is an admissible representation, it further lifts through $\mathbb{B}$, as in the diagram below.

$$\begin{tikzcd}
& \mathbb{B} \arrow[d, "\delta"] \\
& X \arrow[d, "q"] \\
\mathbb{B} \arrow[r, "f"'] \arrow[ur, dotted] \arrow[uur, dotted] & Y
\end{tikzcd}$$

It remains to prove the converse, namely that if a sequential T_0-space X has an admissible representation then there exists an ω-projecting quotient $q \colon Y \to X$. Since X has an admissible representation it has a countable pseudobase $\mathcal{B} = \{B_i \mid i \in \mathbb{N}\}$, by Theorem 4.2.18. Let $q \colon \mathbb{P} \rightharpoonup X$ be a partial map defined by

$$qa = x \iff (\forall\, n \in a \,.\, x \in B_n) \wedge \forall\, U \in \mathcal{O}(X) \,.\, (x \in U \Longrightarrow \exists\, n \in a \,.\, B_n \subseteq U) \ .$$

The map q is well defined because $qa = x$ and $qa = y$ implies that x and y share the same neighborhoods, so they are the same point of the T_0-space X. Furthermore, q is surjective because $\mathcal{B}$ is a pseudobase. To see that p is continuous, suppose $pa = x$ and $x \in U \in \mathcal{O}(X)$. There exists $n \in \mathbb{N}$ such that $x \in B_n \subseteq U$. If $n \in b \in \mathsf{dom}(p)$ then $pb \in B_n \subseteq U$. Therefore, $a \in \uparrow n$ and $p_*(\uparrow n) \subseteq B_n \subseteq U$, which means that p is continuous. Let $Y = \mathsf{dom}(p)$.

Let us show that $q\colon Y \to X$ is ω-projecting. Suppose $f\colon Z \to X$ is a continuous map and $Z \in \omega\mathsf{Top}_0$. Define a map $g\colon Z \to \mathbb{P}$ by

$$gz = \{n \in \mathbb{N} \mid \exists U \in \mathcal{O}(Z)\,.\,(z \in U \wedge f_*(U) \subseteq B_n)\} \ .$$

The map g is continuous almost by definition. Indeed, if $gz \in \uparrow n$ then there exists a neighborhood U of z such that $f_*(U) \subseteq B_n$, but then $g_*(U) \in \uparrow n$. To finish the proof we need to show that $fz = p(gz)$ for all $z \in Z$. If $n \in gz$ then $fz \in B_n$ because there exists $U \in \mathcal{O}(Z)$ such that $z \in U$ and $f_*(U) \subseteq B_n$. If $fz \in V \in \mathcal{O}(X)$ then by Lemma 4.2.17 there exists $U \in \mathcal{O}(Z)$ and $n \in \mathbb{N}$ such that $z \in U$ and $f_*(U) \subseteq B_n \subseteq U$. Hence, $n \in gz$. This proves that $fz = p(gz)$. ∎

The relationships between the categories are summarized by the following diagram:

$$\begin{tikzcd}
 & & \mathsf{Equ} \\
\omega\mathsf{Top}_0 \arrow[r] & \mathsf{PQ}_0 = \mathsf{AdmSeq} \arrow[ur] \arrow[dr] & \\
 & & \mathsf{Mod}(\mathbb{B})
\end{tikzcd} \tag{4.8}$$

All functors are full and faithful, preserve finite limits, and countable coproducts. The inclusion $\omega\mathsf{Top}_0 \to \mathsf{PQ}_0$ preserves all exponentials that happen to exist in $\omega\mathsf{Top}_0$, and the other two functors preserve cartesian closed structure.

We can complete the triangle in (4.8) so that the resulting diagram commutes up to natural isomorphism:

$$\begin{tikzcd}
\mathsf{Equ} \arrow[rr, "D"] & & \mathsf{0Equ} \\
 & \mathsf{PQ}_0 = \mathsf{AdmSeq} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

The functor D is the right adjoint to the inclusion $\mathsf{0Equ} \to \mathsf{Equ}$, as described in Subsection 4.2.1.

The correspondence (4.8) explains why domain-theoretic computational models agree so well with computational models studied by TTE—as long as we only build spaces by taking products, coproducts, exponentials, and regular subspaces, starting from countably based T_0-spaces, we remain in PQ_0.

Proposition 4.2.26 *In $\mathsf{Mod}(\mathbb{B})$, the hierarchy of exponentials $\mathbb{N}$, $\mathbb{N}^{\mathbb{N}}$, $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$, …, built from the natural numbers object $\mathbb{N}$, corresponds to the Kleene-Kreisel countable functionals.*

Proof. All functors in (4.8) preserve the natural numbers object and exponentials, and in Subsection 4.1.4 we showed that the repeated exponentials of the natural numbers object in Equ correspond to the Kleene-Kreisel countable functionals. ∎

In domain theory the real numbers are usually represented by the continuous *interval domain* $\mathsf{I}\mathbb{R}$ [Esc97, ES99b], or the algebraic domain of reals $\mathcal{R}$ [Ber00, Nor98b]. When these domains are viewed as domains with totality, they correspond to the equilogical space $\mathbb{R}_{\mathrm{t}} = (\mathbb{R}, 1_{\mathbb{R}})$, i.e., the *topological reals*. However, in Equ the most natural choice of real numbers is the space of *Cauchy reals* $\mathbb{R}_{\mathrm{c}} = (\mathcal{C}, \approx)$, cf. Section 5.5, where $\mathcal{C}$ is the space of rapidly converging Cauchy sequences of rational numbers, and $\approx$ is the evident coincidence relation. The objects $\mathbb{R}_{\mathrm{t}}$ and $\mathbb{R}_{\mathrm{c}}$ are not isomorphic, since all morphisms $\mathbb{R}_{\mathrm{t}} \to \mathbb{R}_{\mathrm{c}}$ are constant. In the internal logic of Equ the topological reals are not at all well behaved. For instance, $\mathbb{R}_{\mathrm{t}}$ is not linearly ordered, in the sense that the statement $\forall\, x, y, z \in \mathbb{R}_{\mathrm{t}} \,.\, (x < y \longrightarrow z < x \lor y < z)$ is not valid. This is unfortunate, because it means that the well known and successful domain-theoretic models of reals are not amenable to the internal logic of Equ. In fact, one would expect that *because* the topological reals behave badly in the internal logic, they should not be a suitable model of real numbers computation. Why is this not the case?

Proposition 4.2.27 *Under the correspondence* (4.8)*, the topological reals* $\mathbb{R}_{\mathrm{t}}$ *in* Equ *correspond to the Cauchy reals* $\mathbb{R}_{\mathrm{c}}$ *in* $\mathsf{Mod}(\mathbb{B})$*. Therefore, the finitely complete, countably cocomplete cartesian closed subcategory* $\mathcal{C}_{\mathsf{Equ}}$ *of* Equ *generated by* $\mathbb{N}$ *and* $\mathbb{R}_{\mathrm{t}}$ *is equivalent to the finitely complete, countably cocomplete cartesian closed subcategory* $\mathcal{C}_{\mathsf{Mod}(\mathbb{B})}$ *of* $\mathsf{Mod}(\mathbb{B})$ *generated by* $\mathbb{N}$ *and* $\mathbb{R}_{\mathrm{c}}$*.*

Proof. This holds because the quotient map $\mathcal{C} \to (\mathcal{C}/{\approx}) = \mathbb{R}$ is an admissible representation of the reals, and the inclusions in (4.8) preserve finite limits, countably coproducts, and exponentials. ∎

The point of Proposition 4.2.27 is that the domain-theoretic models of reals are successful because they correspond to the object of Cauchy reals in $\mathsf{Mod}(\mathbb{B})$, instead of Equ, as one one would expect in view of Theorem 4.1.21.

As a consequence of Proposition 4.2.27 we obtain the following *transfer principle* from $\mathcal{C}_{\mathsf{Mod}(\mathbb{B})}$ to $\mathcal{C}_{\mathsf{Equ}}$. Suppose $X, Y \in \mathcal{C}_{\mathsf{Mod}(\mathbb{B})}$ and in the internal logic of $\mathsf{Mod}(\mathbb{B})$ we construct a morphism $f \colon X \to Y$. Then there exists a corresponding morphism in $\mathcal{C}_{\mathsf{Equ}}$. For example, in the internal logic of $\mathsf{Mod}(\mathbb{B})$ it is valid that every map $f \colon [0,1]_{\mathrm{c}} \to \mathbb{R}_{\mathrm{c}}$ is uniformly continuous.[6] This makes it possible to define the Riemann integral as an operator

$$\int_0^1 : \mathbb{R}_{\mathrm{c}}^{[0,1]_{\mathrm{c}}} \longrightarrow \mathbb{R}_{\mathrm{c}} \;.$$

Therefore, there exists a corresponding Riemann integral operator $\int_0^1 : \mathbb{R}_{\mathrm{t}}^{[0,1]_{\mathrm{t}}} \to \mathbb{R}_{\mathrm{t}}$ in $\mathcal{C}_{\mathsf{Equ}}$. We could not have constructed the same operator easily in the internal logic of Equ, because in the internal logic of Equ it is not valid that every $f \colon [0,1]]_{\mathrm{t}} \to \mathbb{R}_{\mathrm{t}}$ is continuous.

Note, however, that while we can transfer morphisms from $\mathcal{C}_{\mathsf{Mod}(\mathbb{B})}$ to $\mathcal{C}_{\mathsf{Equ}}$, we cannot transfer their *logical properties* in general. Thus, in the internal logic of $\mathsf{Mod}(\mathbb{B})$ the statement "every function $\mathbb{R}_{\mathrm{c}} \to \mathbb{R}_{\mathrm{c}}$ is continuous" is valid, but it is not valid in Equ. This may seem puzzling because

[6]We use the subscripts $\Box_{\mathrm{c}}$ and $\Box_{\mathrm{t}}$ to denote the Cauchy and topological versions of objects. Thus $[0,1]_{\mathrm{c}} = \{x \in \mathbb{R}_{\mathrm{c}} \mid 0 \le x \le 1\}$ and $[0,1]_{\mathrm{t}} = \{x \in \mathbb{R}_{\mathrm{t}} \mid 0 \le x \le 1\}$.

for each *particular* morphism $f \in \mathsf{Hom}_{\mathsf{Equ}}(\mathbb{R}_t, \mathbb{R}_t)$, it is the case in Equ that f is continuous. This is a difference between the internal and the external interpretation of universal quantifiers.

Pour-El and Richards [PER89] studied computability on Banach spaces, and other mathematical structures, in terms of computable sequences of points. It seems quite probable that their approach fits well within the picture presented in this section because the common subcategory AdmSeq of Equ and $\mathsf{Mod}(\mathbb{B})$ is a subcategory of sequential spaces, and sequential spaces are completely determined by their convergent sequences. We leave further investigations of this subject for another occasion.

4.3 Sheaves on Partial Combinatory Algebras

The purpose of this section is to compare realizability models over partial combinatory algebras by embedding them into sheaf toposes. We use the machinery of Grothendieck toposes and geometric morphisms to study the relationship between realizability models over different partial combinatory algebras. This work is related to Rosolini and Streicher [RS99], where the focus was mainly on the locally cartesian closed structure of realizability models. Here we are also interested in comparison of logical properties. As a reference on topos theory we use Mac Lane and Moerdijk [MM92], and Johnstone and Moerdijk [JM89] as a reference on local map of toposes. In Birkedal's dissertation [Bir99] you can find further information about realizability, and also the theory of local maps of toposes and the corresponding $\sharp$-$\flat$ calculus.

4.3.1 Sheaves over a PCA

We would like to embed $\mathsf{Mod}(\mathbb{A})$ into a sheaf topos. An obvious choice is the topos of sheaves for a subcanonical Grothendieck topology on $\mathsf{Mod}(\mathbb{A})$, which is generated by suitable families of regular epimorphic families. There is an equivalent but much simpler description of this topos, which we look at next.

As the site we take the category $\langle \mathbb{A} \rangle$ whose objects are subsets of $\mathbb{A}$, and morphism are the realized maps between subsets of $\mathbb{A}$. More precisely, if $X, Y \subseteq \mathbb{A}$ then $f \colon X \to Y$ is a morphism if there exists $a \in \mathbb{A}$ such that, for all $b \in X$, $ab\!\downarrow$ and $fb = ab$. As the Grothendieck topology on $\langle \mathbb{A} \rangle$ we take the *coproduct topology* C which is generated by those families $\{f_i \colon Y_i \to X\}_{i \in I}$ for which the coproduct $\coprod_{i \in I} Y_i$ exists and $[f_i]_{i \in I} \colon \coprod_{i \in I} Y_i \to X$ is an isomorphism. The cardinality of the index set I depends on the PCA $\mathbb{A}$. In a typical situation, the PCA $\mathbb{A}$ supports exactly the finite coproducts (the first Kleene algebra, syntactic models of λ-calculus), or exactly the countable coproducts (the second Kleene algebra, domain theoretic models of λ-calculus). Thus, in most situations this amounts to taking either precisely the finite or the countable index sets.

In many cases $\langle \mathbb{A} \rangle$ is equivalent to a well known category. For example, $\langle \mathbb{P} \rangle$ is equivalent to the category $\omega\mathsf{Top}_0$, whereas $\langle \mathbb{B} \rangle$ is equivalent to the category 0Dim.

Definition 4.3.1 The category of sheaves on $\langle \mathbb{A} \rangle$ for the coproduct topology is denoted by $\mathsf{Sh}(\mathbb{A})$.

Observe that the sheaves on $\langle \mathbb{A} \rangle$ are simply those presheaves P that "preserve products", i.e., $P(\coprod_i Y_i) \cong \prod_i PY_i$.

Theorem 4.3.2 *The category* $\mathsf{Sh}(\mathbb{A})$ *is equivalent to the category of sheaves* $\mathsf{Sh}(\mathsf{Mod}(\mathbb{A}), R)$ *for the subcanonical Grothendieck topology* R *generated by those families* $\{f_i \colon B_i \to A\}_{i \in I}$ *for which the coproduct* $\coprod_{i \in I} B_i$ *exists and* $[f_i]_{i \in I} \colon \coprod_{i \in I} B_i \to A$ *is a regular epi.*

Proof. The category $\langle\mathbb{A}\rangle$ is the full subcategory of $\mathsf{Mod}(\mathbb{A})$ on the *canonically separated* modest sets. By Theorem 1.3.4, $\langle\mathbb{A}\rangle$ is equivalent to the category of projective modest sets over $\mathbb{A}$. Therefore we can replace $\langle\mathbb{A}\rangle$ with the category $\mathsf{Proj}(\mathbb{A})$ of projective modest sets.

Let the *jointly-split topology* S on $\mathsf{Proj}(\mathbb{A})$ be generated by those families $\{f_i\colon B_i \to A\}_{i\in I}$ for which the coproduct $\coprod_{i\in I} B_i$ exists and $[f_i]_{i\in I}\colon \coprod_{i\in I} B_i \to A$ splits, i.e., has a right inverse $s\colon A \to \coprod_{i\in I} B_i$. Here the cardinality of I is treated as in the definition of coproduct topology. Let us verify that the jointly split families form a basis for Grothendieck topology:

1. Isomorphisms cover: It is obvious that an isomorphism is covering since it is split by its inverse.

2. Stability under pullbacks: Suppose $\{f_i\colon B_i \to A\}_{i\in I}$ covers A. Consider the pullback along $g\colon C \to A$. Since coproducts in $\langle A\rangle$ are stable, we get a pullback diagram

$$\begin{tikzcd} \coprod_i g^*B_i \arrow[r] \arrow[d, "{[g^*f_i]_i}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \coprod_i B_i \arrow[d, "{[f_i]_i}"'] \\ C \arrow[r, "g"] & A \arrow[u, dashed, bend right, "s"'] \end{tikzcd}$$

The morphism s in the above diagram is the splitting of $[f_i]_i$. We want to show that the left-hand vertical morphism splits, which follows easily from the pullback property of the diagram. Since $[f_i]_i \circ s \circ g = 1_C \circ g$ there exists a unique arrow $t\colon C \to \coprod_i g^*B_i$ such that $1_C = [g^*f_i]_i \circ t$, as required.

3. Transitivity: Suppose $\{f_i\colon B_i \to A\}_{i\in I}$ is a covering family, and for each $i \in I$, the family $\{g_{ij}\colon C_{ij} \to B_i\}_{j\in J_i}$ covers B_i. Then $[f_i]_i$ splits by a morphism s, and $[g_ij]_j$ splits by r_i. The map $[f_i \circ g_{ij}]_{ij}$ splits by $\left(\sum_{i\in I} r_i\right) \circ s$.

Next, we show that the jointly split families generate precisely the coproduct topology C. We need to show that a sieve $\{f_i\colon B_i \to A\}_{i\in I}$ contains an S-cover if, and only if, it contains a C-cover. One direction is easy, since every C-cover is obviously an S-cover. For the converse, if $\{f_j\colon B_j \to A\}_{j\in J}$ is jointly split by $s\colon A \to \coprod_{j\in J} B_j$, then we can decompose A into a coproduct $A \cong \coprod_{j\in J} s^*B_j$, as in the pullback diagram

$$\begin{tikzcd} \coprod_{j\in J} s^*B_j \arrow[r] \arrow[d, "1_A"'] \arrow[dr, phantom, "\lrcorner", very near start] & \coprod_{j\in J} B_j \arrow[d, "1"] \\ A \arrow[r, "s"'] & \coprod_{j\in J} B_j \end{tikzcd}$$

Therefore, if a sieve contains a jointly split family $\{f_j\colon B_j \to A\}_{j\in J}$, then it also contains a family whose coproduct is isomorphic to A.

As in the statement of the theorem, let R be the Grothendieck topology on $\mathsf{Mod}(\mathbb{A})$ generated by those families $\{f_i\colon B_i \to A\}_{i\in I}$ for which the coproduct $\coprod_{i\in I} B_i$ exists and $[f_i]_{i\in I}\colon \coprod_{i\in I} B_i \to A$ is a regular epi. To finish the proof, we apply the Comparison Lemma [MM92, Appendix, Corollary 4.3] to $\mathsf{Sh}(\mathsf{Mod}(\mathbb{A}), R)$ and $\mathsf{Sh}(\mathsf{Proj}(\mathbb{A}), S)$. For this we must check three conditions:

1. Topology R is subcanonical: we chose R to be generated by certain regular-epimorphic families.

2. Every object in $\mathsf{Mod}(\mathbb{A})$ is R-covered by objects in $\mathsf{Proj}(\mathbb{A})$: this is Proposition 1.3.3.

3. A family $\{f_i \colon B_i \to A\}_{i \in I}$ is S-covering in $\mathsf{Proj}(\mathbb{A})$ if, and only if, it is R-covering in $\mathsf{Mod}(\mathbb{A})$: this holds because a morphism $f \colon B \to A$ in $\mathsf{Proj}(\mathbb{A})$ is split if, and only if, it is a regular epi in $\mathsf{Mod}(\mathbb{A})$. Indeed, if it is split then it is a regular epi by a general category theoretic argument. Conversely, suppose $f \colon B \to A$ is a regular epi and $A \in \mathsf{Proj}(\mathbb{A})$. Since A is projective there exists a right inverse $s \colon B \to A$ of f, hence f is split.

■

Corollary 4.3.3 *The Yoneda embedding*

$$\mathsf{Mod}(\mathbb{A}) \xrightarrow{\ y\ } \mathsf{Sh}(\mathbb{A})$$

is full and faithful, preserves the locally cartesian closed structure, regular epis, and coproducts. In terms of categorical logic, it preserves and reflects validity of formulas involving full first-order logic, exponentials, dependent types, disjoint sum types, and quotients of $\neg\neg$-stable equivalence relations. In case $\mathsf{Mod}(\mathbb{A})$ *has countable coproducts, y preserves countably infinite disjunctions and the natural numbers object.*

More precisely, the functor y is defined as follows. If $I \colon \langle \mathbb{A} \rangle \to \mathsf{Mod}(\mathbb{A})$ is the inclusion, then for $S \in \mathsf{Mod}(\mathbb{A})$, $yS = \mathsf{Hom}(I(\square), S)$ where the hom-set is taken in $\mathsf{Mod}(\mathbb{A})$. We do not have to compose with sheafification because the topology is subcanonical.

Example 4.3.4 Countably based equilogical spaces embed via the Yoneda embedding into the topos $\mathsf{Sh}(\mathbb{P}) \simeq \mathsf{Sh}(\omega\mathsf{Top}_0, C_\omega)$, where C_ω is the countable coproducts topology.

Example 4.3.5 Similarly, $\mathsf{Mod}(\mathbb{B})$ embeds into $\mathsf{Sh}(\mathbb{B}) \simeq \mathsf{Sh}(\mathsf{0Dim}, C_\omega)$, where $\mathsf{0Dim}$ is the category of countably based 0-dimensional Hausdorff spaces.

4.3.2 Functors Induced by Applicative Morphisms

A discrete applicative morphism $\rho \colon \mathbb{E} \xrightarrow{\mathsf{PCA}} \mathbb{F}$ induces a functor $\widehat{\rho} \colon \mathsf{Mod}(\mathbb{E}) \to \mathsf{Mod}(\mathbb{F})$, as was shown in Proposition 1.4.4. The functor preserves finite limits and regular epis by Proposition 1.4.5. Suppose that in addition $\widehat{\rho}$ preserves all coproducts that exist in $\mathsf{Mod}(\mathbb{E})$. Recall from [MM92, Section VII.7] that in this case $\widehat{\rho}$ induces a geometric morphism $(\rho^*, \rho_*) \colon \mathsf{Sh}(\mathbb{F}) \to \mathsf{Sh}(\mathbb{E})$ between the corresponding toposes, as in the diagram below.

$$\begin{tikzcd}
\mathsf{Sh}(\mathbb{E}) \arrow[r, shift left, "\rho^*"'] & \mathsf{Sh}(\mathbb{F}) \arrow[l, shift left, "\rho_*"'] \\
\mathsf{Mod}(\mathbb{E}) \arrow[u, "y"] \arrow[r, "\widehat{\rho}"'] & \mathsf{Mod}(\mathbb{F}) \arrow[u, "y"']
\end{tikzcd} \tag{4.9}$$

The inverse image part ρ^* of the geometric morphism in the above diagram makes the evident square commute up to natural isomorphism.

4.3.3 Applicative Retractions Induce Local Maps of Toposes

Let $(\eta \dashv \delta)\colon \mathbb{E} \xrightarrow{\mathsf{PCA}} \mathbb{F}$ be an applicative retraction of discrete applicative morphisms. By Theorem 1.4.12, the induced functors $\widehat{\eta}$ and $\widehat{\delta}$ form an adjoint pair $\widehat{\eta} \dashv \widehat{\delta}$,

$$\begin{tikzcd} \mathsf{Mod}(\mathbb{E}) \arrow[r, shift right, "\widehat{\delta}"'] & \mathsf{Mod}(\mathbb{F}) \arrow[l, shift right, "\widehat{\eta}"'] \end{tikzcd}$$

In addition, $\widehat{\delta} \circ \widehat{\eta} \cong 1_{\mathsf{Mod}(\mathbb{F})}$. Suppose further that $\widehat{\delta}$ preserves whatever coproducts exist, and call such a functor "+-preserving". Combining this with (4.9), we get *three* adjoint functors $\eta^* \dashv \eta_* = \delta^* \dashv \delta_*$,

$$\begin{tikzcd} \mathsf{Sh}(\mathbb{E}) \arrow[r, "\eta_* = \delta^*"] & \mathsf{Sh}(\mathbb{F}) \arrow[l, bend right, "\eta^*"'] \arrow[l, bend left, "\delta_*"] \end{tikzcd}$$

where $\delta^* \circ \eta^* \cong 1_{\mathsf{Sh}(\mathbb{F})}$. Thus, a +-preserving discrete applicative retraction $(\eta \dashv \delta)\colon \mathbb{E} \xrightarrow{\mathsf{PCA}} \mathbb{F}$ induces a local map $\mathsf{Sh}(\mathbb{E}) \to \mathsf{Sh}(\mathbb{F})$. This is a familiar setup from Birkedal [Bir99], from which we obtain a $\sharp$-$\flat$ calculus for the internal logic that can be used to compare realizability in modest sets over $\mathbb{E}$ with that over $\mathbb{F}$.

An applicative retraction does not seem to induce a third adjoint if we use *realizability toposes* $\mathsf{RT}(\mathbb{E})$ and $\mathsf{RT}(\mathbb{F})$ instead. A good conceptual explanation of this phenomenon would be desirable.

4.3.4 A Forcing Semantics for Realizability

The following theorem spells out the Kripke-Joyal semantics in $\mathsf{Sh}(\mathbb{A})$. The interpretation of disjunction, negation and existential quantification is simpler than the usual one due to the simple nature of the coproduct topology on the site $\langle \mathbb{A} \rangle$.

Theorem 4.3.6 *Let $X, Y \in \mathsf{Sh}(\mathbb{A})$, and let*

$$x : X \mid \phi(x) \qquad x : X \mid \psi(x) \qquad x : X, y : Y \mid \rho(x, y)$$

be formulas in the internal language of $\mathsf{Sh}(\mathbb{A})$. Let $A \in \langle \mathbb{A} \rangle$ and $a \in XA$. The Kripke-Joyal forcing relation $\models$ is interpreted as follows:

1. *$A \models \phi(a) \wedge \psi(a)$ if, and only if, $A \models \phi(a)$ and $A \models \psi(a)$.*
2. *$A \models \phi(a) \vee \psi(a)$ if, and only if, there exist $A_1, A_2 \in \langle \mathbb{A} \rangle$ such that $A = A_1 + A_2$, $A_1 \models \phi(a \cdot \iota_1)$ and $A_2 \models \psi(a \cdot \iota_2)$.*
3. *$A \models \phi(a) \longrightarrow \psi(a)$ if, and only if, for all $f\colon B \to A$ in $\langle \mathbb{A} \rangle$, $B \models \phi(a \cdot f)$ implies $B \models \psi(a \cdot f)$.*
4. *$A \models \neg\phi(a)$ if, and only if, for all $f\colon B \to A$ in $\langle \mathbb{A} \rangle$, $B \models \phi(a \cdot f)$ implies $B = \mathsf{0}$.*
5. *$A \models \forall\, y \in Y \,.\, \rho(a, y)$ if, and only if, for all $f\colon B \to A$ in $\langle \mathbb{A} \rangle$ and all $b \in YB$, $B \models \rho(a \cdot f, b)$.*
6. *$A \models \exists\, y \in Y \,.\, \rho(a, y)$ if, and only if, $A = \coprod_{i \in I} A_i$ in $\langle \mathbb{A} \rangle$ and for each $i \in I$ there exist $b_i \in YA_i$ such that $A_i \models \rho(a \cdot \iota_i, b_i)$.*

Proof. We only need to show that the standard interpretations of disjunction, negation, and existential quantification simplify to the forms stated in the theorem. This follows easily from the characterization of the topology via the disjoint sum basis.

Let us first consider disjunction. Suppose $A \models \phi(a) \vee \psi(a)$. Then there exists a family $\{\iota_i : A_i \to A\}_{i \in I}$ such that $[\iota_i]_i : A_1 + \cdots + A_k \to A$ is an isomorphism and, for every $i \in I$, $A_i \models \phi(a \cdot \iota_i)$ or $A_i \models \psi(a \cdot \iota_i)$. Define the sets J and K by

$$J = \left\{ i \in I \;\middle|\; \models \phi(a \cdot \iota_i) \right\} , \qquad K = I \setminus J .$$

Let $A'_1 = \coprod_{j \in J} A_j$ and $A'_2 = \coprod_{k \in K} A_k$. Then it is clear that $A = A'_1 + A'_2$. Let $\kappa_1 = [\iota_j]_{j \in J} : A_1 \to A$ and $\kappa_2 = [\iota_k]_{k \in K} : A_2 \to A$ be the isomorphisms. It is now clear that $A'_1 \models \phi(a \cdot \kappa_1)$ and $A_2 \models \psi(a \cdot \kappa_2)$, as required. The converse holds, since if $A = A_1 + A_2$, $A_1 \models \phi(a \cdot \iota_1)$ and $A_2 \models \psi(a \cdot \iota_2)$, then $A \models \phi(a) \vee \psi(a)$ because the sum of canonical inclusions $[\iota_1, \iota_2] : A_1 + A_2 \to A$ is an isomorphism, thus it covers A.

The interpretation of negation is correct because an object is covered by the empty family $\{\}$ if, and only if, it is the initial object $\mathsf{0}$.

Suppose $A \models \exists y \in Y . \rho(a, y)$. Then there exists a family $\{\iota_i : A_i \to A\}_{i \in I}$ such that $A = \coprod_{i \in I} A_k$, $\iota_i : A_i \to A$ is the canonical inclusion for every $i \in I$, and there exists $b_i \in Y A_i$ such that $A_i \models \rho(a \cdot \iota_i, b_i)$. This proves one direction. The converse is proved easily as well. ■

If $\mathsf{Mod}(\mathbb{A})$ has countable coproducts a clause involving countable disjunctions can be added. The forcing semantics can be restricted to the modest sets, as long as the formulas are restricted to first-order logic with exponentials, dependent types, subset types, and quotients of $\neg\neg$-stable equivalence relations. It is a consequence of Corollary 4.3.3 that such a formula is valid in the forcing semantics if, and only if, it is valid in the realizability interpretation.

4.3.5 A Transfer Principle for Modest Sets

Suppose $(\eta \dashv \delta) : \mathbb{E} \xrightarrow{\mathsf{PCA}} \mathbb{F}$ is an applicative retraction such that $\widehat{\delta}$ is $+$-preserving, which means that it preserves all coproducts that exist in $\langle \mathbb{E} \rangle$. The transfer principle from Awodey et. al. [ABS99] can be applied to the induced local map of toposes,

$$\begin{tikzcd} \mathsf{Sh}(\mathbb{E}) \arrow[r, "\eta_* = \delta^*"] & \mathsf{Sh}(\mathbb{F}) \arrow[l, bend right, "\eta^*"'] \arrow[l, bend left, "\delta_*"] \end{tikzcd} .$$

We say that a formula θ in the internal language of a topos is *local*[7] if it is built from atomic predicates, including equations, and first-order logic, such that for every subformula of the form $\phi \longrightarrow \psi$, ϕ does not contain any $\forall$ or $\longrightarrow$.

If θ is a local sentence in the internal logic of $\mathsf{Sh}(\mathbb{F})$, we write $\mathsf{Sh}(\mathbb{F}) \models \theta$ when the interpretation of θ is valid in $\mathsf{Sh}(\mathbb{F})$. The sentence θ can also be interpreted in $\mathsf{Sh}(\mathbb{E})$, where the types and relations occurring in θ are mapped over to $\mathsf{Sh}(\mathbb{E})$ by η^*. The transfer principle from [ABS99] tells us that for such a local sentence θ

$$\mathsf{Sh}(\mathbb{E}) \models \theta \quad \text{if and only if} \quad \mathsf{Sh}(\mathbb{F}) \models \theta .$$

[7]In [ABS99] such a formula is called "stable".

If only types and relations from $\mathsf{Mod}(\mathbb{F})$ occur in θ then the transfer principle restricts to the categories of modest sets:

$$\mathsf{Mod}(\mathbb{E}) \models \theta \quad \text{if and only if} \quad \mathsf{Mod}(\mathbb{F}) \models \theta \,.$$

Here we interpret θ in $\mathsf{Mod}(\mathbb{E})$ by mapping all types and relations that occur in θ over to $\mathsf{Mod}(\mathbb{E})$ by $\widehat{\eta}$. The notation $\mathsf{Mod}(\mathbb{E}) \models \theta$ means that the sentence θ is valid in the standard realizability interpretation, or equivalently, in the forcing semantics as described in Theorem 4.3.6. The following theorem explains why $\mathsf{Mod}(\mathbb{P})$ and $\mathsf{Mod}(\mathbb{B})$ appear to be very similar, at least as far as simple types are concerned.

Theorem 4.3.7 *Let $(\iota \dashv \delta)\colon \mathbb{P} \xrightarrow{\mathsf{PCA}} \mathbb{B}$ be the applicative retraction from Subsection 1.4.2. Let θ be a local first-order sentence such that all variables occurring in θ have types $\mathbb{N}$, $\mathbb{N}^{\mathbb{N}}$, or $\mathbb{R}$. Then*

$$\mathsf{Mod}(\mathbb{P}) \models \theta \quad \text{if and only if} \quad \mathsf{Mod}(\mathbb{B}) \models \theta \,,$$

where $\mathbb{N}$ is interpreted as the natural numbers object, $\mathbb{N}^{\mathbb{N}}$ is interpreted as the obvious exponential, and $\mathbb{R}$ is interpreted as the real numbers object.

Proof. The theorem holds because the functor $\widehat{\iota}\colon \mathsf{Mod}(\mathbb{B}) \to \mathsf{Mod}(\mathbb{P})$ preserves the natural numbers object $\mathbb{N}$, its function space $\mathbb{N}^{\mathbb{N}}$, and the real numbers object $\mathbb{R}$. The functor $\widehat{\delta}$ preserves coproducts by Theorem 4.2.2. ∎

In Theorem 4.3.7 we cannot allow variables of higher types such as $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$ and $\mathbb{R}^{\mathbb{R}}$ to occur, because it is well known that the following local sentence involving $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$ is valid in $\mathsf{Mod}(\mathbb{B})$ but not in $\mathsf{Mod}(\mathbb{P})$:

$$\forall\, F \in \mathbb{N}^{\mathbb{N}^{\mathbb{N}}} .\, \exists\, \alpha \in \mathbb{N}^{\mathbb{N}} .\, \forall\, \beta \in \mathbb{N}^{\mathbb{N}} .\, F\beta = (\alpha|\beta) \,. \tag{4.10}$$

The sentence states that every functional $F \in \mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$ has an associate $\alpha \in \mathbb{N}^{\mathbb{N}}$ in the sense of Kleene [Kle59]. Here $\alpha|\beta$ is Kleene's *continuous function application*. The statement $n = (\alpha|\beta)$ is equivalent to

$$\exists\, m \in \mathbb{N} .\, \big(\alpha(\overline{\beta}m) = n + 1 \land \forall\, k \in \mathbb{N} .\, \big(k < m \longrightarrow \alpha(\overline{\beta}k) = 0\big)\big) \,.$$

Similarly, a statement that all functions $f \in \mathbb{R}^{\mathbb{R}}$ are continuous holds in $\mathsf{Mod}(\mathbb{B})$ but not in $\mathsf{Mod}(\mathbb{P})$. Thus, in a roundabout way, we obtain the following result.

Proposition 4.3.8 *The functor $\widehat{\iota}\colon \mathsf{Mod}(\mathbb{B}) \to \mathsf{Mod}(\mathbb{P})$ does not preserve exponentials. In particular, $\widehat{\iota}(\mathbb{N}^{\mathbb{N}^{\mathbb{N}}})$ is not isomorphic to the object $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$ in $\mathsf{Mod}(\mathbb{P})$, and $\widehat{\iota}(\mathbb{R}^{\mathbb{R}})$ is not isomorphic to the object $\mathbb{R}^{\mathbb{R}}$ in $\mathsf{Mod}(\mathbb{P})$.*

Proof. We can in fact prove Proposition 4.3.8 directly as follows. Let X be the object of type 2 functionals in $\mathsf{0Equ}$, which is equivalent to $\mathsf{Mod}(\mathbb{B})$, and let Y be the object of type 2 functional in Equ.

Both X and Y are equilogical spaces. The space $|X|$ is a Hausdorff space. The space $|Y|$ is the subspace of the total elements of the Scott domain $D = {\mathbb{N}_\perp}^{\mathbb{N}_\perp^\omega}$. The equivalence relation on $|Y|$ is the consistency relation of D restricted to $|Y|$. Suppose $f\colon |Y| \to |X|$ represented an isomorphism, and let $g\colon |X| \to |Y|$ represent its inverse. Because f is monotone in the specialization order and $|X|$ has a trivial specialization order, $a \equiv_Y b$ implies $fx = fy$. Therefore, $g \circ f\colon |Y| \to |Y|$ is an equivariant retraction. By Proposition 4.1.8, Y is a topological object. By Corollary 4.1.9, this would mean that the topological quotient $|Y|/{\equiv_Y}$ is countably based, but it is not, as is well known. Another way to see that Y cannot be topological is to observe that Y is an exponential of the Baire space, but the Baire space is not exponentiable in $\omega\mathsf{Top}_0$. ∎

Contrast Proposition 4.3.8 with result of Subsection 4.1.4 and Proposition 4.2.26, which claim that the Kleene-Kreisel functionals "correspond to" both the finite type functionals in Equ and $\mathsf{Mod}(\mathbb{B})$. How can this be? The correspondence is only at the level of equivalence classes, or global points. Just because two spaces have the same global points, that does not mean they are isomorphic.

Finally, we remark that statement (4.10) is of course valid in $\mathsf{Mod}(\mathbb{P})$ if $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$ is replaced by $\widehat{\iota}(\mathbb{N}^{\mathbb{N}^{\mathbb{N}}})$. But then it becomes a simple truism, since it can be shown that in $\mathsf{Mod}(\mathbb{P})$ the space $\widehat{\iota}(\mathbb{N}^{\mathbb{N}^{\mathbb{N}}})$ is just the set of those functionals that have an associate:

$$\widehat{\iota}(\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}) = \left\{ F \in \mathbb{N}^{\mathbb{N}^{\mathbb{N}}} \mid \exists\, \alpha \in \mathbb{N}^{\mathbb{N}} \,.\, \forall\, \beta \in \mathbb{N}^{\mathbb{N}} \,.\, F\beta = (\alpha|\beta) \right\} .$$

Proposition 4.3.8 should be contrasted with the fact that there is an epi-mono $\widehat{\iota}(\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}) \to \mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$ that is *not* iso but nevertheless induces a natural bijection between the global points of $\widehat{\iota}(\mathbb{N}^{\mathbb{N}^{\mathbb{N}}})$ and the global points of $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$. Therefore, the finite types over $\mathbb{N}$ in $\mathsf{Mod}(\mathbb{P})$ and $\mathsf{Mod}(\mathbb{B})$ are equivalent as far as the *cartesian closed structure* is concerned, but here we see that they have different *logical* properties.

Chapter 5

Computable Topology and Analysis

In the final chapter we develop a selection of topics from computable topology and analysis in the logic of modest sets. Because the logic of modest sets is an intuitionistic logic with additional valid principles, we can follow existing sources on intutionistic and constructive mathematics [TvD88a, TvD88b, BB85, McC84] and synthetic domain theory [Ros86, vOS98, Hyl92]. A number of constructions simplify because of the Axiom of Stability, Markov's Principle, and Number Choice. The main novelty is the computability operator, which fits seamlessly with the rest of the logic. We emphasize the use of higher function types.

The spaces and constructions that we develop in the logic of modest sets can be interpreted in categories of modest sets. In a number of cases it turns out that we obtain structures that have been discovered before by direct constructions in specific models of modest sets. Let us list a few. Our definition of countably based spaces corresponds to Spreen's effective T_0-spaces in $\mathsf{Mod}(\mathbb{N})$. The standard dominance Σ interpreted in $\mathsf{Mod}(\mathbb{N})$ corresponds to the familiar dominance of r.e. sets in the effective topos, whereas in $\mathsf{Equ}_{\mathsf{eff}}$ and $\mathsf{Mod}(\mathbb{B}, \mathbb{B}_\sharp)$ it corresponds to the standard representation of the Sierpinski space as a quotient of the Cantor space. The real numbers $\mathbb{R}$ interpreted in $\mathsf{Mod}(\mathbb{N})$ are the recursive reals, in $\mathsf{Equ}_{\mathsf{eff}}$ or $\mathsf{Mod}(\mathbb{B}, \mathbb{B}_\sharp)$ they are the well known signed binary digit representation of the reals. Moreover, in $\mathsf{Equ}_{\mathsf{eff}}$ and $\mathsf{Mod}(\mathbb{B}, \mathbb{B}_\sharp)$ the intrinsic topology on $\mathbb{R}$ is the usual metric topology. We can also relate Σ and $\mathbb{R}$ to the domain-theoretic framework, as discussed in Subsection 4.2.4. The standard dominance then turns out to be the Sierpinski space, and the reals turn out to be represented as the maximal elements of the interval domain $\mathsf{I}\mathbb{R}$. The interpretation of metric spaces in $\mathsf{Mod}(\mathbb{N})$ gives the usual notion of effective metric spaces, and in $\mathsf{Equ}_{\mathsf{eff}}$ that of a separable metric space with a computable metric.

These examples should suffice to demonstrate two points. First, modest sets really do provide a unifying framework for a number of approaches to computable topology and analysis. Second, if we develop computable topology and analysis in the logic of modest sets, rather than in one specific model of computation, we cover a large class of important models of computation at once. This way we do not have to redo the work every time we change the underlying model. Of course, sometimes we do want to know more about a specific model. In this case we can use additional reasoning principles that are valid in that model. For example, in $\mathsf{Mod}(\mathbb{N})$ Church's thesis is valid, $\mathsf{Mod}(\mathbb{B}, \mathbb{B}_\sharp)$ enjoys a continuity principle, and $\mathsf{Equ}_{\mathsf{eff}}$ enjoys various choice principles.

5.1 Countable Spaces

In this section we present some basic results about *countable space* and *countable sets*. These notions are important for the theory of countably based spaces and separable metric spaces.

Definition 5.1.1 An *enumeration* of a space A is a map $a_\square \colon \mathbb{N} \to A$ such that for every $x \in A$ there exists $n \in \mathbb{N}$ such that $a_n = x$. In other words, an enumeration is a quotient map whose domain is $\mathbb{N}$. A *countable space* is a space A together with an enumeration $a \colon \mathbb{N} \to A$. A *countable set* is a decidable countable space.

A countable space is always given together with an enumeration. Usually we omit explicit mention of the enumeration and tacitly assume that one is given.

Note that according to this definition a countable space is inhabited. We could include the empty space among the countable spaces if we defined an enumeration of a space A to be a map $a_\square \colon \mathbb{N} \to 1 + A$ such that $\forall\, x \in A \,.\, \exists\, n \in \mathbb{N} \,.\, a_n = x$. However, since we mostly work with inhabited countable spaces, we do not use this definition.

Lemma 5.1.2 *Suppose $f \colon \mathbb{N} \to 2$ is a map and there exists $j \in \mathbb{N}$ such that $fj = 1$. Then there exists a smallest $m \in \mathbb{N}$ such that $fm = 1$.*

Proof. Define a map $g \colon \mathbb{N} \to 2$ by induction as follows:

$$g\,0 = 0\;, \qquad g(n+1) = \texttt{if}\ (gn = 1) \lor (fn = 1)\ \texttt{then}\ 1\ \texttt{else}\ 0\;.$$

We prove that there exists exactly one $n \in \mathbb{N}$ such that $gn = 0$ and $g(n+1) = 1$. A simple proof by induction shows that if $gi = 1$ and $i \le k$, then $gk = 1$, so there can be at most one $n \in \mathbb{N}$ for which $gn = 0$ and $g(n+1) = 1$. Observe that if $gn \neq g(n+1)$, then $gn = 0$ and $g(n+1) = 1$. So we just need to prove that there exists $n \in \mathbb{N}$ such that $gn \neq g(n+1)$. By Markov's Principle, it is sufficient to show that $\neg\forall\, k \in \mathbb{N} \,.\, gk = g(k+1)$. So assume $gk = g(k+1)$ for all $k \in \mathbb{N}$. A simple induction proves that in this case $gk = 0$ for all $k \in \mathbb{N}$. But this is impossible because there is some $j \in \mathbb{N}$ for which $fj = 1$ and so $g(j+1) = 1$.

Now let m be the unique number such that $gm = 0$ and $g(m+1) = 1$. It follows that $fm = 1$. If $n < m$, then $gn = 0$ hence $fn = 0$, which confirms that m is really the smallest number at which f attains the value 1. ∎

Theorem 5.1.3 (Minimization Principle) *Let $o = \lambda n : \mathbb{N} \,.\, 0$ be the constantly zero map, and let $A = 2^{\mathbb{N}} \setminus \{o\} = \{f \in 2^{\mathbb{N}} \mid f \neq o\}$. There exists a map $\boldsymbol{\mu} \colon A \to \mathbb{N}$, called the* minimization *operator, such that, for all $f \in A$, $\boldsymbol{\mu} f$ is the smallest number at which f has value 1.*

Proof. By Lemma 5.1.2, for every $f \in A$ there exists a (unique) smallest $m \in \mathbb{N}$ such that $fm = 1$. Now apply the Unique Choice to obtain the map $\boldsymbol{\mu} \colon A \to 2$. ∎

Lemma 5.1.4 *Suppose $f \colon \mathbb{N} \to A$ is a surjection and A is decidable. Then f is a quotient map.*

Proof. Take any $x \in A$. Because f is surjective there $\neg\neg$-exists $n \in \mathbb{N}$ such that $fn = x$. Because equality on A is decidable, we can employ Markov's Principle to conclude that there exists $n \in \mathbb{N}$ such that $fn = x$. Therefore f is a quotient map. ∎

Recall that A is a *retract* of B when there exist maps $s\colon A \to B$ and $r\colon B \to A$ such that $r \circ s = 1_A$. The map s is the *section* and r is the *retraction*. Every section is an embedding and every retraction is a quotient map. Thus, a retract is a special kind of a quotient. By definition, a space is countable if, and only if, it is isomorphic to a quotient of $\mathbb{N}$. We can also characterize the retract of $\mathbb{N}$.

Theorem 5.1.5 *A space is a countable set if, and only if, it is a retract of* $\mathbb{N}$.

Proof. Suppose S is decidable and $s\colon \mathbb{N} \to S$ is an enumeration. By Lemma 5.1.4, $s\colon \mathbb{N} \to S$ is a quotient map. Because equality on S is decidable, we can define the section $i\colon S \to \mathbb{N}$ by minimization as

$$i\,x = \boldsymbol{\mu}(\lambda n\!:\!\mathbb{N}.\, \mathtt{if}\ s_n = x\ \mathtt{then}\ 1\ \mathtt{else}\ 0)\ .$$

Conversely, if S is a retract of $\mathbb{N}$, then it is a countable subspace of the countable set $\mathbb{N}$, therefore it is decidable. ■

Corollary 5.1.6 *A countable set is projective.*

Proof. By Theorem 5.1.5, a countable set is a retract of the projective space $\mathbb{N}$, therefore a regular subspace of the projective space $\mathbb{N}$. A regular subspace of a projective space is projective. ■

Example 5.1.7 The objects of $\mathsf{Mod}(\mathbb{N})$ are enumerated sets. An enumerated set X is a set $|X|$ with a partial surjection $\delta_X\colon \mathbb{N} \rightharpoonup |X|$. We say that X is *total* when the partial surjection δ_X is total. In $\mathsf{Mod}(\mathbb{N})$ a space is countable if, and only if, it is isomorphic to a total enumerated set. Indeed, if X is countable then it is isomorphic to a quotient of $\mathbb{N}$, and a quotient of $\mathbb{N}$ is clearly a total enumerated set. Conversely, a total enumerated set X is a quotient of $\mathbb{N}$, as is witnessed by the following diagram:

$$\begin{tikzcd}
\mathbb{N} \arrow[r, "1_{\mathbb{N}}"] \arrow[d, "1_{\mathbb{N}}"'] & \mathbb{N} \arrow[d, "\delta_X"] \\
\mathbb{N} \arrow[r, "\delta_X"'] & X
\end{tikzcd}$$

For example, the space $\mathbb{N} \to \mathbb{N}_\bot$ is the modest set of partial recursive functions. It is countable because there exists a total recursive enumeration of Gödel indices of partial recursive functions. On the other hand, the space $\mathbb{N} \to \mathbb{N}$ is the modest set of total recursive functions and it is not countable, since the Gödel indices of total recursive functions cannot be recursively enumerated.

Example 5.1.8 In Equ a countable space is isomorphic to an equilogical space $(\mathbb{N}, \equiv)$, where $\mathbb{N}$ is equipped with the discrete topology. But every such equilogical space is isomorphic to a subspace of $\mathbb{N}$, via a choice function that picks a representative of each equivalence class. Thus, in Equ the interpretation of a countable space is that of a countable set with discrete topology. Every such space is characterized by its cardinality. Moreover, every countable space is decidable, and so in Equ the interpretation of countable spaces and countable sets coincide.

Example 5.1.9 In $\mathsf{Equ}_{\mathsf{eff}}$ countable spaces are more delicate than in Equ. Every countable space is isomorphic to an effective equilogical space $(\mathbb{N}, \equiv)$, but it is *not* the case that every such effective equilogical space is isomorphic to a subspace of $\mathbb{N}$. This is so because there might be no way to computably pick a representative of each equivalence class.

How about countable sets? By Theorem 5.1.5, the countable sets are the retracts of $\mathbb{N}$. Up to isomorphism, a retract of $\mathbb{N}$ is a regular subspace of $\mathbb{N}$. Hence, in $\mathsf{Equ}_{\mathsf{eff}}$ a countable set A is completely described by a pair of maps (r, s) where $r : \mathbb{N} \to \mathbb{N}$ is a total recursive function, $s : \mathbb{N} \rightharpoonup \mathbb{N}$ is a partial recursive function, and $r \circ s = 1_{\mathsf{dom}(s)}$. But if A is described by (r, s) then it is also described by (r, s') where $s' : \mathbb{N} \rightharpoonup \mathbb{N}$ is defined by $s'n = \min\{k \in \mathbb{N} \mid rk = n\}$. Thus, a countable set is described already by a total recursive function $r : \mathbb{N} \to \mathbb{N}$ that enumerates its elements. Conversely, every total recursive function r describes a countable set, namely its range $\mathsf{rng}(r)$. Total recursive maps $q : \mathbb{N} \to \mathbb{N}$ and $r : \mathbb{N} \to \mathbb{N}$ describe isomorphic countable sets exactly when there exist partial recursive functions $a, b : \mathbb{N} \rightharpoonup \mathbb{N}$ such that $\mathsf{dom}(a) = \mathsf{rng}(q)$, $\mathsf{dom}(b) = \mathsf{rng}(r)$, $b \circ a = 1_{\mathsf{rng}(q)}$ and $a \circ b = 1_{\mathsf{rng}(r)}$. In other words, there is a computable bijection between $\mathsf{rng}(q)$ and $\mathsf{rng}(r)$. Thus we have determined that in $\mathsf{Equ}_{\mathsf{eff}}$ the countable sets are interpreted as r.e. sets, and isomorphism of countable sets is interpreted as computable bijective correspondence of r.e. sets.

5.2 The Generic Convergent Sequence

Convergent sequences and their limits play an important role in topology. In the logic of modest sets, the notion of a sequence and its limit is captured by the coinductive type for the functor $1+\square$, which we study in this section.

The map $1 \to 2$ that maps $\star$ to 1 yields the polynomial functor $P_{1\to 2}X = 1 + X$. The *generic convergent sequence* is the coinductive type $\mathbb{N}^+ = \mathsf{M}_{1\to 2}$. Its structure map is an isomorphism $\mathsf{p} : \mathbb{N}^+ \to 1 + \mathbb{N}^+$. We denote the inverse of p by $\mathsf{s} = \mathsf{p}^{-1} : 1 + \mathbb{N}^+ \to \mathbb{N}^+$, and we write $\mathsf{0} = \mathsf{s}\star$. The coinductive principle for $\mathbb{N}^+$ is

$$\big(\forall\, x, y \in \mathbb{N}^+ .\, (\rho(x, y) \longrightarrow ((x = \mathsf{0} \longleftrightarrow y = \mathsf{0}) \wedge (x \neq \mathsf{0} \wedge y \neq \mathsf{0} \longrightarrow \rho(\mathsf{p}x, \mathsf{p}y))))\big) \\ \longrightarrow \forall\, x, y \in \mathbb{N} .\, (\rho(x, y) \longrightarrow x = y) \ .$$

For any map $c : C \to 1 + C$, there is a unique map $h : C \to \mathbb{N}^+$ satisfying

$$hx = \begin{cases} \mathsf{0} & \text{if } cx = \star \,, \\ \mathsf{s}(h(cx)) & \text{otherwise} \,. \end{cases}$$

By Proposition 2.2.10, there exists a unique injection $i : \mathbb{N} \to \mathbb{N}^+$, such that

$$i\,0 = \mathsf{0} \,, \qquad\qquad i(n + 1) = \mathsf{s}n \ .$$

There is a unique map $h : 1 \to \mathbb{N}^+$, defined by corecursion from the map $\mathsf{inr} : 1 \to 1 + 1$. This map satisfies the equation $h\star = \mathsf{s}(h\star)$. The point $\infty = h\star$ is called the *point at infinity* or the *limit point* of $\mathbb{N}^+$. It is the unique point satisfying the equation $\infty = \mathsf{s}(\infty)$. Indeed, if $a = \mathsf{s}b$ and $b = \mathsf{s}b$, we can prove $a = b$ by a straightforward coinduction on the relation $\rho(x, y)$, defined by

$$\rho(x, y) \longleftrightarrow (x = \mathsf{s}x) \wedge (y = \mathsf{s}y) \ .$$

Since s is the inverse of p, the point at infinity is the unique point that satisfies the equation $\mathsf{p}\,\infty = \infty$.

Lemma 5.2.1 *For all $x \in \mathbb{N}^+$, $x = \infty$ if, and only if, for all $n \in \mathbb{N}$, $x \neq in$.*

Proof. Suppose it were the case that $\infty = in$ for some $n \in \mathbb{N}$. Then we would have $i(n+1) = \mathsf{s}(in) = \mathsf{s}\infty = \infty = in$, and since i is injective, $n = n+1$. This is impossible, hence ∞ is not of the form in. Conversely, suppose that $a \neq in$ for all $n \in \mathbb{N}$. We show that $a = \infty$ by coinduction on the relation

$$\rho(x, y) \longleftrightarrow \forall\, n \in \mathbb{N}\,.\,(x \neq in \land y \neq in)\ .$$

Suppose $\rho(x, y)$. Then $x \neq \mathsf{0} = i0$ and $y \neq \mathsf{0} = i0$. Consider any $n \in \mathbb{N}$. If $\mathsf{p}x = in$, then $x = \mathsf{s}(\mathsf{p}x) = \mathsf{s}(in) = i(n+1)$, which would contradict $\rho(x, y)$. Therefore, $\mathsf{p}x \neq in$. A similar argument shows that $\mathsf{p}y \neq in$. This proves the coinduction step. Now, since $\rho(a, \infty)$ holds, we conclude that $a = \infty$. ■

Lemma 5.2.1 tells us that $\mathbb{N}^+$ can be pictured as follows:

$$\bullet \qquad\qquad \bullet \qquad \bullet \quad \bullet \quad \bullet \;\; \bullet \; \bullet \; \bullet \; \bullet \cdots\cdots\cdots \infty$$

The map s is like the successor map on natural numbers, except that it maps the point at infinity to itself.

Theorem 5.2.2 *The generic convergent sequence is a retract of $2^{\mathbb{N}}$.*

Proof. We exhibit $\mathbb{N}^+$ as a retract of $2^{\mathbb{N}}$ by identifying a retract of $2^{\mathbb{N}}$ which satisfies the universal property of $\mathbb{N}^+$, so it must be isomorphic to $\mathbb{N}^+$. Let N be the subspace

$$N = \left\{ f \in 2^{\mathbb{N}} \mid f0 = 0 \land \forall\, n \in \mathbb{N}\,.\,(fn = 1 \longrightarrow f(n+1) = 1) \right\}\ .$$

The subspace N is a retract of $2^{\mathbb{N}}$, as is witnessed by the retraction $r \colon 2^{\mathbb{N}} \to N$, defined by

$$(rf)0 = 0\ , \qquad\qquad (rf)(n+1) = (\mathtt{if}\ fn = 1\ \mathtt{then}\ 1\ \mathtt{else}\ (rf)n)\ .$$

It is obvious that $rf \in N$ for every $f \in 2^{\mathbb{N}}$, and a straightforward proof by induction shows that $rf = f$ for all $f \in N$. Let $p \colon N \to 1 + N$ be defined by

$$pf = (\mathtt{if}\ f1 = 1\ \mathtt{then}\ \star\ \mathtt{else}\ \lambda n{:}\mathbb{N}\,.\,f(n+1))\ .$$

Let us prove that (N, p) satisfies the same universal property that $(\mathbb{N}^+, \mathsf{p})$ does. Suppose $c \colon C \to 1 + C$ is a $P_{1\to 2}$ coalgebra. We need to prove that there exists a unique $h \colon C \to N$ such that $(1 + h) \circ c = p \circ h$. The map h can be defined recursively as

$$\begin{aligned} hx0 &= 0\ , \\ hx(n+1) &= (\mathtt{if}\ cx = \star\ \mathtt{then}\ 1\ \mathtt{else}\ h(cx)n)\ . \end{aligned}$$

It is easy to check that $(1 + h) \circ c = p \circ h$. Suppose $g \colon C \to N$ also satisfies the equation $(1 + g) \circ c = p \circ g$. We show that $gxn = hxn$ for all $x \in C$ and $n \in \mathbb{N}$ by induction on n. Because $gx \in N$, $gx0 = 0 = hx0$, which takes care of the base case. For the induction step, suppose $gxn = hxn$ for all $x \in C$. If $cx = \star$ then $gx(n+1) = 1 = hx(n+1)$, otherwise

$$gx(n+1) = p(gx)n = g(cx)n = h(cx)n = hx(n+1)\ ,$$

where we used the induction hypothesis in the second to the last step. This completes the induction. We can write out explicitly the isomorphism $\nu\colon \mathbb{N}^+ \to N$:

$$\begin{aligned} \nu x 0 &= 0 \,, \\ \nu x(n+1) &= (\texttt{if}\ \mathtt{p}\,x = \star\ \texttt{then}\ 1\ \texttt{else}\ \nu(\mathtt{p}\,x)n) \,. \end{aligned}$$

We see that the point at infinity corresponds to the constant map $\lambda n\!:\!\mathbb{N}\,.\,0$. Observe that if $rf = \lambda n\!:\!\mathbb{N}\,.\,0$, where r is the retraction from $2^{\mathbb{N}}$ onto N, then $f = \lambda n\!:\!\mathbb{N}\,.\,0$. ■

Corollary 5.2.3 *The inclusion $i\colon \mathbb{N} \to \mathbb{N}^+$ is an embedding.*

Proof. Let $j\colon \mathbb{N} \to 2^{\mathbb{N}}$ be defined by

$$\begin{aligned} j\,m\,0 &= 0 \,, \\ j\,0\,(n+1) &= 1 \,, \\ j\,(m+1)\,(n+1) &= jmn \,. \end{aligned}$$

It is not hard to see that, under the isomorphism ν from the proof of Theorem 5.2.2, $i\colon \mathbb{N} \to \mathbb{N}^+$ corresponds to $j\colon \mathbb{N} \to N$. Thus, i is an embedding if, and only if, j is. We show that j is an embedding by proving that, for all $f \in N$,

$$(\neg\neg\exists\, n \in \mathbb{N}\,.\, f = jn) \longrightarrow \exists\, n \in \mathbb{N}\,.\, f = jn \,. \tag{5.1}$$

Suppose that for $f, g \in \mathbb{N}$ and $k \in \mathbb{N}$, $fk = gk = 0$ and $f(k+1) = g(k+1) = 1$. Then for all $m > k$, $fm = 1 = gm$, and for all $m \leq k$, $fm = 0 = gk$, therefore $f = g$. We see that if $fn = 0$ and $f(n+1) = 1$, then $f = jn$. Thus, (5.1) is equivalent to

$$(\neg\neg\exists\, n \in \mathbb{N}\,.\,(fn = 0 \land f(n+1) = 1)) \longrightarrow \exists\, n \in \mathbb{N}\,.\,(fn = 0 \land f(n+1) = 1) \,. \tag{5.2}$$

But (5.2) holds by Markov's Principle. ■

By Corollary 5.2.3, we may identify $\mathbb{N}$ with its image in $\mathbb{N}^+$ and think of $\mathbb{N}$ as a regular subspace of $\mathbb{N}^+$, i.e., $\mathbb{N} \subseteq \mathbb{N}^+$.

Example 5.2.4 In $\mathsf{Equ}_{\mathsf{eff}}$ the interpretation of $\mathbb{N}^+$ is, as expected, the one-point compactification of the natural numbers. This is most easily seen by using Theorem 5.2.2. According to the proof of the theorem, $\mathbb{N}^+$ is isomorphic to the space N

$$N = \left\{ f \in 2^{\mathbb{N}} \mid f0 = 0 \land \forall\, n \in \mathbb{N}\,.\,(fn = 1 \longrightarrow f(n+1) = 1) \right\} \,.$$

Since the defining predicate is a negative formula we can read the definition set-theoretically: $[\![N]\!]$ is the subspace of the Cantor space consisting of those infinite sequences that start with a 0 and change the value at most once. Therefore, $[\![N]\!]$ is isomorphic to the one-point compactification of $\mathbb{N}$.

Example 5.2.5 Like in the previous example we can use Theorem 5.2.2 to compute the interpretation of $\mathbb{N}^+$ in $\mathsf{Mod}(\mathbb{N})$. Up to isomorphism, $[\![\mathbb{N}^+]\!]$ turns out to be the modest set whose underlying set $|\mathbb{N}^+|$ is the set $\mathbb{N} \cup \{\infty\}$, an element $n \neq \infty$ is realized by codes of Turing machines that terminate after exactly n steps, and ∞ is realized by codes of Turing machines that never terminate.

5.3 Semidecidable Predicates

Recall from Section 2.3.2 that a decidable predicate $\phi(x : A)$ is one that has a characteristic map $f : A \to 2$. The space $2 = \{0, 1\}$ can be thought of as the space of Boolean values, as it has the structure of a Boolean algebra. From a computational point of view, a decidable predicate is one that can be computed by a program, which on input x outputs either 1 or 0, depending on whether $\phi(x)$ holds or not.

A *semidecidable* predicate $\phi(x : A)$ is a predicate for which there exists a program that detects that $\phi(x)$ holds, but cannot necessarily detect $\neg\phi(x)$. An example of this is the predicate $\exists\, n \in \mathbb{N} .\, fn = 1$ defined for $f \in \mathbb{N} \to 2$. If there is $n \in \mathbb{N}$ such that $fn = 1$, then we can find it by a simple search. However, if there is no such n, then the search will not terminate, and at no point can we know whether running the search longer would result in termination. In general, there might be no way to establish that a given function is not constantly zero, since such a decision procedure would entail decidability of the Halting Problem.[1] The topological intuition is that the semidecidable predicates are the open subspaces. In fact, the whole subject has a double nature—we can talk about semidecidable predicates, or about open subspaces. We prefer to use the topological terminology.

We can list some properties that we would expect the semidecidable predicates to have. Clearly, the empty and the total predicates ought to be semidecidable. The semidecidable predicates should be closed under conjunction. In addition, if ϕ is a semidecidable predicate on A, and ψ is a semidecidable predicate on $\{x \in A \mid \phi(x)\}$, then ψ should be a semidecidable predicate on A.

We also make the assumption that semidecidable predicates are closed under existential quantification over $\mathbb{N}$, i.e., if $\phi(x : A, n : \mathbb{N})$ is semidecidable, then so is $\exists\, n \in \mathbb{N} .\, \phi(x, n)$. From a topological point of view, this is clearly a reasonable requirement since a countable union of open sets is open. Speaking computationally, this requirement presumes that we can execute in parallel, or interleave the execution of, infinitely many processes.

The idea is to find a space Σ that has the structure that reflects the desired properties of semidecidable predicates, and define the semidecidable predicates on A as those that are classified by maps $A \to \Sigma$. What we need is the notion of a *dominance* Σ, which comes from synthetic domain theory [Ros86, vOS98, Hyl92].

A dominance has two points, called *bottom* and *top*:

$$\bot \in \Sigma\,, \qquad\qquad \top \in \Sigma\,. \tag{Σ0}$$

A predicate $\phi(x : A)$ is said to be *semidecidable* when it is classified by a map $f : A \to \Sigma$, i.e., for all $x \in A$,

$$\phi(x) \longleftrightarrow fx = \top\,.$$

Similarly, we say that a subspace $U \subseteq A$ is *semidecidable*, or *open*, when it is the inverse image of $\top$ for some map $u : A \to \Sigma$:[2]

$$U = u^* \top = \{x \in A \mid ux = \top\}\,.$$

A subspace $F \subseteq A$ is said to be *co-semidecidable*, or *closed*, when there exists a map $f : A \to \Sigma$ such that $F = f^* \bot$. We say that f classifies the closed subspace F. For the rest of the section, we

[1] In the general case the underlying PCA $\mathbb{A}_\sharp$ could be powerful enough to actually decide the Halting Problem, see Example 5.3.19.

[2] Here and always in this section, equality between subspaces should be understood as an isomorphism.

use the topological terminology and speak of open and closed subspaces, rather than semidecidable and co-semidecidable predicates.

The first axiom we consider tells us that there are no other points in Σ. We must be careful how this is stated. For example, if we required that $\forall x \in \Sigma . (x = \top \vee x = \bot)$, that would force $\Sigma = \{\bot\} + \{\top\}$ and the dominance would be isomorphic to 2. Instead, we require

$$\forall x \in \Sigma . (x \neq \bot \longrightarrow x = \top) \ . \tag{$\Sigma 1$}$$

It follows immediately from the Axiom of Stability that

$$\forall x \in \Sigma . (x \neq \top \longrightarrow x = \bot) \ . \tag{$\Sigma 1'$}$$

Another consequence is that $\bot \neq \top$. More importantly, we can deduce that, up to isomorphism of subspaces, there is a bijective correspondence between open (closed) subspaces of A and maps $A \to \Sigma$.

Lemma 5.3.1 *For all $x, y \in \Sigma$, if $x = \top \longleftrightarrow y = \top$ then $x = y$.*

Proof. Assume that $x = \top \longleftrightarrow y = \top$. We derive a contradiction from the assumption that $x \neq y$. If $x = \top$ then $y = \top$, hence $x = y$ which contradict $x \neq y$. Thus, $x \neq \top$. If $x \neq \top$ then $y \neq \top$, hence $x = \bot$ and $y = \bot$ by Axiom $\Sigma 1$, which again implies $x = y$. Thus, $x = \top$. We have proved both $x \neq \top$ and $x = \top$, which is a contradiction. Therefore, $\neg(x \neq y)$ and by the Axiom of Stability $x = y$. ∎

Theorem 5.3.2 *An open (closed) subspace of A is classified by exactly one map $A \to \Sigma$.*

Proof. Suppose u and v both classify the open subspace $U \subseteq A$. Then, for all $x \in A$, $ux = \top \longleftrightarrow vx = \top$, and by Lemma 5.3.1 this implies $ux = vx$. Therefore $u = v$ by Extensionality. The proof for closed subspaces is analogous. ∎

The first axiom implies that semidecidable predicates are stable.

Proposition 5.3.3 *Every open (closed) subspace is a regular subspace.*

Proof. For any $u \in X \to \Sigma$, the subspace $u^*\top = \{x \in X \mid ux = \top\}$ is regular because its defining predicate is stable, by the Axiom of Stability. ∎

The *complement* of a subspace $A \subseteq B$ is the subspace $B \setminus A = \{x \in B \mid x \notin A\}$.[3] We expect the complement of an open set to be closed, and vice versa.

Proposition 5.3.4 *A subspace is closed (open) if, and only if, it is the complement of an open (closed) subspace.*

Proof. This follows from the observation that, for all $x \in \Sigma$, $x = \bot$ if, and only if, $x \neq \top$. ∎

[3] Recall that in this case $x \notin A$ is an abbreviation for $\neg \exists y \in A . \mathrm{i}_A y = x$.

From now on we identify open subspaces of X with the elements of Σ^X. We write $\mathcal{O}(X) = \Sigma^X$ and call $\mathcal{O}(X)$ the *intrinsic topology* of X. If $U \in \mathcal{O}(X)$ and $x \in X$, we abbreviate $Ux = \top$ by $x \in U$, and $Ux = \bot$ by $x \notin U$.

Another consequence of the first axiom is that all maps are continuous in the intrinsic topology.

Proposition 5.3.5 *Every map $f\colon X \to Y$ is continuous in the intrinsic topology, i.e., the inverse image of an open subspace is open.*

Proof. If $U \subseteq Y$ is open and classified by $u\colon Y \to \Sigma$, then f^*U is classified by $u \circ f$. ■

The second axiom states that semidecidable subspaces compose:

$$U \in \mathcal{O}(X) \wedge V \in \mathcal{O}(U) \longrightarrow \exists\, W \in \mathcal{O}(X) \,.\, \forall x \in X \,.\, ((x \in U \wedge x \in V) \longleftrightarrow x \in W) \ . \tag{$\Sigma 2$}$$

The open subspace W is uniquely determined, so we usually abuse notation and denote it simply by V. From the second axiom we can derive a map $\Box \wedge \Box\colon \Sigma \times \Sigma \to \Sigma$, called the *meet* operation, such that, for all $x, y \in \Sigma$,

$$x \wedge y = \top \longleftrightarrow (x = \top \text{ and } y = \top) \ . \tag{5.3}$$

To see this, apply the second axiom to the open subspaces $U = \{\langle x, y\rangle \in \Sigma \times \Sigma \mid x = \top\}$ and $V = \{\langle x, y\rangle \in U \mid y = \top\}$. By the second axiom there is a unique open subspace $W\colon \Sigma \times \Sigma$ with the property that $\langle x, y\rangle \in W$ if, and only if, $\langle x, y\rangle \in U$ and $\langle x, y\rangle \in V$, in other words $x = \top$ and $y = \top$. Therefore, W is the desired meet operation.

Recall that the *intersection* of subspaces $A, B \subseteq X$ is defined by

$$A \cap B = \{x \in X \mid x \in A \wedge x \in B\} \ .$$

Theorem 5.3.6 *The intersection of two open subspaces is open.*

Proof. For any $U, V \in \mathcal{O}(X)$,

$$\begin{aligned} U \cap V &= \{x \in X \mid x \in U \wedge x \in V\} \\ &= \{x \in X \mid Ux = \top \wedge Vx = \top\} = \{x \in X \mid (Ux) \wedge (Vx) = \top\} = \{x \in X \mid (U \wedge V)x = \top\} \ , \end{aligned}$$

where $U \wedge V\colon X \to \Sigma$ is defined by $\lambda x\!:\!X \,.\, ((Ux) \wedge (Vx))$. ■

From now on we interchangeably write $U \cap V$ and $U \wedge V$ to denote the intersection of U and V. Define the relation $\leq$ on Σ by

$$x \leq y \longleftrightarrow (x = \top \longrightarrow y = \top) \ .$$

This is the *intrinsic order* on the dominance. According to this definition, we get

$$\bot \leq \top \ , \qquad\qquad \top \not\leq \bot \ .$$

The space Σ can be drawn as the poset

$$\begin{tikzcd} \top \arrow[d, dash] \\ \bot \end{tikzcd}$$

Proposition 5.3.7 *A dominance is partially ordered by $\leq$ and $\wedge$ is the greatest lower bound operation.*

Proof. It is obvious that $\leq$ is reflexive and transitive. To see that it is antisymmetric, just observe that $x \leq y$ and $y \leq x$ is equivalent to $x = \top \longleftrightarrow y = \top$, then apply Lemma 5.3.1.

It is obvious that $x \wedge y$ is below both x and y. Suppose $z \leq x$ and $z \leq y$. Then $z = \top$ implies $x = \top$ and $y = \top$, hence $z = \top \longrightarrow (x \wedge y = \top)$, and so $z \leq x \wedge y$. ∎

We can extend the partial order $\leq$ on Σ to arbitrary intrinsic topologies by defining, for $U, V \in \mathcal{O}(X)$,

$$U \leq V \longleftrightarrow \forall x \in X . (Ux \leq Vx) \ .$$

Clearly, $U \leq V$ if, and only if, $U \subseteq V$. Recall that $U \subseteq V$ means that the inclusion $U \hookrightarrow X$ factors through the inclusion $V \hookrightarrow X$. From now on we interchangeably write $U \leq V$ and $U \subseteq V$. The relation $\leq$ is called the *inclusion order* on the topology $\mathcal{O}(X)$.

As the third axiom of dominance, we postulate that there exists a map $\bigvee \colon \Sigma^{\mathbb{N}} \to \Sigma$ such that, for all $U \colon \mathbb{N} \to \Sigma$,

$$\bigvee U = \top \longleftrightarrow \exists\, n \in \mathbb{N} . (Un = \top) \ . \tag{$\Sigma 3$}$$

The immediate consequence of this is that a union of open sets is open.

Proposition 5.3.8 *The union of a family $U \colon \mathbb{N} \to \mathcal{O}(X)$ of open sets is open.*

Proof. The union of the family U is the least subspace that contains Un for every $n \in \mathbb{N}$, i.e., the subspace

$$\bigcup U = \left\{ x \in X \mid \exists\, n \in \mathbb{N} . (x \in Un) \right\} \ .$$

This is an open subspace of X because it equals $\left\{ x \in X \mid \bigvee(\lambda n : \mathbb{N} .\ (Unx)) = \top \right\}$. ∎

Proposition 5.3.9 *The union of two open sets is open.*

Proof. Suppose $U, V \in \mathcal{O}(X)$. Their union is the subspace

$$U \cup V = \left\{ x \in X \mid x \in U \vee x \in V \right\} \ .$$

Define a map $W \colon \mathbb{N} \to \mathcal{O}(X)$ by induction as follows:

$$W0 = U \ , \qquad\qquad W(n+1) = V \ .$$

For every $x \in X$, $x \in U \vee x \in V$ if, and only if, $x \in Wn$ for some $n \in \mathbb{N}$. Thus, the union $\bigcup W$, which is open by Proposition 5.3.8, equals $U \cup V$. ∎

In order to get a theory in which open and closed subsets are not interchangeable, we need to break the symmetry between $\perp$ and $\top$. We would like to make sure that there is no "twist" map $\perp \mapsto \top$, $\top \mapsto \perp$. This is the consequence of the fourth axiom, which is known as *Phoa's Principle*:

$$x \leq y \longleftrightarrow \exists\, f \in \Sigma^{\Sigma} . (f\perp = x \text{ and } f\top = y) \ . \tag{$\Sigma 4$}$$

The power of this axiom is evident from the following string of consequences.

Proposition 5.3.10 *There does not exist a map $t \colon \Sigma \to \Sigma$ such that $t\perp = \top$ and $t\top = \perp$.*

Proof. If this were such a map, then $\top \leq \bot$ would follow by (Σ4). But we already know that $\top \not\leq \bot$. ∎

Lemma 5.3.11 *Let $f, g \colon \Sigma \to \Sigma$. If $f\bot = g\bot$ and $f\top = g\top$, then $f = g$.*

Proof. Consider any $x \in \Sigma$. Suppose $fx \neq gx$. Then $x \neq \bot$ and $x \neq \top$, which is impossible. Therefore $\neg(fx \neq gx)$, and by the Axiom of Stability it follows that $fx = gx$. ∎

Proposition 5.3.12 *If $x \leq y$, where $x, y \in \Sigma$, then there exists exactly one $f \colon \Sigma \to \Sigma$ such that $f\bot = x$ and $f\top = y$.*

Proof. Assume $f\bot = x = g\bot$ and $f\top = y = g\top$, and apply Lemma 5.3.11. ∎

Proposition 5.3.13 *Every map $f \colon \Sigma \to \Sigma$ is monotone, i.e., $x \leq y \longrightarrow fx \leq fy$.*

Proof. Suppose $x \leq y$. By (Σ4), there exists $g \colon \Sigma \to \Sigma$ such that $g\bot = x$ and $g\top = y$. Therefore $fx = (f \circ g)\bot$ and $fy = (f \circ g)\top$, so by (Σ4) we get $fx \leq fy$. ∎

Another way to express Proposition 5.3.13 is

$$x \leq y \longleftrightarrow \forall\, f \in \Sigma^{\Sigma} \,.\, (fx \leq fy) \ .$$

For any space X we can define the *intrinsic preorder* on X by

$$x \leq y \ \longleftrightarrow \forall\, U \in \mathcal{O}(X) \,.\, (x \in U \longrightarrow y \in U) \ .$$

The relation $\leq$ is always reflexive and transitive, but it is not necessarily antisymmetric, so it may fail to be a partial order.

Definition 5.3.14 A space is an *intrinsic T_0-space* when its intrinsic preorder is a partial order.

Proposition 5.3.15 *Every map $f \colon X \to Y$ is monotone in the intrinsic preorder.*

Proof. Assume $x \leq y$. Suppose $fx \in U$ for some $U \in \mathcal{O}(Y)$. Then $x \in f^*U$ and $f^*U \in \mathcal{O}(X)$, therefore $y \in f^*U$ by assumption, and we conclude $fy \in U$. ∎

Note that the intrinsic order on $\mathcal{O}(X)$ and the inclusion order on $\mathcal{O}(X)$ have different definitions.

Proposition 5.3.16 *The intrinsic order and the inclusion order on $\mathcal{O}(X)$ coincide.*

Proof. We need to prove that for any $U, V \in \Sigma^X$,

$$(\forall\, x \in X \,.\, Ux \leq Vx) \longleftrightarrow \forall\, \phi \in \Sigma^{\Sigma^X} \,.\, (\phi U \leq \phi V) \ .$$

The implication from right to left follows when we take, for every $x \in X$, $\phi_x = \lambda U \colon \Sigma^X \,.\, Ux$. For the converse, suppose $Ux \leq Vx$ for all $x \in X$. By Proposition 5.3.12, there exists a unique map $[Ux, Vx] \colon \Sigma \to \Sigma$ such that $[Ux, Vx]\bot = Ux$ and $[Ux, Vx]\top = Vx$. By Unique Choice, there exists a unique map $F \colon X \to \Sigma^{\Sigma}$ such that $Fx = [Ux, Vx]$. Consider the map $F' \colon \Sigma \to \Sigma^X$, defined by $F'sx = Fxs$. It has the property that $F'\bot = U$ and $F'\top = V$. Now, for any $\phi \colon \Sigma^X \to \Sigma$, we have

$$\phi U = \phi(F'\bot) = (\phi \circ F')\bot \ , \qquad\qquad \phi V = \phi(F'\top) = (\phi \circ F')\top \ .$$

If we apply (Σ4) to the map $\phi \circ F'$, we obtain $\phi U \leq \phi V$. ∎

What are the maps $\Sigma \to \Sigma$? We can identify three maps: the constant $\bot$ map, the constant $\top$ map, and the identity. There are no others.

Proposition 5.3.17 *For all $f : \Sigma \to \Sigma$, it is* not *the case that $f \neq \lambda x : \Sigma \,.\, \bot$ and $f \neq \lambda x : \Sigma \,.\, \top$ and $f \neq 1_\Sigma$.*

Proof. Suppose $f : \Sigma \to \Sigma$ is a map such that $f \neq \lambda x : \Sigma \,.\, \bot$ and $f \neq \lambda x : \Sigma \,.\, \top$ and $f \neq 1_\Sigma$. Suppose $f\bot = \bot$. Then $f\top \neq \bot$, otherwise $f = 1_\Sigma$ by Lemma 5.3.11, but also $f\top \neq \top$, otherwise $f = \lambda x : \Sigma \,.\, \bot$ by the same lemma. We derived a contradiction from the assumption that $f\bot = \bot$, therefore $f\bot \neq \bot$. A similar argument shows that $f\bot \neq \top$. This is impossible. ∎

We emphasize that what we proved is *not* equivalent to the statement that every map $\Sigma \to \Sigma$ equals identity, constant $\bot$ map, or constant $\top$ map. In fact, we proved the double negation of that statement.

The last axiom relates the dominance to computability:

$$\bot \in \#\Sigma \,, \qquad \bot \in \#\Sigma \,, \qquad \bigvee \in \#\left(\Sigma^{\mathbb{N}} \to \Sigma\right) . \tag{$\Sigma 5$}$$

The axiom reflects the intuition that $\top$ corresponds to a terminating *computation*, and $\bot$ corresponds to a non-terminating one.

Recall that if $U \in \mathcal{O}(A)$ and $V \in \mathcal{O}(U)$, then there exists a unique $W \in \mathcal{O}(A)$ such that $x \in W \longleftrightarrow (x \in U \land x \in V)$. By Unique Choice, we obtain a map

$$E : \textstyle\sum_{U \in \mathcal{O}(A)} \mathcal{O}(U) \to \mathcal{O}(A)$$

with the property that $E\langle U, V\rangle = W$, where U, V and W are as above. By Theorem 2.3.1, E is computable, and since the meet operation $\land$ is defined in terms of E, we conclude that

$$\land \in \#(\Sigma \times \Sigma \to \Sigma) .$$

An obvious question is whether there are any dominances. We can use the idea from the beginning of the section, namely, that an infinite binary sequence $f : \mathbb{N} \to 2$ represents a terminating computation if it contains a 1, and represents a non-terminating computation if it does not. We can effectively detect that there is $n \in \mathbb{N}$ such that $fn = 1$, but cannot detect that there is not one. Thus, the constant sequence $o = \lambda n : \mathbb{N} \,.\, 0$ should represent $\bot$, and all other sequences should represent $\top$. This suggest that we define $\Sigma = 2^{\mathbb{N}}/{\sim}$, where $\sim$ is defined by, for all $f, g \in 2^{\mathbb{N}}$,

$$f \sim g \longleftrightarrow (f = o \longleftrightarrow g = o) .$$

As it turns out, in general Σ satisfies all axioms, except Phoa's Principle ($\Sigma 4$). The reason for this is that the underlying PCA of the category of modest sets might be powerful enough to decide Π^1_1 statements. In this case, the symmetry between $\bot$ and $\top$ is not broken, and that is why Phoa's principle fails.

Let $o = \lambda n : \mathbb{N} \,.\, 0$ be the constantly zero map. The idea that there is no way to distinguish o among all the maps in $2^{\mathbb{N}}$ is expressed formally by the statement

$$\forall\, H \in 2^{\mathbb{N}} \to 2 \,.\, \Big((\forall\, f \in 2^{\mathbb{N}} \,.\, (f \neq o \longrightarrow Hf = 1)) \longrightarrow Ho = 1 \Big) . \tag{WCP}$$

In words, if H is a decidable predicate on $2^{\mathbb{N}}$ that holds for all $f \neq o$, then it holds for o as well. We call this statement the "weak continuity principle" (WCP). In view of the retraction constructed in the proof of Theorem 5.2.2, WCP is equivalent to

$$\forall\, f \in \mathbb{N}^{+} \to 2\,.\,((\forall\, n \in \mathbb{N}\,.\, fn = 1) \longrightarrow f\,\infty = 1)\ . \qquad \text{(WCP)}$$

In words, if a sequence $f0$, $f1$, ..., is constantly equal to 1 then its limit $f\infty$ is equal to 1. From the topological point of view, WCP asserts that ∞ is not an isolated point of $\mathbb{N}^{+}$.

Proposition 5.3.18 WCP *holds if, and only if,* $2^{\mathbb{N}}$ *is* not *decidable.*

Proof. Assume WCP holds. If $2^{\mathbb{N}}$ were decidable then the characteristic map of comparison with o would violate WCP, therefore $2^{\mathbb{N}}$ is not decidable. Conversely, assume $2^{\mathbb{N}}$ is not decidable and let $H\colon 2^{\mathbb{N}} \to 2$ be a map such that, for all $f \neq o$, $Hf = 1$. If $Ho = 0$, then we would have

$$\forall\, f \in 2^{\mathbb{N}}\,.\,(f = o \vee f \neq o)\ ,$$

which would imply that $2^{\mathbb{N}}$ is decidable. Therefore $Ho = 1$, which proves WCP. ∎

Example 5.3.19 All categories of modest sets considered so far, Equ, $\mathsf{Equ}_{\mathsf{eff}}$, 0Equ, $\mathsf{Mod}(\mathbb{B}, \mathbb{B}_{\sharp})$, $\mathsf{Mod}(\mathbb{V})$, Mod(RE), and $\mathsf{Mod}(\mathbb{N})$, satisfy WCP. This is most easily proved by looking at what it means for $2^{\mathbb{N}}$ to be a decidable space. In the models built on topological PCAs $\mathbb{P}$, $\mathbb{B}$, and $\mathbb{U}$, it means that $2^{\mathbb{N}}$ is a discrete space. This is not the case because $2^{\mathbb{N}}$ is interpreted as the Cantor space. In Mod(RE) and $\mathsf{Mod}(\mathbb{N})$ the decidability of $2^{\mathbb{N}}$ is equivalent to the decidability of the Halting Problem. But the Halting Problem is not decidable, so neither is equality on $2^{\mathbb{N}}$.

There is a PCA $\mathbb{J}$ such that WCP is not valid in $\mathsf{Mod}(\mathbb{J})$. The idea is to use infinite time Turing machines by Hamkins and Lewis [HL00]. An infinite Turing machine is like an ordinary Turing machine whose running time is allowed to be any ordinal number. Infinite time Turing machines have their Gödel codes, just like the ordinary ones, and form a PCA in the same way that the ordinary Turing machines do. It is not hard to show that an infinite time Turing machine can decide any Π^1_1 statement [HL00, Corollary 2.3], therefore it can decide the statement $\forall\, n \in \mathbb{N}\,.\, fn = 1$, given a realizer $a \in \mathbb{J}$ for f.

Theorem 5.3.20 (The Standard Dominance) *Let* $o = \lambda n{:}\mathbb{N}\,.\,0$ *and let* $\sim$ *be an equivalence relation on* $2^{\mathbb{N}}$ *defined by*

$$f \sim g \longleftrightarrow (f = o \longleftrightarrow g = o)\ ,$$

The space $\Sigma = 2^{\mathbb{N}}/{\sim}$ *is called the* standard dominance. *It satisfies axioms* ($\Sigma0$), ($\Sigma1$), ($\Sigma2$), ($\Sigma3$), *and* ($\Sigma5$). *Furthermore,* Σ *satisfies Phoa's Principle* ($\Sigma4$) *if, and only if,* WCP *holds.*

Proof. It is obvious that $\sim$ is an equivalence relation. Observe that by Markov's Principle $f \neq o$ is equivalent to $\exists\, n \in \mathbb{N}\,.\,(fn = 1)$. The bottom element is $\bot = [o]$, and the top element is $\top = [\lambda n{:}\mathbb{N}\,.\,1]$. They are both computable because the maps o and $\lambda n{:}\mathbb{N}\,.\,1$ are computable, and so is the quotient map $[\square]_{\sim}$.

To prove ($\Sigma1$), suppose $[f] \neq \bot$. Then $\neg(f \sim o)$, which simplifies to $\neg\neg\exists\, n \in \mathbb{N}\,.\, fn = 1$. By Markov's Principle, $\exists\, n \in \mathbb{N}\,.\, fn = 1$. Therefore $f \sim \lambda n{:}\mathbb{N}\,.\,1$, and $[f] = \top$, as required.

Next, we validate ($\Sigma2$). Suppose $u\colon A \to \Sigma$ and $v\colon u^{*}\top \to \Sigma$. We show that for every $x \in A$ there is a unique $t \in \Sigma$ such that $t = \top$ if, and only if, $ux = \top$ and $vx = \top$. Note that vx is defined only when $ux = \top$. Informally, the proof goes as follows. Let $[f] = ux$. We construct a map $h \in 2^{\mathbb{N}}$ such that h outputs zeroes for as long as f does. If and when $fn = 1$, we know that $x \in u^*x$. At that point, let $[g] = vx$. Now we let h output the values of g, i.e., $hn = g0$, $h(n+1)g = g1$, and so on. If f never attains the value 1, then h does not either. If f attains value 1, then h attains 1 if, and only if, g does. It is clear that $[h]$ does not depend on the choice of representatives f and g. We must be careful, though, to choose the representative g only once in the entire construction of h. Now we proceed with a rigorous proof.

Let $s\colon 2^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ be the map

$$\begin{aligned}(s\,f)0 &= (\texttt{if } f0 = 0 \texttt{ then } 0 \texttt{ else } 1)\ ,\\ (s\,f)(n+1) &= (\texttt{if } (s\,f\,n \neq 0) \texttt{ then } s\,f\,n \texttt{ else } (\texttt{if } fn = 1 \texttt{ then } n+1 \texttt{ else } 0))\ .\end{aligned}$$

It has the following property: if there is $n \in \mathbb{N}$ such that $fn = 1$ then $(s\,f)m = 0$ for all $m < \boldsymbol{\mu} f$, and $(s\,f)m = 1 + \boldsymbol{\mu} f$ for all $m \geq \boldsymbol{\mu} f$. Here $\boldsymbol{\mu}$ is the minimization operator, described in Theorem 5.1.3. If $fn = 0$ for all $n \in \mathbb{N}$, then $(s\,f)n = 0$ for all $n \in \mathbb{N}$.

Consider any $x \in A$. There exists $f\colon \mathbb{N} \to 2$ such that $[f] = ux$. Define a map $g_2\colon \mathbb{N} \to 1 + 2^{\mathbb{N}}$ recursively as follows. Let $g_2 0 = \star$. To define $g_2(n+1)$, consider the value of $g_2 n$. If $g_2 n \neq \star$, then let $g_2(n+1) = g_2 n$. Otherwise, consider fn. If $fn = 0$, let $g_2(n+1) = \star$. If $fn = 1$, then $x \in u^*\top$, therefore there exists $g\colon \mathbb{N} \to 2$ such that $[g] = vx$. Let $g_2(n+1) = \mathsf{inr}\, g$. The map g_2 is well defined by Number Choice. It has the following property: if there is $n \in \mathbb{N}$ such that $fn = 1$, then $g_2 m = \star$ for all $m < 1 + \boldsymbol{\mu} f$, and $g_2 m = g$ for all $m \geq 1 + \boldsymbol{\mu} f$, where $g\colon \mathbb{N} \to 2$ is such that $[g] = vx$. If $fn = 0$ for all $n \in \mathbb{N}$ then $g_2 n = \star$ for all $n \in \mathbb{N}$.

We define a map $h\colon \mathbb{N} \to 2$ as follows. If $(s\,f)n = 0$, let $hn = 0$. If $(s\,f)n = k \neq 0$, then $f(k-1) = f(\boldsymbol{\mu} f) = 1$, and because $n \geq \boldsymbol{\mu} f$ we have $g_2(n+1) \neq \star$. Set $hn = (g_2(n+1))(n+1-k)$. This is well defined because $n \geq \boldsymbol{\mu} f$, hence $1 + n \geq 1 + \boldsymbol{\mu} f = k$. The map h has the following property:

$$[h] = \top \longleftrightarrow ux = \top \wedge vx = \top\ . \tag{5.4}$$

Indeed, suppose $[h] = \top$. There exists $n \in \mathbb{N}$ such that $hn = 1$. Then $\exists\, k \in \mathbb{N}\,.\,(fk = 1)$, hence $ux = [f] = \top$. Furthermore, $1 = hn = (g_2(n+1))(n+1-k)$, hence $g_2(n+1) \neq \star$, $[g_2(n+1)] = \top$, and so $vx = [g_2(n+1)] = \top$. The converse is equally easy.

By Lemma 5.3.1, there is at most one $[h] \in \Sigma$ that satisfies (5.4). Thus, we have proved that for every $x \in A$ there exists exactly one $[h] \in \Sigma$ such that (5.4) holds. By Unique Choice, there exists a map $w\colon A \to \Sigma$ such that, for all $x \in A$,

$$wx = \top \longleftrightarrow ux = \top \wedge vx = \top\ .$$

This is what (Σ2) states.

Let us proceed to (Σ3). Let $p\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be an isomorphism, say, $p\,\langle n, m\rangle = 2^n(2m+1) - 1$. Define a map $I\colon (2^{\mathbb{N}})^{\mathbb{N}} \to 2^{\mathbb{N}}$ by

$$(I\,f)n = \widetilde{f}(p^{-1}n)\ .$$

The map I interleaves a sequence of sequences $f0, f1, \ldots$ into a single binary sequence in such a way that every value $f\,i\,j$ appears exactly once in the interleaved sequences. Therefore,

$$(\exists\, i, j \in \mathbb{N}\,.\, f\,i\,j = 1) \longleftrightarrow \exists\, n \in \mathbb{N}\,.\,(I\,f)n = 1\ .$$

Suppose $u\colon \mathbb{N} \to \Sigma$. Because $\mathbb{N}$ is projective, there exists a map $u_0\colon \mathbb{N} \to 2^{\mathbb{N}}$ such that $un = [u_0 n]$ for all $n \in \mathbb{N}$. Let $x = [I u_0]$. We claim that

$$x = \top \longleftrightarrow (\exists\, n \in \mathbb{N} .\, un = \top) \;. \tag{5.5}$$

Suppose $un = \top$. Then $[u_0 n] = \top$, therefore $u_0 n k = 1$ for some $k \in \mathbb{N}$. But then $(Iu_0)(p\langle n, k\rangle) = 1$, therefore $x = \top$. The converse is proved similarly. By Lemma 5.3.1, there is at most one $x \in \Sigma$ that satisfies (5.5). By Unique Choice, there exists a unique map $\bigvee\colon \Sigma^{\mathbb{N}} \to \Sigma$ such that

$$\bigvee u = \top \longleftrightarrow \exists\, n \in \mathbb{N} .\, un = \top \;,$$

as is easily verified. The map $\bigvee$ is computable by Theorem 2.3.1.

Next, we prove that WCP implies Phoa's Principle for Σ. Suppose $x, y \in \Sigma$ and $x \leq y$. There exist $f, g \in \mathbb{N} \to 2$ such that $x = [f]$ and $y = [g]$. Let $H\colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ be defined by

$$(H\,k)n = (\texttt{if}\ (s\,k)n = 0\ \texttt{then}\ fn\ \texttt{else}\ g(n + 1 - ((s\,k)n))) \;,$$

where s has been defined previously in the proof. The sequence Hk is equal to f as long as the value of k is 0. Once k attains the value 1, Hk starts enumerating the sequence g. Therefore, if $k \sim o$ then $Hk \sim f$, and if $k \sim \lambda n \colon \mathbb{N} .\, 1$ then $Hk \sim g$. It is not hard to see that $k \sim k'$ implies $Hk \sim Hk'$. We obtain a map $h\colon \Sigma \to \Sigma$ that satisfies $h[k] = [Hk]$, $h\bot = [f] = x$ and $h\top = [g] = y$.

Conversely, suppose $h\colon \Sigma \to \Sigma$ is a map such that $h\bot = \top$. It suffices to show that $h\top = \bot$ entails a contradiction. So assume $h\top = \bot$. Let $H\colon 2^{\mathbb{N}} \to \Sigma$ be the map defined by $Hf = h[f]$. It has the property that $Hf = \top$ if, and only if, $f \sim o$. We claim that, for all $f \in 2^{\mathbb{N}}$, $Hf = \top$ or $Hf = \bot$. Consider an arbitrary $f \in 2^{\mathbb{N}}$. There exists $g \in 2^{\mathbb{N}}$ such that $Hf = [g]$. Define $k\colon \mathbb{N} \to \mathbb{N}$ by

$$kn = \begin{cases} 0 & \text{if } fn = 0 \text{ and } gn = 0 \;, \\ 1 & \text{if } fn = 1 \text{ and } gn = 0 \;, \\ 2 & \text{if } fn = 0 \text{ and } gn = 1 \;, \\ 3 & \text{if } fn = 1 \text{ and } gn = 1 \;. \end{cases}$$

It is not the case that $kn = 0$ for all $n \in \mathbb{N}$, because that would imply $h\bot = h[f] = Hf = [g] = \bot$. By Markov's principle, there exists $n \in \mathbb{N}$ such that $kn \neq 0$. We consider three cases: (a) if $kn = 1$ then $[f] = \top$, hence $Hf = [g] \neq \top$, and so $Hf = \bot$; (b) if $kn = 2$ then $Hf = [g] = \top$; (c) if $kn = 3$ then $[f] = [g] = \top$, hence $h\top = Hf = [g] = \top$, which is impossible, hence this case never happens. This proves the claim. Now we can define a map $G\colon 2^{\mathbb{N}} \to 2$ by cases:

$$Gf = \begin{cases} 0 & \text{if } Hf = \top \;, \\ 1 & \text{if } Hf = \bot \;. \end{cases}$$

This is well defined because we just proved that $\forall\, f \in 2^{\mathbb{N}} .\, (Hf = \top \lor Hf = \bot)$, and clearly it cannot happen that $Hf = \top$ and $Hf = \bot$ at the same time. The map G contradicts WCP because it has the property that $Gf = 0$ if, and only if, $f \neq o$. This concludes the proof that Σ satisfies Phoa's Principle if WCP is valid.

Lastly, let us derive WCP from Phoa's Principle. For convenience we switch the roles of 0 and 1 in the statement of WCP. Suppose $H\colon 2^{\mathbb{N}} \to 2$ is a map such $f \neq o$ implies $Hf = 0$. Define a map $h\colon \Sigma \to \Sigma$ by

$$h[f] = [\lambda n \colon \mathbb{N} .\, (Hf)] \;.$$

This is well defined because $f \sim f'$ implies $Hf = Hf'$. Indeed, suppose $f \sim f'$. If $Hf = 1$ then $f = o$ by assumption on H, and since $f' \sim f$ we get $f' = o$, hence $Hf' = 1 = Hf$. Similarly, if $Hf' = 1$ then $Hf = 1$. This leaves us with the case $Hf = Hf' = 0$, but here nothing needs to be proved.

Observe that $h\top = h[\lambda n : \mathbb{N} . 1] = [o] = \bot$. By Phoa's principle, $h\bot = \top \longrightarrow h\top = \top$, so we get $h\bot = \top \longrightarrow \mathsf{false}$, from which we conclude $h\bot = \bot$. Now this implies that $Ho = 0$, which confirms WCP. ∎

Example 5.3.21 In $\mathsf{Equ}_{\mathsf{eff}}$ the standard dominance is interpreted as the equilogical space $(2^{\mathbb{N}}, \sim)$ where $2^{\mathbb{N}}$ is the Cantor space and $\sim$ is defined by

$$f \sim g \iff (f = o \iff g = o) \; .$$

Because $2^{\mathbb{N}}$ is a 0-dimensional space $(2^{\mathbb{N}}, \sim)$ is a 0-equilogical space. In fact, $(2^{\mathbb{N}}, \sim)$ is an admissible representation of the Sierpinski space (the two-point lattice with the Scott topology). This means that the domain-theoretic interpretation of the standard dominance is the Sierpinski space, cf. Subsection 4.2.4.

Example 5.3.22 In $\mathsf{Mod}(\mathbb{N})$ the standard dominance is interpreted, up to isomorphism, as the modest set whose underlying set is $\{\bot, \top\}$, and the existence predicate is

$$\mathsf{E}_{\Sigma}\top = H \; , \qquad\qquad \mathsf{E}_{\Sigma}\bot = \mathbb{N} \setminus H \; ,$$

where H is the halting set,

$$H = \{ n \in \mathbb{N} \mid \text{the } n\text{-th Turing machine halts} \} \; .$$

This is the familiar dominance from the effective topos. Let us also compute the interpretation of $\Sigma^{\mathbb{N}}$ in $\mathsf{Mod}(\mathbb{N})$. By Corollary 5.3.32, $\Sigma^{\mathbb{N}}$ is a quotient of $\mathbb{N}^{\mathbb{N}}$ by an equivalence relation $\sim$ defined by

$$f \sim g \longleftrightarrow \forall n \in \mathbb{N} . \, ((\exists\, m \in \mathbb{N} . \, fm = n + 1) \longleftrightarrow (\exists\, m \in \mathbb{N} . \, gm = n + 1)) \; . \qquad (f, g : \mathbb{N}^{\mathbb{N}})$$

The interpretation of $\mathbb{N}^{\mathbb{N}}$ is the modest set of total recursive functions. By Markov's principle, the statement $f \sim g$ is equivalent to a negative formula, therefore we can interpret it set-theoretically. It says that f and g enumerate the same set. We now see that the interpretation of $\Sigma^{\mathbb{N}}$ in $\mathsf{Mod}(\mathbb{N})$ is the modest set of r.e. sets $(\mathsf{RE}, \Vdash_{\mathsf{RE}})$, where $n \Vdash_{\mathsf{RE}} U$ if, and only if, $\mathsf{W}_n = U$. In words, the realizers of an r.e. set U are Gödel codes of those Turing machines that enumerate U.

Proposition 5.3.23 *For any map $f \colon \mathbb{N} \to \Sigma$,*

$$(\neg\neg\exists\, k \in \mathbb{N} . \, fk = \top) \longrightarrow \exists\, k \in \mathbb{N} . \, fk = \top \; .$$

Proof. Because $\mathbb{N}$ is projective there exists a map $g \colon \mathbb{N} \to 2^{\mathbb{N}}$ such that $fn = [gn]_{\Sigma}$ for all $n \in \mathbb{N}$. Let $p \colon \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ be a computable isomorphism, say $p(2^i(2j + 1) - 1) = \langle i, j \rangle$. Define a map $h \colon \mathbb{N} \to 2$ by

$$hn = \widetilde{g}(pn) \; .$$

The map h has the property that $h(p^{-1}\langle i, j \rangle) = gij$. By assumption, there $\neg\neg$-exists $i \in \mathbb{N}$ such that there exists $j \in \mathbb{N}$ such that $gij = 1$. Therefore, there $\neg\neg$-exists $n \in \mathbb{N}$ such that $hn = 1$. By Markov's principle, there exists $n \in \mathbb{N}$ such that $hn = 1$. Let $\langle k, j \rangle = pn$. Then $hn = gkj = 1$, hence $fk = [gk] = \top$. ∎

Proposition 5.3.24 *The standard dominance* Σ *is isomorphic to* $\mathbb{N}^+/\!\sim$ *where* $\sim$ *is defined by* $x \sim y \longleftrightarrow (x = \infty \longleftrightarrow y = \infty)$, *for all* $x, y \in \mathbb{N}^+$.

Proof. By Theorem 5.2.2 there is a section $s\colon \Sigma \to 2^{\mathbb{N}}$ and a retraction $r\colon 2^{\mathbb{N}} \to \Sigma$ such that $rf = \infty$ if, and only if, $f = (\lambda n{:}\mathbb{N}.\,0)$. The maps s and r induce isomorphisms between Σ and $\mathbb{N}^+/\!\sim$. ∎

Corollary 5.3.25 *If there exists a map* $f\colon \mathbb{N}^+ \to \Sigma$ *such that, for all* $x \in \mathbb{N}^+$, $fx = \top \longleftrightarrow x = \infty$, *then* $\neg\mathsf{WCP}$.

Proof. If there were such a map $f\colon \mathbb{N}^+ \to \Sigma$, the induced map $\Sigma \to \Sigma$ would violate Phoa's Principle, hence WCP would entail a contradiction. ∎

Proposition 5.3.26 *Recall that the order relation* $\le$ *on* $\mathbb{N} \times \mathbb{N}$ *is decidable. There is a decidable extension* $\le_{\mathbb{N}^+}$ *on* $\mathbb{N} \times \mathbb{N}^+$ *such that, for all* $n, m \in \mathbb{N}$,

$$n \le_{\mathbb{N}^+} m \longleftrightarrow n \le m \;,$$

and $n \le \infty$ *for all* $n \in \mathbb{N}$.

Proof. The characteristic map $c\colon \mathbb{N} \times \mathbb{N}^+ \to 2$ of $\le_{\mathbb{N}^+}$ is defined by

$$c\,\langle 0, y\rangle = 1 \;, \qquad\qquad c\,\langle n+1, y\rangle = (\texttt{if}\ y = 0\ \texttt{then}\ 0\ \texttt{else}\ c\,\langle n, \mathsf{p}\,y\rangle) \;.$$

∎

We write $\le$ instead of $\le_{\mathbb{N}^+}$.

Proposition 5.3.27 *Recall that the strict order relation* $<$ *on* $\mathbb{N} \times \mathbb{N}$ *is decidable, hence semidecidable. There is a semidecidable extension* $<_{\mathbb{N}^+}$ *on* $\mathbb{N}^+ \times \mathbb{N}^+$ *such that, for all* $n, m \in \mathbb{N}$,

$$n <_{\mathbb{N}^+} m \longleftrightarrow n < m \;,$$

and, for all $x \in \mathbb{N}^+$,

$$\infty \not< x \;, \qquad\qquad x < \infty \longleftrightarrow x \ne \infty \;.$$

Proof. Define a map $f\colon \mathbb{N}^+ \times \mathbb{N}^+ \to \mathbb{N}^+$ by corecursion:

$$f\,\langle x, y\rangle = (\texttt{if}\ y = 0\ \texttt{then}\ \infty\ \texttt{else}\ (\texttt{if}\ x = 0\ \texttt{then}\ 0\ \texttt{else}\ \mathsf{s}(f\langle \mathsf{p}\,x, \mathsf{p}\,y\rangle))) \;.$$

The characteristic map of $<_{\mathbb{N}^+}$ is the composition $[\Box]_\Sigma \circ f\colon \mathbb{N}^+ \times \mathbb{N}^+ \to \mathbb{N}^+ \to \Sigma$. ∎

We write $<$ instead of $<_{\mathbb{N}^+}$.

Theorem 5.3.28 *The principle* WCP *is equivalent to*

$$\forall\, f \in \Sigma^{\mathbb{N}^+} .\, \big((\forall\, x \in \mathbb{N}^+ .\, fx \le f(\mathsf{s}x)) \wedge f\infty = \top \longrightarrow \exists\, n \in \mathbb{N}.\, fn = \top\big) \;. \tag{5.6}$$

Proof. In words, (5.6) states that every monotonic $f\colon \mathbb{N}^+ \to \Sigma$ that attains $\top$ at infinity, attains it at some finite argument already. First we show that WCP implies (5.6). Suppose $f\colon \mathbb{N}^+ \to \Sigma$ is monotonic and $f\infty = \top$. If $\neg\exists\, n \in \mathbb{N}\,.\, fn = \top$ were the case, then we would get $\forall n \in \mathbb{N}\,.\, fn = \bot$ and $f\infty = \top$, which contradicts WCP by Corollary 5.3.25. Therefore, $\neg\neg\exists\, n \in \mathbb{N}\,.\, fn = \top$. By Proposition 5.3.23, there exists $n \in \mathbb{N}$ such that $fn = \top$. This proves (5.6).

Conversely, suppose (5.6) holds and let $f\colon \mathbb{N}^+ \to 2$ be a map such that $\forall n \in \mathbb{N}\,.\, fn = 1$. Let $l\colon \mathbb{N}^+ \times \mathbb{N}^+ \to \Sigma$ be the characteristic map of $<$, as in Proposition 5.3.27. Define $g\colon \mathbb{N}^+ \to \Sigma$ by

$$gx = (\texttt{if}\ fx = 1\ \texttt{then}\ \bot\ \texttt{else}\ \top)\ .$$

Define $h\colon \mathbb{N}^+ \to \Sigma$ by

$$hx = gx \vee \bigvee \lambda y : \mathbb{N}^+\, .\ (l\langle y, x\rangle \wedge gy)\ .$$

For all $x \in \mathbb{N}^+$, if $fx = 0$ then $hx = \top$. Also, if $hx = \top$ then $h(\mathsf{s}\,x) = \top$, hence by (5.6),

$$h\infty = \top \longrightarrow \exists\, n \in \mathbb{N}\,.\, hn = \top\ ,$$

from which it follows that

$$\forall n \in \mathbb{N}\,.\, hn = \bot \longrightarrow h\infty = \bot\ .$$

Since $fn = 0$ for all $n \in \mathbb{N}$, $hn = \bot$ for all $n \in \mathbb{N}$, therefore $h\infty = \bot$. But this means that $g\infty = \bot$, therefore $f\infty = 1$. ∎

5.3.1 Σ-partial Maps and Lifting

A *partial map* $f\colon A \rightharpoonup B$ is a map $f\colon U \to B$ where $U \subseteq A$. We say that U is the *support* of f, and denote it by $\|f\|$. An important special case is a Σ*-partial map*, which is a partial map whose support is an open subspace. We show that there is an operation, called *lifting*, which assigns to a space B a space $B_\bot$, called "B bottom", such that $A \to B_\bot$ corresponds to the space of Σ-partial maps.

Consider the polynomial functor $P_\top$ for the constant map $\top\colon 1 \to \Sigma$. For a space A, define

$$A_\bot = P_\top A = \textstyle\sum_{s \in \Sigma} A^{\top^* s}\ .$$

To see what the space $A_\bot$ is like, let us compute the inverse images $\mathsf{fst}^*\bot$ and $\mathsf{fst}^*\top$ for the canonical projection $\mathsf{fst}\colon A_\bot \to \Sigma$. If for a point $\langle s, f\rangle \in A_\bot$ it is the case that $s = \bot$, then $f \in A^0$ because $\top^*\bot = 0$. Thus, there is exactly one point $\bot_A \in A_\bot$, called the "bottom of $A_\bot$", such that $\mathsf{fst}\,\bot_A = \bot$. On the other hand, the preimage $\mathsf{fst}^*\top$ is isomorphic to A, since $\top^*\top = 1$, and so

$$\langle s, x\rangle \in \mathsf{fst}^*\top \longleftrightarrow s = \top \wedge x \in A^{\top^*\top} \longleftrightarrow s = \top \wedge x \in A\ .$$

Thus, we can think of A as an open subspace of $A_\bot$, where the inclusion $\eta_A\colon A \to A_\bot$ maps $x \in A$ to $\eta_A x = \langle \top, x\rangle$. We usually omit the embedding η_A and write x instead of $\eta_A x$. Thus, if $x \in A_\bot$, the meaning of $x \in A$ is $\exists!\, x' \in A\,.\, \eta_A x' = x$.

In a the category $\mathsf{Mod}(\mathbb{A}, \mathbb{A}_\sharp)$, the lifting $A_\perp$ can be described alternatively as the dependent product of $!_A \colon A \to 1$ along the constant map $\top \colon 1 \to \Sigma$:

$$\begin{tikzcd} A \arrow[d, "!_A"'] & \prod_\top !_A = A_\perp \arrow[d, "\mathsf{fst}"] \\ 1 \arrow[r, "\top"'] & \Sigma \end{tikzcd}$$

Since the dependent product is the right adjoint to the pullback functor, it follows that there is a bijective correspondence

$$\frac{f \colon A \to B_\perp}{g \colon U \to B} \qquad U \in \mathcal{O}(A)$$

More precisely, for every $f \colon A \to B_\perp$ there exists a unique map $f' \colon \|f\| \to B$, where $\|f\| = f^*B = (f \circ \mathsf{fst})^*\top$, such that $f'x = fx$ for all $x \in \|f\|$. Conversely, for every Σ-partial map $g \colon U \to B$, where $U \in \mathcal{O}(A)$, there is a unique $g' \colon A \to B_\perp$ such that $g'x = gx$ for all $x \in U$, and $g'x = \perp_B$ for all $x \notin U$. Therefore, $A \to B_\perp$ really does correspond to the space of Σ-partial maps.

Proposition 5.3.29 *(a) The bottom element $\perp_A$ is the least element of $A_\perp$ in the intrinsic order. (b) For all $x, y \in A$, $x \leq_A y$ if, and only if, $x \leq_{A_\perp} y$.*

Proof. (a) Because $\{\top\} \times A_\perp \subseteq \Sigma \times A_\perp$ is an open subspace of $\Sigma \times A_\perp$, there exists a unique map $f \colon \Sigma \times A_\perp \to A_\perp$ such that $f\langle\top, x\rangle = x$ for all $x \in A_\perp$, and $\langle\perp, x\rangle = \perp_A$ for all $x \in A_\perp$. Now suppose $U \in \mathcal{O}(A_\perp)$ and $\perp_A \in U$. Because $\perp \leq \top$ in Σ, we get for every $x \in A_\perp$,

$$\langle\perp, x\rangle \leq \langle\top, x\rangle \ ,$$

therefore by monotonicity of $U \circ f$,

$$U\perp_A = U(f\langle\perp, x\rangle) \leq U(f\langle\top, x\rangle) = Ux \ .$$

Thus, if $\perp_A \in U$ then $x \in U$.

(b) Let $x, y \in A$ and suppose $x \leq_A y$. If $U \in \mathcal{O}(A_\perp)$ and $x \in U$, then $x \in U \cap A$, and $y \in U \cap A$, hence $y \in U$, which implies $x \leq_{A_\perp} y$. Conversely, suppose $x \leq_{A_\perp} y$, $V \in \mathcal{O}(A)$ and $x \in V$. There exists a unique $V' \in \mathcal{O}(A)$ such that $V' = A \cap V$. Because $x \in V$, we get $x \in V'$, hence $y \in V'$, and so $y \in V$, which implies $x \leq_A y$. ■

The polynomial functor $P_\top$ transforms a map $f \colon A \to B$ to a map $f_\perp \colon A_\perp \to B_\perp$. The map $f_\perp$ is characterized by identities

$$f_\perp(\eta_A x) = \eta_B(fx) \ , \qquad\qquad f_\perp(\perp_A) = \perp_B \ .$$

We next prove a theorem which corresponds to the recursion-theoretic theorem that a non-empty set is r.e. if, and only if, it is the range of a total recursive function.

Theorem 5.3.30 *A subspace $S \subseteq \mathbb{N}$ is open if, and only if, there exists a map $f \colon \mathbb{N} \to \mathbb{N}$ such that*

$$S = \left\{n \in \mathbb{N} \mid \exists\, m \in \mathbb{N} .\, fm = n + 1\right\} \ .$$

We say that f lists *S.*

Proof. The trick with adding 1 to n ensures that the theorem holds for the empty subspace. Let $f\colon \mathbb{N} \to \mathbb{N}$ be a map. Then

$$\{n \in \mathbb{N} \mid \exists\, m \in \mathbb{N}.\, fm = n+1\} = \bigcup_{m \in \mathbb{N}} \{n \in \mathbb{N} \mid fm = n+1\} \;,$$

which is a countable union of open subspaces, therefore it is open. Conversely, suppose $S\colon \mathbb{N} \to \Sigma$. Because $\mathbb{N}$ is projective there exists $g\colon \mathbb{N} \to 2^{\mathbb{N}}$ such that $Sn = [gn]_\Sigma$ for all $n \in \mathbb{N}$. Let $\langle \Box, \Box \rangle\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a computable isomorphism, for example $\langle m, n\rangle = 2^m(2n+1) - 1$. Define the map $f\colon \mathbb{N} \to \mathbb{N}$ by

$$f\langle m, n\rangle = (\texttt{if}\ gmn = 1\ \texttt{then}\ m\ \texttt{else}\ 0) \;.$$

If $f\langle m, n\rangle \neq 0$ then $gmn = 1$, therefore $Sm = [gm]_\Sigma = \top$. If $Sm = \top$ then there exists $n \in \mathbb{N}$ such that $gmn = 1$, therefore $f\langle m, n\rangle = 1$. ∎

Corollary 5.3.31 *Every inhabited open subspace of* $\mathbb{N}$ *is countable.*

Proof. Immediate. ∎

Corollary 5.3.32 $\Sigma^{\mathbb{N}}$ *is a quotient of* $\mathbb{N}^{\mathbb{N}}$.

Proof. Define a map $q\colon \mathbb{N}^{\mathbb{N}} \to \Sigma^{\mathbb{N}}$ by

$$q\,f\,n = [\lambda m{:}\mathbb{N}.\ (\texttt{if}\ fm = n+1\ \texttt{then}\ 1\ \texttt{else}\ 0)]_\Sigma \;.$$

By Theorem 5.3.30, for every $S\colon \mathbb{N} \to \Sigma$ there exists $f \in \mathbb{N}^{\mathbb{N}}$ such that $qf = S$. Therefore q is a quotient map. ∎

Definition 5.3.33 We say that a space A *has a bottom* when in the intrinsic preorder on A there exists a smallest point, called *a bottom* of A.

When A is an intrinsically T_0-space the bottom, if it exists, is unique.

Proposition 5.3.34 *Suppose the embedding* $\eta_A\colon A \to A_\bot$ *has a left inverse* $\mu\colon A_\bot \to A$, *i.e.,* $\mu \circ \eta_A = 1_A$. *Then* $\mu\bot_A$ *is a bottom of* A.

Proof. For any $x \in A$, $\bot_A \leq \eta_A x$, therefore $\mu\bot_A \leq \mu(\eta_A x) = x$ and so $\mu\bot_A$ is indeed the bottom of A. ∎

A space of the form $A_\bot$ has a left inverse $\mu_A\colon (A_\bot)_\bot \to A_\bot$ to $\eta_{A_\bot}\colon A_\bot \to (A_\bot)_\bot$. Indeed, since $A \subseteq (A_\bot)_\bot$ is an open subspace, there exists a unique map $\mu_A\colon (A_\bot)_\bot \to A_\bot$ that corresponds to the map $1_A\colon A \to A$ viewed as a Σ-partial map $(A_\bot)_\bot \rightharpoonup A$. Then for every $x \in A$, $\mu_A(\eta_A x) = 1_A x = x$.

A space $A_\bot$ has a unique bottom, namely $\bot_A$. For suppose $\langle s, x\rangle \in A_\bot$ is a smallest element. Then $\langle s, x\rangle \leq \bot_A$ therefore $s = \mathsf{fst}\,\langle s, x\rangle \leq \mathsf{fst}\,\bot_A = \bot$, hence $s = \bot$ and $\langle s, x\rangle = \bot_A$.

If A and B have bottoms then so does $A \times B$, because the intrinsic preorder on $A \times B$ is coordinate-wise. Moreover, if $\mu\colon A_\bot \to A$ and $\nu\colon B_\bot \to B$ are left inverses to η_A and η_B, respectively, then a left inverse to $\eta_{A \times B}$ is the composition of maps

$$(A \times B)_\bot \xrightarrow{\langle \mathsf{fst}_\bot, \mathsf{snd}_\bot \rangle} A_\bot \times B_\bot \xrightarrow{\mu \times \nu} A \times B \;.$$

Partial Booleans

Let us show that the space of *partial Booleans* $2_\perp$ is isomorphic to the space

$$B = \{\langle x, y\rangle \in \Sigma \times \Sigma \mid x \wedge y = \perp\} \;.$$

First we prove the following proposition.

Proposition 5.3.35 *The space* $T = \{\langle x, y\rangle \in \Sigma \times \Sigma \mid x \neq y\}$ *is isomorphic to* 2.

Proof. In one direction the isomorphism is the map $i \colon 2 \to T$, defined by $i0 = \langle\perp, \top\rangle$, $i1 = \langle\top, \perp\rangle$. The inverse $j \colon T \to 2$ is defined as follows. Suppose $\langle x, y\rangle \in T$. There exist $f, g \in 2^{\mathbb{N}}$ such that $x = [f]_\Sigma$ and $y = [g]_\Sigma$. Define $h \colon \mathbb{N} \to 2$ by

$$h(2n) = fn \;, \qquad h(2n+1) = gn \;.$$

Because $x \neq y$, it is not the case that $f = g = o$. Therefore $h \neq o$ and so $n = \boldsymbol{\mu} h$ is well defined. Now set $j\langle x, y\rangle$ to the value 0 if n is odd and 1 if n is even. It is not hard to check that $j \circ i = 1_2$ and $i \circ j = 1_T$. ∎

If we view the isomorphism $j \colon T \to 2$ from Proposition 5.3.35 as a Σ-partial map $j \colon B \rightharpoonup 2$, we obtain the corresponding total map $f \colon B \to 2_\perp$. We construct the inverse $g \colon 2_\perp \to B$ as the composition

$$2_\perp \xrightarrow{g'_\perp} B_\perp \xrightarrow{\mu} B$$

where $g' \colon 2 \to B$ is the map defined by $g'0 = \langle\perp, \top\rangle$ and $g'1 = \langle\top, \perp\rangle$, and μ is the left inverse to the inclusion $\eta_B \colon B \to B_\perp$, as in Proposition 5.3.34. The map $\mu \colon B_\perp \to B$ is the restriction of $\mu \colon (\Sigma \times \Sigma)_\perp \to \Sigma \times \Sigma$ to the subspace B. The maps f and g are inverses of each other. Indeed, since for all $x \in 2_\perp$ it is not the case that $x \neq 0$, $x \neq 1$, and $x \neq \perp_2$, it is sufficient to verify these three cases:

$$f(g0) = f\,\langle\perp, \top\rangle = 0 \;, \qquad f(g1) = f\,\langle\top, \perp\rangle = 1 \;, \qquad f(g\perp_2) = f\,\langle\perp, \perp\rangle = \perp_2 \;.$$

The equality $g \circ f = 1_B$ is established analogously.

5.4 Countably Based Spaces

Various definitions of topological notions that are classically equivalent give inequivalent notions in the logic of modest sets, and in constructive logic in general. In this section we define two versions of countably based spaces—a pointwise one and a point-free one. We focus on countably based topological spaces, with the additional assumption that the element-hood relation is semidecidable.[4] It turns out that the pointwise topology is not as well behaved as the point-free version. Thus, we eventually abandon the pointwise version and study only the point-free one. This way we obtain a reasonably well-behaved theory of open and closed subspaces, and continuous maps.

[4] If we wanted to study general topology we would have to be able to speak of arbitrary families of subspaces, and that would require us to pass to the topos $\mathsf{RT}(\mathbb{A}, \mathbb{A}_\sharp)$. Instead, we focus on what can be done in the logic of modest sets.

Let X be a space. A *prebasis* on X is a countable space B with a semidecidable relation $\in_S \subseteq X \times B$. We can think of $\in_S$ as a map $\square \in_S \square \colon X \times B \to \Sigma$. For each $U \in B$ we define $|U| = \{x \in X \mid x \in_B U\}$. Since $\in_B$ is semidecidable, $|U|$ is open in the intrinsic topology $\mathcal{O}(X)$. Thus, every $U \in B$ defines a map $U \colon X \to \Sigma$, $U = \lambda x : X \,.\, (x \in U)$.

We define the inclusion relation $\subseteq_B$, for $U, V \in B$, by

$$U \subseteq_B V \longleftrightarrow \forall\, x \in X \,.\, (x \in_B U \longrightarrow x \in_B V) \ .$$

Because $\in_B$ is stable, $\subseteq_B$ is a stable relation. When it is also semidecidable, we say that B has *semidecidable inclusion.*

Definition 5.4.1 (Pointwise Topology) Let B be a prebasis on a space X. A subspace $S \subseteq X$ is *pointwise open* with respect to B when

$$\forall\, x \in S \,.\, \exists\, U \in B \,.\, (x \in U \wedge |U| \subseteq S) \ .$$

A prebasis B is a *pointwise basis* when

(1) X is pointwise open with respect to B.

(2) For every $U, V \in B$, $|U| \cap |V|$ is pointwise open with respect to B.

When B is a pointwise basis for X we say that (X, B) is *pointwise countably based.*

A map $f \colon X \to Y$ between pointwise countably based spaces (X, B) and (Y, C) is *pointwise continuous* when, for every $V \in C$, the inverse image $f^*|V| = \{x \in X \mid fx \in_C V\}$ is pointwise open with respect to B.

Definition 5.4.2 (Point-free Topology) Let B be a prebasis on a space X, and let $U \colon \mathbb{N} \to B$ be the enumeration of B. A subspace $S \subseteq X$ is *open* with respect to B when it is a countable union of elements of B, which means that there exists $c \colon \mathbb{N} \to \mathbb{N}$, called a *countable union map* for S, such that

$$\forall\, x \in X \,.\, \big(x \in S \longleftrightarrow \exists\, n \in \mathbb{N} \,.\, \big(cn \neq 0 \wedge x \in U_{(cn)-1}\big)\big) \ .$$

A prebasis B is a *basis* when

(1) X is open with respect to B.

(2) For all $V, W \in B$, $|V| \cap |W|$ is open with respect to B.

When B is a basis for X we say that (X, B) is *countably based.*

A map $f \colon X \to Y$ between countably based spaces (X, B) and (Y, C) is *continuous* when, for every $V \in C$, the inverse image $f^*|V| = \{x \in X \mid fx \in_C V\}$ is open with respect to B.

If B is a (pointwise) basis on X and $x \in X$, we denote by B_x the *basic neighborhood filter* of x, which is the space

$$B_x = \{U \in B \mid x \in U\} \ .$$

The only difference between pointwise and point-free topology is that the notion of an open subspace is defined differently. Let us compare these two definitions.

Proposition 5.4.3 *If a subspace $S \subseteq X$ is open with respect to B, then it is pointwise open with respect to B. Therefore, a countably based space is also pointwise countably based, and a continuous map is also pointwise continuous.*

Proof. Let $U\colon \mathbb{N} \to B$ be an enumeration of B. Suppose S is open with respect to B. Let $c\colon \mathbb{N} \to \mathbb{N}$ be a countable union map for S. If $x \in S$ then there exists $n \in \mathbb{N}$ such that $x \in U_{(cn)-1}$, and $|U_{(cn)-1}| \subseteq S$ holds because $y \in |U_{(cn)-1}|$ implies $y \in S$. ■

Proposition 5.4.4 *Let B be a prebasis on X and let $U\colon \mathbb{N} \to B$ be the enumeration of B. A subspace $S \subseteq X$ is open with respect to B if, and only if, there exists $C \in \mathcal{O}(\mathbb{N})$ such that, for all $x \in X$,*

$$x \in S \longleftrightarrow \exists\, n \in C\,.\, x \in U_n\;.$$

The space C is called a countable union predicate *for S.*

Proof. Suppose $S \subseteq X$ is open with respect to B, and let $c\colon \mathbb{N} \to \mathbb{N}$ be a countable union map for S. Let $C \in \mathcal{O}(\mathbb{N})$ be the subspace listed by c, as in Theorem 5.3.30:

$$C = \left\{ n \in \mathbb{N} \;\middle|\; \exists\, m \in \mathbb{N}\,.\, cm = n+1 \right\}\;.$$

Now we have

$$x \in S \longleftrightarrow \exists\, m \in \mathbb{N}\,.\, \left(cm \neq 0 \wedge x \in U_{(cm)-1}\right) \longleftrightarrow \exists\, n \in C\,.\, x \in U_n\;.$$

For the converse, use the converse of Theorem 5.3.30 to obtain a map c that lists a given $C \in \mathcal{O}(\mathbb{N})$. ■

Definition 5.4.5 Let B be a prebasis on X. A *strong inclusion* for B is a semidecidable binary relation $\prec$ on B such that

$$\forall\, U, V \in B\,.\, (U \prec V \longrightarrow U \subseteq_B V)\;,$$

and

$$\forall\, U, V \in B\,.\, \forall\, x \in |U| \cap |V|\,.\, \exists\, W \in B\,.\, (x \in W \wedge W \prec U \wedge W \prec V)\;.$$

Corollary 5.4.6 *Let B be a prebasis on X. If X is open with respect to B and B has a strong inclusion then B is a basis.*

Proof. Let $U\colon \mathbb{N} \to B$ be an enumeration of B, and let $V, W \in B$. Define $C \in \mathcal{O}(\mathbb{N})$ by

$$n \in C \longleftrightarrow (U_n \prec V \wedge U_n \prec W)\;.$$

By Proposition 5.4.4, C is a union predicate for the subspace $T = \left\{ x \in X \;\middle|\; \exists\, n \in C\,.\, x \in U_n \right\}$ which is open with respect to B. We just need show that $x \in T$ if, and only if, $x \in |V| \cap |W|$. If $x \in T$ then there exists $n \in C$ such that $x \in U_n$, $U_n \prec V$, and $U_n \prec W$, therefore $x \in |V| \cap |W|$. Conversely, suppose $x \in |V| \cap |W|$. Then there exists $n \in \mathbb{N}$ such that $x \in U_n$, $U_n \prec V$ and $U_n \prec W$, therefore, $n \in C$, hence $x \in T$. ■

The preceding definition and corollary are important because two important classes of countably based spaces, namely separable metric spaces and effective domains, have a strong inclusion.

If a prebasis B has semidecidable inclusion then the pointwise and point-free open sets for B coincide. However, a prebasis having a strong inclusion is a very strong requirement, which we prefer not to make. The following is an important property of point-free topology that does not hold for the pointwise version. Because of this the point-free topology is preferable.

Proposition 5.4.7 *Let B be a prebasis on X. Every subspace that is open with respect to B is open in the intrinsic topology $\mathcal{O}(X)$.*

Proof. Let $U\colon \mathbb{N} \to B$ be the enumeration of B. Suppose $S \subseteq X$ is open with respect to B. Let $C \in \mathcal{O}(\mathbb{N})$ be a countable union predicate for S. For all $x \in X$,

$$x \in S \longleftrightarrow \exists\, n \in \mathbb{N}\,.\, (Cn = \top \wedge x \in_B U_n)\ .$$

The right-hand side is semidecidable since both $Cn = \top$ and $x \in_B U_n$ are. ■

The following two propositions hold for pointwise and point-free topology.

Proposition 5.4.8 *Let (X, B) be (pointwise) countably based. A countable union of (pointwise) open subspaces is (pointwise) open. More precisely, if $\{S_n \mid n \in \mathbb{N}\}$ is a dependent type such that, for all $n \in \mathbb{N}$, $S_n \subseteq X$ is (pointwise) open, then $S = \{x \in X \mid \exists\, n \in \mathbb{N}\,.\, x \in S_n\}$ is a (pointwise) open subspace of X.*

Proof. The point-free version: For every $n \in \mathbb{N}$ there exists $C \in \mathcal{O}(\mathbb{N})$ such that C is a countable union predicate for S_n. By Number Choice there exists a map $C_\square\colon \mathbb{N} \to \mathcal{O}(\mathbb{N})$ such that, for every $n \in \mathbb{N}$, C_n is a countable union predicate for S_n. Let $D = \bigcup_{n \in \mathbb{N}} C_n$. Then D is a countable union predicate for S. The pointwise version is even easier. ■

Proposition 5.4.9 *(a) The identity map and every constant map are (pointwise) continuous. (b) The composition of (pointwise) continuous maps is (pointwise) continuous.*

Proof. (a) Obvious. (b) Hint: use Proposition 5.4.8 for the point-free version. ■

Definition 5.4.10 A *(pointwise) homeomorphism* $h\colon (X, B) \to (Y, C)$ is a (pointwise) continuous isomorphism whose inverse is also (pointwise) continuous.

Definition 5.4.11 Let (X, B) be (pointwise) countably based. A subspace $S \subseteq X$ is *(pointwise) closed* when its complement $X \setminus S = \{x \in X \mid \neg(x \in S)\}$ is (pointwise) open. A subspace that is both closed and open is *clopen*.

We study only the point-free topology from now on. First, let us look at how a countably based space can be generated from a subbasis.

Proposition 5.4.12 *A* subbasis *on a space X is a countable space S with a semidecidable relation $\in_S \subseteq X \times S$. Every subbasis generates a basis $B = \mathsf{List}_S$ with the relation*

$$x \in_B [U_0, \ldots, U_{k-1}] \longleftrightarrow x \in_S U_0 \wedge \cdots \wedge x \in_S U_{k-1}\ .$$

Here $x \in_B [\,]$ is interpreted as true.

Proof. In other words, the basis generated by a subbasis S consists of finite sequences of elements from S. This definition results in a basis because $|[\,]| = X$, and if $[U_0, \ldots, U_{k-1}] \in B$ and $[V_0, \ldots, V_{n-1}] \in B$, then

$$\big|[U_0, \ldots, U_{k-1}]\big| \cap \big|[V_0, \ldots, V_{n-1}]\big| = \big|[U_0, \ldots, U_{k-1}, V_0, \ldots, V_{n-1}]\big|\ .$$

■

Proposition 5.4.13 *Let (X, B) be countably based. Let B' be the basis generated from B. Then 1_X is a homeomorphism between (X, B) and (X, B').*

Proof. The inverse image of $U \in B$ is open with respect to B' because it equals $[U]$. The inverse image of $[U_0, \ldots, U_{k-1}]$ is open with respect to B because it equals $|U_0| \cap \cdots \cap |U_{k-1}|$. ∎

The basis B' generated from a basis B is closed under intersections, and there is a basic open in B' that covers the whole space X, namely the empty list $[\,]$. Thus, we may always assume without loss of generality that a basis is closed under finite intersections, and that there exists a basic open that covers the whole space.

Proposition 5.4.14 *Let (X, B) be a countably based space, and let $U \colon \mathbb{N} \to B$ be an enumeration of B. Define a prebasis B' on X to be the space $B' = \mathbb{N}$ with the element-hood relation $\in_{B'}$, defined by*

$$x \in n \longleftrightarrow x \in U_n \ .$$

Then B' is a basis, and the identity map 1_X is a homeomorphism between (X, B) and (X, B').

Proof. Let $V \in B$. There exists $n \in \mathbb{N}$ such that $V = U_n$. The inverse image $1_X^* V$ is equal to $|n|$, where n is viewed as an element of B'. Conversely, the inverse image of $n \in B$ is equal to $|U_n|$. ∎

Definition 5.4.15 A countably based space (X, B) is:

(1) A *T_0-space* when, for all $x, y \in X$,

$$(\forall\, U \in B \,.\, (x \in U \longleftrightarrow y \in U)) \longrightarrow x = y \ .$$

(2) A *Hausdorff space* when, for all $x, y \in X$,

$$x \neq y \longrightarrow \exists\, U \in B_x \,.\, \exists\, V \in B_y \,.\, |U| \cap |V| = 0 \ .$$

Remark 5.4.16 The definition of a Hausdorff space is phrased using the inequality relation. A more constructive approach would be to use an *apartness* relation on X, cf. Definition 5.5.4.

Definition 5.4.17 Let (X, B) be countably based. We say that a sequence $a \colon A \to X$ *converges to* $x \in X$, written $\langle a_n \rangle_{n \in \mathbb{N}} \to x$, when

$$\forall\, U \in B_x \,.\, \exists\, n \in \mathbb{N} \,.\, (\forall\, m \in \mathbb{N} \,.\, a_{n+m} \in U) \ .$$

We say that x is the *limit of* $\langle a_n \rangle_{n \in \mathbb{N}}$, and that $\langle a_n \rangle_{n \in \mathbb{N}}$ is a *convergent* sequence.

Proposition 5.4.18 *Let (X, B) and (Y, C) be countably based, $\langle a_n \rangle_{n \in \mathbb{N}} \to x$ in X, and let $f \colon X \to Y$ be a continuous map. Then $\langle f a_n \rangle_{n \in \mathbb{N}} \to f x$ in Y.*

Proof. Let $V \in C_{fx}$. There exists $U \in B_x$ such that $f_*|U| \subseteq |V|$. Because $\langle a_n \rangle_{n \in \mathbb{N}} \to x$, there exists $n \in \mathbb{N}$ such that $a_{n+m} \in U$ for all $m \in \mathbb{N}$, but then $f a_{n+m} \in V$ for all $m \in \mathbb{N}$. ∎

The Space of Continuous Maps

Let (X, B) and (Y, C) be countably based spaces. Let $U\colon \mathbb{N} \to B$ and $V: \mathbb{N} \to C$ be the enumerations of B and C, respectively. The space of continuous map $\mathcal{C}((X, B), (Y, C))$, usually written as $\mathcal{C}(X, Y)$, is

$$\mathcal{C}(X,Y) = \left\{ f\colon X \to Y \;\middle|\; \forall\, m \in \mathbb{N} \,.\, \exists\, R \in \Sigma^{\mathbb{N}} \,.\, \forall\, x \in X \,.\, (fx \in V_m \longleftrightarrow \exists\, n \in R \,.\, x \in U_n) \right\} \,.$$

Here we used characterization of open subspaces from Proposition 5.4.4. By Proposition 5.3.23, $\exists\, n \in R \,.\, x \in U_n$ is stable, therefore the statement $\forall\, x \in X \,.\, (fx \in V_m \longleftrightarrow \exists\, n \in R \,.\, x \in U_n)$ is stable. By Proposition 3.6.6, $\mathcal{C}(X, Y)$ is isomorphic to the quotient

$$\left\{ \langle f, R\rangle \in Y^X \times (\Sigma^{\mathbb{N}})^{\mathbb{N}} \;\middle|\; \forall\, x \in X \,.\, (fx \in V_m \longleftrightarrow \exists\, n \in R \,.\, x \in U_n) \right\} \Big/ {\sim} \,, \tag{5.7}$$

where $\langle f, R\rangle \sim \langle g, S\rangle$ if, and only if, $f = g$.

A similar argument shows that the space $\mathcal{C}_{\mathrm{p}}(X, Y)$ of *pointwise* continuous maps is isomorphic to the quotient

$$\begin{aligned} &\big\{ \langle f, r\rangle \in Y^X \times {\mathbb{N}_\perp}^{\mathsf{r}X \times \mathbb{N}} \;\big| \\ &\qquad \forall\, a \in \mathsf{r}X \,.\, \forall\, m \in \mathbb{N} \,.\, \big(r\langle a, m\rangle \in \mathbb{N} \longrightarrow ([a]_{\mathsf{r}} \in U_{r\langle a,m\rangle} \wedge |U_{r\langle a,m\rangle}| \subseteq f^*|V_m|)\big) \big\} \big/ {\sim} \,. \end{aligned} \tag{5.8}$$

5.4.1 Countably Based Spaces in $\mathsf{Mod}(\mathbb{N})$

We look at the interpretation of pointwise and point-free topology in the category $\mathsf{Mod}(\mathbb{N})$. It turns out that the interpretations of pointwise T_0-spaces agree with Spreen's definitions of effective T_0-spaces [Spr98], whereas the interpretations of point-free T_0-spaces agree with the RE-T_0-spaces, defined in this section. Moreover, the RE-T_0-spaces are equivalent to the category of projective modest sets in $\mathsf{Mod}(\mathsf{RE})$.

AS a reference on recursion theory we use [Soa87]. We denote a standard numbering of partial recursive functions by φ_n, $n \in \mathbb{N}$, and a standard numbering of r.e. sets by W_n, $n \in \mathbb{N}$.

Spreen T_0-spaces

First we overview the basic definitions of effective T_0-spaces by Spreen [Spr98]. His work inspired the present definitions of pointwise and point-free topology. The objects of $\mathsf{Mod}(\mathbb{N})$ are *numbered sets* $X = (|X|, x\colon \mathbb{N} \rightharpoonup |X|)$, where the *numbering* x is a partial surjection. A numbered set is said to be *total* when x is a total function.

A *Spreen T_0-space* (X, B) is a T_0-space $|X|$ with a countable basis $|B|$, where $X = (|X|, x)$ is a numbered set and $B = (|B|, b)$ is a total numbered set, satisfying the following conditions:

1. There is a transitive relation $\prec$ on $\mathbb{N} \times \mathbb{N}$, called the *strong inclusion*, such that $m \prec n$ implies $b_m \subseteq b_n$.

2. The enumeration b is an *effective strong base*, which means that there exists a partial recursive function $s\colon \mathbb{N}^3 \rightharpoonup \mathbb{N}$ such that for all $m, n, i \in \mathbb{N}$, if x_i is defined and $x_i \in b_m \cap b_n$, then $x_i \in b_{s(m,n,i)}$, $s(m, n, i) \prec m$ and $s(m, n, i) \prec n$.

3. The relation $x_i \in b_m$ is completely r.e. in $\langle i, m\rangle \in \mathbb{N} \times \mathbb{N}$, which means that there exists an r.e. set $E \subseteq \mathbb{N} \times \mathbb{N}$ such that $x_i \in b_m$ is equivalent to $\langle i, m\rangle \in E$ for all $m \in \mathbb{N}$ and for all $i \in \mathsf{dom}(x)$.

Remark 5.4.19 In the definition of Spreen T_0-spaces we could replace strong inclusion with ordinary subset inclusion, and the resulting definition would be equivalent to the original one. The point of having a strong inclusion is that in practice it is often the case that the strong inclusion is much better behaved than subset inclusion.

Another point that may seem puzzling at first is that a Spreen T_0-space is necessarily countable, because the numbering x is a surjection from a subset of $\mathbb{N}$ onto X. However, we must not forget that in the internal logic of $\mathsf{Mod}(\mathbb{N})$ the meaning of 'countable' changes to 'effectively countable', cf. Example 5.1.7.

Let (X, x, b) and (Y, y, c) be Spreen T_0-spaces. A map $f\colon X \to Y$ is *effectively continuous* when there exists a total recursive function $h\colon \mathbb{N} \to \mathbb{N}$ so that for all $n \in \mathbb{N}$

$$f^*(c_n) = \bigcup \{b_m \mid m \in \mathsf{W}_{hn}\} \ .$$

A map $f\colon X \to Y$ is *effectively pointwise continuous* when there exists a partial recursive function $g\colon \mathbb{N} \times \mathbb{N} \rightharpoonup \mathbb{N}$ such that, for all $i \in \mathsf{dom}(x)$ and $n \in \mathsf{dom}(c)$ with $fx_i \in c_n$, $g(i,n)\downarrow \in \mathsf{dom}(b)$, $x_i \in b_{g(i,n)}$, and $b_{f(i,n)} \subseteq f^*c_n$.

Spreen T_0-spaces X and Y are said to be *(pointwise) homeomorphic* when there exist (pointwise) continuous maps $f\colon X \to Y$ and $g\colon Y \to X$ that are inverses of each other.

RE-T_0-spaces

We define a version of effective T_0-spaces that corresponds to the projective modest sets of $\mathsf{Mod}(\mathsf{RE})$, as will be proved in Theorem 5.4.23. For lack of a better term, we call these spaces "RE-T_0-spaces", to indicate what underlying PCA they arise from. Let $\mathsf{finset} : \mathbb{N} \to \mathbb{P}_0$ be a canonical numbering of finite set of natural numbers, as defined in Section 1.1.3.

An *RE-T_0-space* (X, B) is a T_0-space $|X|$ with a countable basis $|B|$, where $X = (|X|, x)$ is a numbered set and $B = (|B|, b)$ is a total numbered set, satisfying the following conditions:

1. There is a total recursive function $r\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that for all $m, n \in \mathbb{N}$

$$b_m \cap b_n = \bigcup \{b_j \mid j \in \mathsf{W}_{r(m,n)}\} \ .$$

2. The relation $x_i \in b_m$ is completely r.e., which means that there exists an r.e. set $E \subseteq \mathbb{N} \times \mathbb{N}$ such that $x_i \in b_m$ is equivalent to $\langle i, m\rangle \in E$ for all $m \in \mathbb{N}$ and for all $i \in \mathsf{dom}(x)$.

The definition of *effectively (pointwise) continuous* functions between RE-T_0-spaces is the same as for the Spreen T_0-spaces.

Proposition 5.4.20 *Every RE-T_0-space is also a Spreen T_0-space.*

Proof. We only need to show an RE-T_0-space (X, x, b) has an effective strong base, which is easy if we take subset inclusion to be the the strong inclusion. ∎

To find a Spreen T_0-space which is not a RE-T_0-space, we need the following bit of knowledge from recursion theory.[5]

Proposition 5.4.21 *There exists an infinite and coinfinite r.e. set M such that whenever $M \subseteq K$ and K is r.e., then $K \setminus M$ or $\mathbb{N} \setminus K$ is finite. Such an r.e. set M is called a* maximal r.e. set.

Proof. See for example [Soa87, Chap. X, Sect. 3]. ∎

Theorem 5.4.22 *There is a Spreen T_0-space that is not an* RE-T_0*-space.*

Proof. We are going to construct a subspace X of $\mathbb{N}$ in such a way that two of its basic open sets have a very complicated intersection. Let M be a maximal r.e. set, as in Proposition 5.4.21. Let I be a superset of M that is coinfinite and not r.e. There are coinfinite sets A and B that are not r.e. such that $A \cup B$ is coinfinite, $I = A \cap B$, $A \setminus I$ is infinite, and $B \setminus I$ is infinite. Let X be a superset of $A \cup B$ that is not r.e., is coinfinite, and $X \setminus (A \cup B)$ is infinite. The sets are depicted in Figure 5.4.1. The space (X, x, b) is defined as follows. The topology on X is the discrete topology.

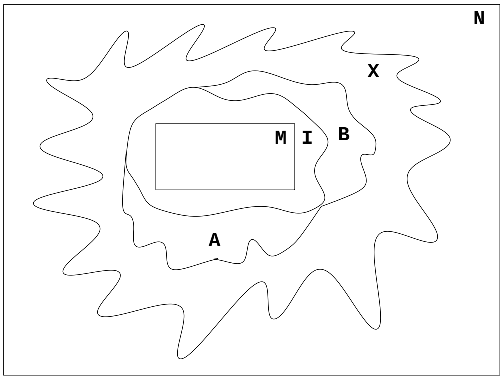

Figure 5.1: $M \subset I \subset A, B \subset X \subset \mathbb{N}$

The numbering $x\colon \mathbb{N} \rightharpoonup X$ is defined by

$$\begin{aligned} x(4n) = n &\iff n \in X \setminus (A \cup B)\ , \\ x(4n+1) = n &\iff n \in A\ , \\ x(4n+2) = n &\iff n \in B\ , \\ x(4n+3) = n &\iff n \in A \cap B\ . \end{aligned}$$

If $0 \leq j \leq 3$ and $n \notin X$ then $x(4n+j)$ is undefined. The base b is defined by

$$b_0 = A\ , \qquad b_1 = B\ , \qquad b_{n+2} = \{n\} \cap X\ .$$

[5] I thank Douglas Cenzer for noticing that the notion of a maximal r.e. sets was exactly what I needed to finishing off the proof of Theorem 5.4.22.

Let us verify that (X, x, b) is a Spreen T_0-space. The relation $x_i \in b_n$ is completely r.e. because, for all $n \in \mathbb{N}$, all $k \in \mathbb{N}$, and all $i, j \in \{0,1\}$ such that $4k + 2j + i \in \mathsf{dom}(x)$,

$$\begin{aligned} x_{4k+2j+i} \in b_n \iff\ & (n = 0 \Longrightarrow i = 1) \land \\ & (n = 1 \Longrightarrow j = 1) \land \\ & (n \geq 2 \Longrightarrow k = n - 2) \ . \end{aligned}$$

The right-hand side of the above equivalence is a recursive relation. For the strong inclusion on X we can take the ordinary set inclusion. Let $s \colon \mathbb{N}^3 \rightharpoonup \mathbb{N}$ be the function defined for $i, j \in \{0,1\}$, $k, m, n \in \mathbb{N}$ by

$$s(4k + 2j + i, m, n) = k + 2 \ .$$

Suppose $x_{4k+2j+i}$ is defined and $x_{4k+2j+i} \in b_m \cap b_n$. Then clearly we have

$$x_{4k+2j+1} = k \in \{k\} = b_{s(4j+2j+i,m,n)} \subseteq b_m \cap b_n \ .$$

This shows that b is an effective strong base.

Suppose (X, x, b) were an RE-T_0-space. Then for some total recursive function $r \colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ we would have

$$I = A \cap B = b_0 \cap b_1 = \bigcup \left\{ b_k \bigm| k \in \mathsf{W}_{r(0,1)} \right\} \ .$$

Since I does not contain A or B, it follows that $0 \notin \mathsf{W}_{r(0,1)}$ and $1 \notin \mathsf{W}_{r(0,1)}$. The set

$$C = \left\{ k - 2 \bigm| k \in \mathsf{W}_{r(0,1)} \right\}$$

is an r.e. set and $M \subseteq I \subseteq C$. Since $I \setminus M$ is infinite, $C \setminus M$ is infinite. It follows from maximality of M that $\mathbb{N} \setminus C$ is finite, but this is only possible if $C \cap (X \setminus I) \neq \emptyset$. Pick some $k \in C \cap (X \setminus I)$. Then on one hand $k \in X \setminus I$, and on the other $k \in b_{k+2} \in I$ because $k + 2 \in \mathsf{W}_{r(0,1)}$. This is a contradiction. ∎

Theorem 5.4.23 *The category of* RE-T_0*-spaces and effectively continuous maps is equivalent to the full subcategory of the projective modest sets in* $\mathsf{Mod}(\mathsf{RE})$*.*

Proof. Let $\mathcal{S}$ be the category of RE-T_0-spaces and effectively continuous maps. Let $\mathcal{T}$ be the full subcategory of $\mathsf{Mod}(\mathsf{RE})$ on the canonically separated modest sets. The category $\mathcal{T}$ is equivalent to the category of projective modest sets in $\mathsf{Mod}(\mathsf{RE})$. The objects of $\mathcal{T}$ are simply the subsets of RE. Every regular projective in $\mathsf{Mod}(\mathsf{RE})$ is isomorphic to an object in $\mathcal{T}$. A morphism $f \colon A \to B$ in $\mathcal{T}$ is a function $f \colon A \to B$ such that there exists an r.e. extension $\overline{f} \colon \mathsf{RE} \to \mathsf{RE}$ which makes the following diagram commute

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, tail] & B \arrow[d, tail] \\ \mathsf{RE} \arrow[r, "\overline{f}"] & \mathsf{RE} \end{tikzcd}$$

We define functors $F \colon \mathcal{T} \to \mathcal{S}$ and $G \colon \mathcal{S} \to \mathcal{T}$, and show that they form an equivalence.

Given a subset $A \subseteq \mathsf{RE}$, let $FA = (A, a, b)$ be the RE-T_0-space, where A is equipped with the subspace topology of RE, and the partial enumeration $a \colon \mathbb{N} \rightharpoonup A$ is defined as follows: a_i is defined and its value is W_i if, and only if, $\mathsf{W}_i \in A$. The base $b \colon \mathbb{N} \to \mathcal{O}(A)$ is defined by

$$b_n = A \cap \uparrow(\mathsf{finset}\, n) \ .$$

For any $n \in \mathbb{N}$ we have

$$\begin{aligned}\bigcap \left\{ b_k \mid k \in \mathsf{finset}\, n \right\} &= A \cap \bigcap \left\{ \uparrow(\mathsf{finset}\, k) \mid k \in \mathsf{finset}\, n \right\} \\ &= A \cap \uparrow\left(\bigcup \left\{ \mathsf{finset}\, k \mid k \in \mathsf{finset}\, n \right\} \right) = b_{rn} \ ,\end{aligned}$$

where $r \colon \mathbb{N} \to \mathbb{N}$ is a suitable total recursive function that satisfies

$$\mathsf{finset}\,(rn) = \bigcup \left\{ \mathsf{finset}\, k \mid k \in \mathsf{finset}\, n \right\} \ .$$

The relation $x_i \in b_n$ is completely r.e. because for all $n \in \mathbb{N}$ and all $i \in \mathsf{dom}(a)$

$$a_i \in b_n \iff \mathsf{finset}\, n \subseteq \mathsf{W}_i \ .$$

Thus FA is indeed an RE-T_0-space.

Given a morphism $f \colon A \to A'$ in $\mathcal{T}$, let $Ff = f$. To see that this is a well defined morphism, we need to find a total recursive function $h \colon \mathbb{N} \to \mathbb{N}$ such that

$$f^*(b'_n) = \bigcup \left\{ b_m \mid m \in \mathsf{W}_{hn} \right\} \ .$$

By assumption, f has an r.e. extension $\overline{f} \colon \mathsf{RE} \to \mathsf{RE}$. Thus, the relation $\uparrow(\mathsf{finset}\, m) \subseteq \overline{f}^*(\uparrow(\mathsf{finset}\, n))$ is r.e. in $\langle m, n \rangle \in \mathbb{N} \times \mathbb{N}$. From this it follows easily that there is a total recursive function $h \colon \mathbb{N} \to \mathbb{N}$ such that

$$\uparrow(\mathsf{finset}\, m) \subseteq \overline{f}^*(\uparrow(\mathsf{finset}\, n)) \iff m \in \mathsf{W}_{hn} \ .$$

Since $\uparrow(\mathsf{finset}\, m) \subseteq \overline{f}^*(\uparrow(\mathsf{finset}\, n))$ implies $b_m \subseteq f^*(b'_n)$, it follows that

$$f^*(b'_n) \supseteq \bigcup \left\{ b_m \mid m \in \mathsf{W}_{hn} \right\} \ .$$

On the other hand, if $ft \in b'_n$ for some $t \in A$, then $\overline{f}t = ft \in b'_n \subseteq \uparrow(\mathsf{finset}\, n)$. There exists $m \in \mathbb{N}$ such that $t \in \uparrow(\mathsf{finset}\, m) \subseteq \overline{f}^*(\uparrow(\mathsf{finset}\, n))$, hence $m \in \mathsf{W}_{hn}$ and $t \in b_m \subseteq f^*(b'_n)$. Therefore,

$$f^*(b'_n) \subseteq \bigcup \left\{ b_m \mid m \in \mathsf{W}_{hn} \right\} \ .$$

We showed that Ff is well-defined.

Let us now define the functor $G \colon \mathcal{S} \to \mathcal{T}$. Given an RE-T_0-space (X, x, b), let $\nu \colon X \to \mathcal{P}\mathbb{N}$ be defined by

$$\nu(t) = \left\{ n \in \mathbb{N} \mid t \in b_n \right\} \ .$$

Since the relation $x_i \in b_n$ is completely r.e. and x is surjective, $\nu(t) \in \mathsf{RE}$ for all t. Define $G(X, x, b)$ to be the subset $GX = \left\{ \nu(t) \mid t \in X \right\}$. Given a morphism $f \colon (X, x, b) \to (Y, y, c)$ let $Gf \colon GX \to GY$ be defined by

$$(Gf)(\nu t) = \nu(ft) \ .$$

We need to find an r.e. extension g of Gf. By assumption, there exists a total recursive function $h\colon \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}$

$$f^*(c_n) = \bigcup \{b_m \mid m \in \mathsf{W}_{hn}\} \ .$$

Let $g\colon \mathsf{RE} \to \mathsf{RE}$ be defined by

$$gx = \{n \in \mathbb{N} \mid \exists\, m \in x \,.\, m \in \mathsf{W}_{hn}\} \ .$$

For any $t \in X$ we have

$$\begin{aligned} g(\nu t) = g(\{m \in \mathbb{N} \mid t \in b_m\}) &= \{n \in \mathbb{N} \mid \exists\, m \in \mathbb{N} \,.\, (t \in b_m \wedge m \in \mathsf{W}_{hn})\} \\ &= \{n \in \mathbb{N} \mid t \in f^*(c_n)\} = \{n \in \mathbb{N} \mid ft \in c_n\} = \nu(ft) \ . \end{aligned}$$

Thus, g is an r.e. extension of Gf. Next, we verify that $G(FA) \cong A$ for any $A \in \mathcal{T}$. Let $FA = (A, a, b)$. Then

$$\begin{aligned} G(FA) &= \{\{n \in \mathbb{N} \mid t \in b_n\} \mid t \in A\} \\ &= \{\{n \in \mathbb{N} \mid t \in \uparrow(\mathsf{finset}\, n) \mid t \in A\}\} = \{\{n \in \mathbb{N} \mid \mathsf{finset}\, n \subseteq t\} \mid t \in A\} \ . \end{aligned}$$

The sets A and $G(FA)$ are isomorphic via the r.e. isomorphisms $f\colon \mathsf{RE} \to \mathsf{RE}$ and $g\colon \mathsf{RE} \to \mathsf{RE}$ defined by

$$fx = \{n \in \mathbb{N} \mid \mathsf{finset}\, n \subseteq x\} \ , \qquad gx = \bigcup \{\mathsf{finset}\, n \mid n \in x\} \ .$$

Finally, let us verify that $F(G(X, x, b)) \cong X$ for any $X \in \mathcal{S}$. Let $(Y, y, c) = F(G(X, x, b))$. Then

$$\begin{aligned} Y &= \{\{n \in \mathbb{N} \mid t \in b_n\} \mid t \in X\} \ , \\ y_i &= \begin{cases} \mathsf{W}_i & \text{if } \mathsf{W}_i \in Y \ , \\ \bot & \text{otherwise} \end{cases} \\ c_n &= Y \cap \uparrow(\mathsf{finset}\, n) \\ &= \bigcup \Big\{\{k \in \mathbb{N} \mid t \in b_k\} \mid t \in \bigcap \{b_i \mid i \in \mathsf{finset}\, n\}\Big\} \ . \end{aligned}$$

Define $f\colon X \to Y$ and $g\colon Y \to X$ by

$$ft = \{n \in \mathbb{N} \mid t \in b_n\} \ , \qquad gx = \text{the } t \in X \text{ such that } x = \{n \in \mathbb{N} \mid t \in b_n\} \ .$$

The map g is well defined because X is a T_0-space, hence $\{n \in \mathbb{N} \mid t \in b_n\} = \{n \in \mathbb{N} \mid t' \in b_n\}$ implies $t = t'$. It is obvious that f and g are inverses of each other. We need to establish that they are morphisms in $\mathcal{S}$. To see that g is a morphism, observe that

$$g^*(b_m) = \{y \in Y \mid m \in y\} = Y \cap \uparrow\{m\} = c_{\mathsf{finset}^{-1}\{m\}} \ .$$

For f we have

$$f^*(c_n) = f^*(Y \cap \uparrow(\mathsf{finset}\, n)) = \bigcap \{b_k \mid k \in \mathsf{finset}\, n\} \ .$$

We easily obtain a total recursive map $r\colon \mathbb{N} \to \mathbb{N}$ such that

$$\bigcap \{b_k \mid k \in \mathsf{finset}\, n\} = \bigcup \{b_j \mid j \in \mathsf{W}_{rn}\} \ ,$$

which means that f is a morphism. ∎

Countably Based Spaces in $\mathsf{Mod}(\mathbb{N})$

Let us unravel the interpretation of pointwise and point-free countably based spaces in $\mathsf{Mod}(\mathbb{N})$.

Proposition 5.4.24 *Let X and B be spaces. Then (X, B) is pointwise countably based in the internal logic of $\mathsf{Mod}(\mathbb{N})$ if, and only if, the interpretation of (X, B) in $\mathsf{Mod}(\mathbb{N})$ is a Spreen T_0-space.*

Proof. The space X is interpreted as an enumerated set $X = (|X|, x\colon \mathbb{N} \rightharpoonup |X|)$. The basis B is interpreted as an enumerated set $B = (|B|, b\colon \mathbb{N} \rightharpoonup |B|)$. As we saw in Example 5.1.7, requiring B to be countable is the same, up to isomorphism, as requiring the numbering $b\colon \mathbb{N} \rightharpoonup |B|$ to be total. It is not hard to see that the relation $\in_B$ is semidecidable if, and only if, $x_i \in b_n$ is completely r.e. Next, a realizer for the statement

$$\forall\, U, V \in B\,.\, \forall\, x \in |U| \cap |V|\,.\, \exists\, W \in B\,.\, (x \in W \wedge |W| \subseteq |U| \cap |V|)$$

is a partial recursive map $s\colon \mathbb{N}^3 \to \mathbb{N}$ such that for all $m, n \in \mathsf{dom}(b)$, $i \in \mathsf{dom}(x)$, if $x_i \in b_m \cap b_n$ then $s\langle m, n, i\rangle\downarrow$, $x_i \in b_{s\langle m,n,i\rangle}$ and $b_{s\langle i,m,n\rangle} \subseteq b_m \cap b_n$. This is exactly the definition of an effective strong base, if we take the subset inclusion as the strong inclusion. By Remark 5.4.19, we can assume that strong inclusion always coincides with subset inclusion, at least for the purposes of defining Spreen T_0-spaces. Definition 5.4.15 of a T_0-space is a negative formula, therefore its realizability interpretation is the same as the set-theoretic interpretation. Lastly, we need to show that if (X, B) is a Spreen T_0-space then X is pointwise open with respect to B, which means that

$$\forall\, x \in X\,.\, \exists\, m \in \mathbb{N}\,.\, x \in U_m\ , \tag{5.9}$$

where $U\colon \mathbb{N} \to B$ is an enumeration of B. By Proposition 5.3.23, $\exists\, m \in \mathbb{N}\,.\, x \in U_m$ is equivalent to $\neg\forall\, m \in \mathbb{N}\,.\, \neg(x \in U_m)$, therefore (5.9) is equivalent to a negative formula. It holds in the logic of modest sets if, and only if, it holds when interpreted classically, which it does. ∎

Proposition 5.4.25 *A pair of spaces (X, B) is a countably based space if, and only if, its interpretation in $\mathsf{Mod}(\mathbb{N})$ is an RE-T_0-space.*

Proof. The proof goes along the same lines as the proof of Proposition 5.4.24. Let $U\colon \mathbb{N} \to B$ be an enumeration of B. Most of the proof is left as an exercise. We only compute the realizer for the statement

$$\forall\, n, m \in \mathbb{N}\,.\, \Big(\exists\, R \in \Sigma^{\mathbb{N}}\,.\, \forall\, x \in X\,.\, (x \in |U_n| \cap |U_m| \longleftrightarrow \exists\, k \in R\,.\, x \in U_k)\Big)\ . \tag{5.10}$$

By Proposition 5.3.23, $\exists\, k \in R\,.\, x \in U_k$ is equivalent to a negative formula, hence

$$\forall\, x \in X\,.\, (x \in |U_n| \cap |U_m| \longleftrightarrow \exists\, k \in R\,.\, x \in U_k)$$

is equivalent to a negative formula and can be interpreted set-theoretically. Recall from Example 5.3.22 that $\Sigma^{\mathbb{N}}$ is interpreted as the modest set of r.e. sets. Therefore, a realizer for (5.10) amounts to a total recursive function $r\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that

$$|U_m| \cap |U_n| = \bigcup \big\{ |U_k| \bigm| k \in \mathsf{W}_{r(m,n)} \big\}\ .$$

This matches exactly the definition of RE-T_0-spaces. ∎

Proposition 5.4.26 *(a) In* $\mathsf{Mod}(\mathbb{N})$ *a map is effectively pointwise continuous in the sense of Definition 5.4.1 if, and only if, it is effectively pointwise continuous in the sense of Spreen* T_0*-spaces.* *(b) In* $\mathsf{Mod}(\mathbb{N})$ *a map is effectively continuous in the sense of Definition 5.4.2 if, and only if, it is effectively continuous in the sense of* RE-T_0*-spaces.*

Proof. The proposition can be read off (5.7) and (5.8), which characterize a representation of continuous and pointwise continuous maps, respectively. ∎

Example 5.4.27 We could formulate the theory of equilogical spaces in the logic of modest sets. We define an equilogical space to be a countably based T_0-space (X, B) with a stable equivalence relation $\equiv \subseteq X \times X$, and an equivariant map to be a continuous map that preserves equivalence relations. What do we get if we interpret this *internal* version of equilogical spaces in a category of modest sets?

Proposition 5.4.25 tells that the internal equilogical spaces interpreted in $\mathsf{Mod}(\mathbb{N})$ correspond to RE-T_0-spaces with equivalence relations—but that is just the category $\mathsf{Mod}(\mathsf{RE})$. Thus, the internal theory of equilogical spaces in $\mathsf{Mod}(\mathbb{N})$ is the theory of modest sets over RE.

5.5 Real Numbers

In this section we construct the real numbers as equivalence classes of Cauchy sequences.[6] This is a standard construction in classical analysis, and it works just as well in the logic of modest sets as it does classically. The resulting space $\mathbb{R}$, when interpreted in categories of modest sets, corresponds to well known constructions of reals in various constructive settings.

5.5.1 Integers and Rational Numbers

Let us first build the spaces of integers $\mathbb{Z}$ and rational numbers $\mathbb{Q}$, starting from the natural numbers $\mathbb{N}$. An integer can be thought of as a difference of two natural numbers. Thus, we define $\mathbb{Z} = (\mathbb{N} \times \mathbb{N})/\!\sim$ where

$$\langle a, b\rangle \sim \langle c, d\rangle \longleftrightarrow a + d = b + c \; .$$

We write $[a, b]$ instead of $[\langle a, b\rangle]$ to denote the equivalence class of $\langle a, b\rangle$. The basic arithmetic operations on $\mathbb{Z}$ are defined by

$$0 = [0,0] \; , \qquad 1 = [1,0] \; , \qquad -[a,b] = [b,a] \; ,$$
$$[a,b] + [c,d] = [a+c, b+d] \; ,$$
$$[a,b] \cdot [c,d] = [ac + bd, ad + bc] \; .$$

The space $\mathbb{Z}$ is an ordered ring with a decidable order relation, defined by

$$[a,b] < [c,d] \longleftrightarrow a + d < b + c \; .$$

Every natural number n can be thought of as the integer $[n, 0]$. Hence, the natural numbers are a regular subspace of $\mathbb{Z}$, and the embedding preserves the ordered semi-ring structure of $\mathbb{N}$. The

[6] In a realizability topos Number Choice is valid, from which it follows that the Dedekind reals and the Cauchy completion of the rational numbers coincide, see [TvD88a, Proposition V.5.10]. The Cauchy reals have the advantage that their construction can be stated without reference to the powerset of rational numbers.

space $\mathbb{Z}$ is a countable set, as is easily seen since $\mathbb{N}\times\mathbb{N}$ is a countable set and $\sim$ is a decidable relation. In fact, $\mathbb{Z}$ is isomorphic to $\mathbb{N}$, but the isomorphism does not preserve any algebraic structure.

A rational number can the thought of as a pair, p/q, where $p \in \mathbb{Z}$ and $q \in \mathbb{N}$, $q \neq 0$. Thus, we define

$$\mathbb{Q} = (\mathbb{Z} \times (\mathbb{N} \setminus \{0\}))/{\sim} \ ,$$

where $\sim$ is defined by

$$\langle p, q\rangle \sim \langle s, t\rangle \longleftrightarrow p \cdot t = s \cdot q \ .$$

We write a/b instead of $[\langle a, b\rangle]$. It is obvious that $\mathbb{Q}$ is a countable set, since $\mathbb{N} \times \mathbb{N}$ is a countable set and $\sim$ is a decidable relation. In fact, $\mathbb{Q}$ is isomorphic to $\mathbb{N}$, but the isomorphism does not preserve any algebraic structure. The rational numbers form a field:

$$0 = 0/1 \qquad\qquad 1 = 1/1$$

$$-(a/b) = (-a)/b \qquad\qquad (a/b)^{-1} = \begin{cases} b/a & \text{if } a \geq 0 \\ (-b)/(-a) & \text{if } a < 0 \end{cases}$$

$$(a/b) + (c/d) = (ad + bc)/(bd) \qquad\qquad (a/b) \cdot (c/d) = (ac)/(bd)$$

The rationals are an order field, with a decidable order relation, defined by

$$a/b < c/d \longleftrightarrow ad < bc \ .$$

The integers are a regular subspace of the rationals, where an integer $k \in \mathbb{Z}$ is thought of as the fraction $k/1$. The embedding is an ordered ring homomorphism. In fact, $\mathbb{Q}$ is the usual field of fractions generated by $\mathbb{Z}$.

For every rational $q \in \mathbb{Q}$ there exists a unique pair $\langle a, b\rangle$, where $a \in \mathbb{Z}$, $b \in \mathbb{N} \setminus \{0\}$, such that a and b are relatively prime and $p = a/b$. We say that the fraction a/b is in *lowest terms*. In other words, there is a "lowest terms" map $\mathbb{Q} \to \mathbb{Z} \times (\mathbb{N} \setminus \{0\})$.

5.5.2 The Construction of Cauchy Reals

We denote a sequence $a\colon \mathbb{N} \to \mathbb{Q}$ of rational numbers by $\langle a_n\rangle_{n\in\mathbb{N}}$, or just by $\langle a_n\rangle$. A *Cauchy sequence* $\langle a_n\rangle$ is a sequence that satisfies

$$\forall\, q \in \mathbb{Q}\,.\,(q > 0 \longrightarrow \exists\, n \in \mathbb{N}\,.\,\forall\, m, p \in \mathbb{N}\,.\,|a_{n+m} - a_{n+p}| < q) \ .$$

We call this the *Cauchy convergence test*, or shortly the *Cauchy test*. It is equivalent to the simpler statement

$$\forall\, q \in \mathbb{Q}\,.\,(q > 0 \longrightarrow \exists\, n \in \mathbb{N}\,.\,\forall\, m \in \mathbb{N}\,.\,|a_{n+m} - a_n| < q) \ ,$$

which is further equivalent to the still simpler statement

$$\forall\, k \in \mathbb{N}\,.\,\exists\, n \in \mathbb{N}\,.\,\forall\, m \in \mathbb{N}\,.\,|a_{n+m} - a_n| < 1/k \ ,$$

because for every $q \in \mathbb{Q}$, such that $q > 0$, there exists $k \in \mathbb{N}$ such that $1/k < q$. The space of Cauchy sequences is

$$\mathcal{C} = \left\{a \in \mathbb{Q}^{\mathbb{N}} \mid \forall\, k \in \mathbb{N}\,.\,\exists\, n \in \mathbb{N}\,.\,\forall\, m \in \mathbb{N}\,.\,|a_{n+m} - a_n| < 1/k\right\} \ .$$

We define the *coincidence relation* $\approx$ on $\mathcal{C}$ by

$$\langle a_n \rangle \approx \langle b_n \rangle \longleftrightarrow \forall\, k \in \mathbb{N}\,.\, \exists\, n \in \mathbb{N}\,.\, \forall\, m, p \in \mathbb{N}\,.\, |a_{n+m} - b_{n+p}| < 1/k \;.$$

It is obvious that $\approx$ is symmetric, and it is not hard to see that it is also reflexive and transitive. We prove that $\approx$ is stable. First, we show that

$$\langle a_n \rangle \approx \langle b_n \rangle \longleftrightarrow \forall\, k, j \in \mathbb{N}\,.\, \exists\, n \in \mathbb{N}\,.\, |a_{j+n} - b_{j+n}| < 1/k \;.$$

Indeed, the implication from left to right is easy. As for the converse, suppose that the right-hand side holds. For every $k \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that, for all $m, p \in \mathbb{N}$,

$$|a_{n+m} - a_{n+p}| < 1/(3k) \;, \qquad |b_{n+m} - b_{n+p}| < 1/(3k) \;.$$

By assumption, there exists $r \in \mathbb{N}$ such that $|a_{n+r} - b_{n+r}| < 1/(3k)$. From this we get

$$\begin{aligned} |a_{n+m} - b_{n+p}| &\leq |a_{n+m} - a_{n+r} + a_{n+r} - b_{n+r} + b_{n+r} - b_{n+p}| \leq \\ &\qquad |a_{n+m} - a_{n+r}| + |a_{n+r} - b_{n+r}| + |b_{n+r} - b_{n+p}| < 1/k \;. \end{aligned}$$

Therefore, $\langle a_n \rangle \approx \langle b_n \rangle$. Now it follows quickly that $\approx$ is stable: if $\neg\neg(\langle a_n \rangle \approx \langle b_n \rangle)$ then, for all $k, j \in \mathbb{N}$, $\neg\neg\exists\, n \in \mathbb{N}\,.\, |a_{j+n} - b_{j+n}| < 1/k$, and by Markov's Principle, $\exists\, n \in \mathbb{N}\,.\, |a_{j+n} - b_{j+n}| < 1/k$, hence $\langle a_n \rangle \approx \langle b_n \rangle$.

The space of *Cauchy reals* is the space $\mathbb{R} = \mathcal{C}/{\approx}$. We usually refer to it just as *reals* or *real numbers*. We denote the real number represented by the Cauchy sequence $\langle a_n \rangle_{n \in \mathbb{N}}$ by $[a_n]_{n \in \mathbb{N}}$ or just $[a_n]$. The rational numbers are a subspace of $\mathbb{R}$, since a rational number q can be represented as the constant sequence $\langle q \rangle$.

Lemma 5.5.1 *Suppose $a \in \mathcal{C}$ and $f \colon \mathbb{N} \to \mathbb{N}$ is a map such that $fn \geq n$ for all $k \in \mathbb{N}$. Then $a \circ f \in \mathcal{C}$ and $a \approx a \circ f$.*

Proof. First, let us show that $a \circ f$ is a Cauchy sequence. Let $k \in \mathbb{N}$. There exists $n \in \mathbb{N}$ such that for all $m, p \in \mathbb{N}$, $|a_{n+m} - a_{n+p}| < 1/k$. In particular, set $m = f(n+m') - n$ and $p = f(n+p') - n$ where $m', p' \in \mathbb{N}$ are arbitrary. Then $|a_{f(n+m)'} - a_{f(n+p')}| < 1/k$, hence $a \circ f \in \mathcal{C}$.

Let $k \in \mathbb{N}$. There exists $n \in \mathbb{N}$ such that $|a_{n+m} - a_{n+p}| < 1/k$ for all $m, p \in \mathbb{N}$. If we take $p = f(n + p') - n$, where $p' \in \mathbb{N}$ is arbitrary, we get $|a_{n+m} - a_{f(n+p')}| < 1/k$, hence $a \approx a \circ f$. ∎

The Rapidly Converging Reals

The Cauchy reals, as defined above, are equivalence classes of arbitrary Cauchy sequences. Sometimes it is convenient to take only the rapidly converging sequences. Thanks to Number Choice, the space of Cauchy reals does not change if we take just the rapidly converging sequence, which we prove next. A sequence $\langle a_n \rangle_{n \in \mathbb{N}}$ is said to be *rapidly converging* when

$$\forall\, k \in \mathbb{N}\,.\, |a_{k+1} - a_k| < 2^{-k} \;.$$

Notice that, for all $m, n \in \mathbb{N}$ such that $m > n$, we have

$$|a_m - a_n| \leq \sum_{k=n}^{m-1} |a_{k+1} - a_k| < \sum_{k=n}^{m-1} 2^{-k} = \frac{1}{2^{n-1}} - \frac{1}{2^{m-1}} < \frac{1}{2^{n-1}} \;.$$

The *coincidence relation* $\sim$ is defined on rapidly converging sequences by

$$\langle a_n \rangle \sim \langle b_n \rangle \longleftrightarrow \forall\, k \in \mathbb{N} \,.\, |a_k - b_k| \leq 2^{-k+2} \;.$$

It is obviously reflexive and symmetric. To see that it is transitive, suppose $\langle a_n \rangle \sim \langle b_n \rangle$ and $\langle b_n \rangle \sim \langle c_n \rangle$. For all $n, m \in \mathbb{N}$ such that $m > n$,

$$\begin{aligned} |a_n - c_n| &= |a_n - a_m + a_m - b_m + b_m - c_m + c_m - c_n| \\ &\leq |a_n - a_m| + |a_m - b_m| + |b_m - c_m| + |c_m - c_n| \\ &< \frac{1}{2^{n-1}} - \frac{1}{2^{m-1}} + \frac{1}{2^{m-2}} + \frac{1}{2^{m-2}} + \frac{1}{2^{n-1}} - \frac{1}{2^{m-1}} = \frac{1}{2^{n-2}} + \frac{3}{2^{m-1}} \;. \end{aligned}$$

Hence, for all $m > n$, $|a_n - c_n| < 2^{-n+2} + 3 \cdot 2^{-m+1}$, which is only possible if $|a_n - c_n| \leq 2^{-n+2}$. Therefore, $\sim$ is a transitive relation. Lastly, $\sim$ is stable because $\neg\neg\forall\, k \in \mathbb{N} \,.\, |a_k - b_k| < 2^{-k+2}$ is equivalent to $\forall\, k \in \mathbb{N} \,.\, \neg\neg(|a_k - b_k| < 2^{-k+2})$ as $<$ is decidable on $\mathbb{Q}$.

Let $\mathcal{C}_{\mathrm{r}}$ be the space of rapidly converging sequences,

$$\mathcal{C}_{\mathrm{r}} = \left\{ a \in \mathbb{Q}^{\mathbb{N}} \;\middle|\; \forall\, k \in \mathbb{N} \,.\, |a_{k+1} - a_k| < 2^{-k} \right\} \;.$$

and let $\mathbb{R}_{\mathrm{r}} = \mathcal{C}_{\mathrm{r}}/{\sim}$ be the space of rapidly converging reals.

Proposition 5.5.2 *The spaces $\mathbb{R}$ and $\mathbb{R}_{\mathrm{r}}$ are canonically isomorphic, and the isomorphism fixes the rational numbers.*

Proof. By "canonically isomorphic" we mean that there is an isomorphism that is the identity on the rational numbers and preserves the algebraic structure. Define a relation ρ on $\mathbb{R} \times \mathbb{R}_{\mathrm{r}}$ by

$$\rho(x, y) \longleftrightarrow \exists\, a \in \mathcal{C} \,.\, \exists\, b \in \mathcal{C}_{\mathrm{r}} \,.\, (x = [a] \wedge y = [b] \wedge a \approx b) \;.$$

It suffices to show that for every $x \in \mathbb{R}$ there exists a unique $y \in \mathbb{R}_{\mathrm{r}}$ such that $\rho(x, y)$, and vice versa, that for every $y \in \mathbb{R}_{\mathrm{r}}$ there exists a unique $x \in \mathbb{R}$ such that $\rho(x, y)$. The by Unique Choice we obtain the desired isomorphisms.

Uniqueness is easy to establish. Suppose $\rho(x, y)$ and $\rho(x, y')$. Then there exist $a, a' \in \mathcal{C}$ and $b, b' \in \mathcal{C}_{\mathrm{r}}$ such that $x = [a] = [a']$, $y = [b]$, $y' = [b']$, $a \approx b$, and $a' \approx b'$. Because $a \approx a'$, we get $b \approx b'$ by transitivity of $\approx$, hence $y = [b] = [b'] = y'$. The other half of uniqueness is proved the same way.

If $y \in \mathbb{R}_{\mathrm{r}}$ then there exists $a \in \mathcal{C}_{\mathrm{r}}$ such that $y = [a]$. Because every rapidly converging sequence is a Cauchy sequence, it is also the case that $a \in \mathcal{C}$, hence $\rho([a], y)$.

The only non-obvious part is to show that for every $x \in \mathbb{R}$ there exists $y \in \mathbb{R}_c$ such that $\rho(x, y)$. In other words, we need to show that for every $x \in \mathbb{R}$ there exists a rapidly converging sequence a such that $x = [a]$. We know that there exists $b \in \mathcal{C}$ such that $x = [b]$. The idea is to "speed up" the sequence b. Because b is a Cauchy sequence it is the case that

$$\forall\, k \in \mathbb{N} \,.\, \exists\, n \in \mathbb{N} \,.\, |b_{n+1+k} - b_{n+k}| < 2^{-k} \;.$$

Thus, by Number Choice there exists a function $f \colon \mathbb{N} \to \mathbb{N}$ such that $fk \geq k$ and $|b_{(fk)+1} - b_{fk}| < 2^{-k}$ for all $k \in \mathbb{N}$. The sequence $a = b \circ f$ is rapidly converging, and it only remains to be seen that $a \approx b$. This follows from Lemma 5.5.1. ∎

Signed Binary Digit Representation

We demonstrate how the realizability interpretation of the real numbers tells us which concrete implementations of real number arithmetic are good, and which are bad.

An implementation of real number arithmetic in terms of arbitrary Cauchy sequences and their rates of convergence would very likely turn out to be quite inefficient, since the rates of convergence could be arbitrarily slow. It makes sense to require a fixed rate of convergence, say a geometric one. This is how we defined the rapidly converging reals. We can simplify the construction still further. There are many ways to do this. Lester used rapidly converging sequences of dyadic rational numbers, Edalat and coworkers [EP97, ES99b, EK99] represented real numbers as streams of linear fractional transformations, and there are still other possibilities [GL00].

The logic of modest sets can be used to distinguish the good implementations of reals from the bad ones: an acceptable implementation must be computably isomorphic to the Cauchy reals, because only then does it have the expected *logical* properties of the real numbers. We demonstrate how this works by looking at a well known representations of the reals, the signed binary digit representation. This is a simple example, but the idea behind it should prove useful for more complicated examples.

The *signed binary representation* is defined as follows. Let $D = \{-1, 0, 1\}$ be the set of signed binary digits. The set D is isomorphic to $1 + 1 + 1$. Let $\mathbb{R}_{\mathrm{sb}} = (\mathbb{Z} \times D^{\mathbb{N}})/{\sim}$, where $\sim$ is defined, for all $\langle m, a\rangle, \langle n, b\rangle \in \mathbb{Z} \times D^{\mathbb{N}}$, by

$$\langle m, a\rangle \sim \langle n, b\rangle \longleftrightarrow \forall\, k \in \mathbb{N} \,.\, |S_{\langle m,a\rangle}\, k - S_{\langle n,b\rangle} k| < 2^{\max(m,n)-k+2} \;, \tag{5.11}$$

where $S_{\langle m,a\rangle} k$ is the *k-th approximation*, defined by

$$S_{\langle m,a\rangle} k = 2^m \cdot \sum_{i=0}^{k} \frac{a_k}{2^{k+1}} \;.$$

The number m is called the *exponent* and the sequence a is the *mantissa* of the *signed binary digit expansion* $\langle m, a\rangle$.

Proposition 5.5.3 *The Cauchy reals* $\mathbb{R}$ *and the signed binary reals* $\mathbb{R}_{\mathrm{sb}}$ *are canonically isomorphic.*

Proof. The proof goes the same way as the proof of Proposition 5.5.2. Every signed binary real is a Cauchy real, since $S_{\langle m,a\rangle}$ converts a signed binary digit expansion into a converging Cauchy sequence. The only interesting part of the proof is how to obtain a signed binary digit representation $\langle m, a\rangle$ from a rapidly converging Cauchy sequence $\langle b_n\rangle_{n\in\mathbb{N}}$. There is an integer $m \geq 0$ such that $|b_0| < 2^{m-1}$, which we pick to be the exponent. The mantissa $a \in D^{\mathbb{N}}$ is defined inductively in such a way that $|S_{\langle m,a\rangle} k - b_k| \leq 2^{m-k-1}$ for all $k \in \mathbb{N}$:

$$a_0 = 0 \;, \qquad a_{k+1} = \begin{cases} 1 & \text{if } b_{k+1} - 2^m \sum_{i=0}^{k} (a_k/2^k) \geq 2^{m-k-1} \;, \\ -1 & \text{if } b_{k+1} - 2^m \sum_{i=0}^{k} (a_k/2^k) \leq -2^{m-k-1} \;, \\ 0 & \text{otherwise} \;. \end{cases}$$

We prove by induction that $|S_{\langle m,a\rangle} k - b_k| < 2^{m-k-1}$ for all $k \in \mathbb{N}$. The base case is obvious. For the induction step, there are three cases to consider. We only spell out the details of the case when

$a_{k+1} = 1$, since the case $a_{k+1} = -1$ is analogous and $a_{k+1} = 0$ is easy. First observe that by the induction hypothesis we have

$$|b_{k+1} - S_k| \leq |b_{k+1} - b_k| + |b_k - S_k| < 2^{-k} + 2^{m-k-1} < 2^{m-k} .$$

From this we get

$$|S_{k+1} - b_{k+1}| = |b_{k+1} - S_k| - |S_{k+1} - S_k| < 2^{m-k} - 2^{m-k-1} = 2^{m-k-1} .$$

■

5.5.3 The Algebraic Structure of Reals

We now study the algebraic structure of real numbers. Instead of proving the properties of $\mathbb{R}$ directly, we first present the basic theory of intuitionistic ordered fields. The main result is that every Cauchy complete Archimedean field is isomorphic to the reals. Because the logic of modest sets has computability built in, we automatically obtain results about the computable structure of ordered fields.

An important difference between classical and intuitionistic algebra is the notion of apartness relation in the intuitionistic setting, which does not occur in the classical setting.

Definition 5.5.4 An *apartness relation* on a space A is a relation $\lessgtr$ such that, for all $x, y \in A$:

(1) $\neg(x \lessgtr y) \longleftrightarrow x = y$.

(2) $x \lessgtr y \longrightarrow y \lessgtr x$.

(3) $x \lessgtr y \longrightarrow \forall\, z \in A\,.\,(x \lessgtr z \lor z \lessgtr y)$.

The apartness relation $\lessgtr$ is a positive version of inequality. Observe that $x \lessgtr y$ implies $x \neq y$. In an ordered field $x \lessgtr y$ is always defined as $x < y \lor y < x$, which explains the notation. Intuitively, the difference between $x \neq y$ and $x \lessgtr y$ is that when $x \lessgtr y$ holds, we know explicitly why x and y are different, for example, an explicit lower bound for $|x-y|$ can be found. On the other hand, when $x \neq y$ then we might not know of such an explicit bound, and all we know is that the assumption $x = y$ leads to a contradiction. In the field of real numbers, apartness and inequality turn out to coincide, because of Markov's Principle.

Definition 5.5.5 A *field* $(F, 0, 1, +, \cdot, -, {}^{-1}, \lessgtr)$ is a space F with points $0, 1 \in F$, binary operations $+$ and $\cdot$, and unary operations $-\Box \colon F \to F$ and $\Box^{-1} \colon \{x \in F \mid x \lessgtr 0 \to F\}$, and an apartness relation $\lessgtr$, such that:

$$\begin{array}{cc}
0 + x = x & 1 \cdot x = x \\
x + (y + z) = (x + y) + z & x \cdot (y \cdot z) = (x \cdot y) \cdot z \\
x + y = y + x & x \cdot y = y \cdot x \\
x + (-x) = 0 & x \lessgtr 0 \longrightarrow x \cdot x^{-1} = 1
\end{array}$$

$$(x + y) \cdot z = x \cdot z + y \cdot z$$

A field is *computable* when 0, 1, $+$, $\cdot$, $-$, and $^{-1}$ are computable, and $\lessgtr$ is semidecidable and computable. A field *has characteristic zero* when for all $n \in \mathbb{N}$, $n \cdot 1 \lessgtr 0$, where $n \cdot 1$ is defined recursively by $0 \cdot 1 = 0$, $(n+1) \cdot 1 = n \cdot 1 + 1$. In particular, $1 \lessgtr 0$. We only consider fields with characteristic zero, and refer to them simply as fields.

We usually denote a field $(F, 0, 1, +, \cdot, -, ^{-1}, \lessgtr)$ simply by F. Sometimes we deal with many fields at once, in which case we may equip the basic operations with subscripts in order to distinguish them.

Definition 5.5.6 An *ordered field* is a field F with a relation $<$ such that, for all $x, y, z \in F$:

$$\neg(x < y \wedge y < x) \tag{OF1}$$

$$x < y \longrightarrow (x < z \vee z < y) \tag{OF2}$$

$$\neg(x < y \vee y < x) \longrightarrow x = y \tag{OF3}$$

$$x < y \longrightarrow x + z < y + z \tag{OF4}$$

$$0 < x \wedge 0 < y \longrightarrow 0 < x \cdot y \tag{OF5}$$

$$0 < x \longrightarrow 0 < x^{-1} \tag{OF6}$$

We define $x \leq y$ to mean $\neg(y < x)$. The transposes $>$ and $\geq$ have the obvious meaning: $x > y$ means $y < x$, and $x \geq y$ means $y \leq x$. In an ordered field we always define the apartness relation $\lessgtr$ by

$$x \lessgtr y \longleftrightarrow (x < y \vee y < x) \ .$$

It is easy to check that this really is an apartness relation. An ordered field is *computable* when it is a computable field and the order relation is semidecidable and computable. An ordered field F is *Archimedean* when, for all $x, y \in F$, if $x > 0$ then there exists $n \in \mathbb{N}$ such that $n \cdot x > y$.

We can write (OF3) in a more familiar form as

$$(x \leq y \wedge y \leq x) \longrightarrow x = y \ . \tag{OF3'}$$

Note that we did *not* require the classical law of trichotomy, $x < y \vee x = y \vee x > y$. Instead, we replaced it with its intuitionistic version (OF1) & (OF2).

Proposition 5.5.7 *Let F be an ordered field and $x, y \in F$. If for all $z \in F$, $z < x \longleftrightarrow z < y$, then $x = y$.*

Proof. Assume that $z < x \longleftrightarrow z < y$ for all $z \in F$. If $x < y$ then $x < x$ by assumption, but this is impossible by (OF1), therefore $y \leq x$. A similar argument shows that $x \leq y$, hence $x = y$ by (OF3'). ∎

Proposition 5.5.8 *Let F be an ordered field. Then for all $x, y, z \in F$:*

(1) Transitivity: $x < y \wedge y < z \longrightarrow x < z$.

(2) If $x > 0$ then $-x < 0$ and vice versa.

(3) If $x > 0$ and $y < z$ then $x \cdot y < x \cdot z$.

(4) If $x < 0$ and $y < z$ then $x \cdot y > x \cdot z$.

(5) If $x \lessgtr 0$ then $x \cdot x > 0$. In particular, $1 > 0$.

(6) If $0 < x < y$ then $0 < 1/y < 1/x$.

Proof. (1) Suppose $x < y$ and $y < z$. By (OF2), $x < z$ or $z < y$. However, $z < y$ is impossible by (OF1), hence $x < z$. (2) If $x > 0$ then $0 = -x + x > -x + 0$, so that $-x < 0$. If $x < 0$ then $0 = -x + x < -x + 0$, so that $-x > 0$. (3) Since $z > y$, we have $z - y > y - y = 0$, hence $x(z - y) > 0$ by (OF5), and therefore $xz = x(z - y) + xy > 0 + xy = xy$. (4) By (2) and (3) we get $-(x(z - y)) = (-x)(z - y) > 0$, so that $x(z - y) < 0$, hence $xz < xy$. (5) If $x > 0$ then $x \cdot x > 0$ by (OF5). If $x < 0$ then $-x > 0$ by (2), hence $x \cdot x = (-x) \cdot (-x) > 0$. (6) Since $x > 0$ and $y > 0$, $x^{-1} > 0$ and $y^{-1} > 0$ by (OF6). If we multiply $0 < x < y$ by the positive quantity $x^{-1}y^{-1}$, we get $0 < y^{-1} < x^{-1}$. ∎

Proposition 5.5.9 *An ordered field F is Archimedean if, and only if, for every $x \in F$ there exists $n \in \mathbb{N}$ such that $x < n$.*

Proof. If F is Archimedean, there exists $n \in \mathbb{N}$ such that $x < n \cdot 1 = n$. Conversely, if $x > 0$ and $y \in F$, then there exists $n \in \mathbb{N}$ such that $y/x < n$. Multiply by x on both sides to get $y < n \cdot x$. ∎

Proposition 5.5.10 *The rational numbers form a computable Archimedean field.*

Proof. Left as an exercise. ∎

Proposition 5.5.11 *Let F be a field. There is a unique homomorphism of fields $\mathbb{Q} \to F$. If F is an ordered field, then the unique homomorphism preserves order. If F is a computable field, then the unique homomorphism is computable.*

Proof. Define multiplication of $x \in F$ by a natural number $n \in \mathbb{N}$ by

$$0 \cdot x = 0 \qquad (n + 1) \cdot x = x + n \cdot x \;,$$

and extend it to multiplication by an integer by

$$[a, b] \cdot x = a \cdot x - b \cdot x \;,$$

where $[a, b] \in \mathbb{Z}$. Because F has characteristic zero, $n \cdot 1 = 0$ is equivalent to $n = 0$, so we can extend the multiplication to rational numbers by

$$(p/q) \cdot x = (p \cdot x)/(q \cdot 1).$$

This is well defined because $q \neq 0$, therefore $q \cdot 1 \neq 0$. It is easy to check that the map $i \colon \mathbb{Q} \to F$, defined by

$$i(p/q) = (p/q) \cdot 1 \;,$$

is a homomorphism of fields, and that it preserves order when F is ordered. Uniqueness of i follows from the fact that every homomorphism of fields preserves 0, 1, $+$, $-$ and $/$, and every element of $\mathbb{Q}$ can be expressed using these basic operations. Since i is defined in terms of the basic operations on F, it is clear that it is computable when the basic operations are. ∎

It is straightforward to check that a homomorphism of fields is an injective map. Thus, we can think of the rational numbers $\mathbb{Q}$ as a subspace of a field F, and omit explicit mention of the homomorphism $\mathbb{Q} \to F$.

Proposition 5.5.12 *Let F be an Archimedean field. For every $x \in F$ and every $k \geq 1$ there exist $a, b \in \mathbb{Q}$ such that $a < x < b$ and $b - a < 1/k$.*

Proof. First we prove that there always exist integers $a, b \in \mathbb{Z}$ such that $a < x < b$ and $b - a = 2$. We do this by constructing sequences of integers $\langle a_n \rangle_{n \in \mathbb{N}}, \langle b_n \rangle_{n \in \mathbb{N}}$ so that, for all $n \in \mathbb{N}$, $a_n < x < b_n$, $b_n - a_n \geq 2$, and if $b_n - a_n > 2$ then $b_n - a_n = b_{n+1} - a_{n+1} + 1$.

Because F is Archimedean, there exists $b_0 \in \mathbb{Z}$ such that $x < b_0$, and there exists $a_0 \in \mathbb{Z}$ such that $-x < -a_0$. This gives us $a_0 < x < b_0$. If $b_0 - a_0 < 2$, increase b_0 by 2. For the inductive step, suppose we have found a_n and b_n such that $a_n < x < b_n$ and $b_n - a_n \geq 2$. If $b_n - a_n = 2$ then let $a_{n+1} = a_n$ and $b_{n+1} = b_n$. Otherwise $a_n + 1 < b_n - 1$, and so we have $a_n + 1 < x$ or $x < b_n - 1$. By Number Choice, we can now define a_{n+1} and b_{n+1} so that

$$
\begin{aligned}
a_n + 1 < x &\longrightarrow (a_{n+1} = a_n + 1 \land b_{n+1} = b_n) \;, \\
x < b_n - 1 &\longrightarrow (a_{n+1} = a_n \land b_{n+1} = b_n - 1) \;.
\end{aligned}
$$

This completes the definition of $\langle a_n \rangle_{n \in \mathbb{N}}$ and $\langle b_n \rangle_{n \in \mathbb{N}}$. Now let $j = b_0 - a_0$. Then $b_j - a_j = 2$ and $b_j < x < a_j$. To finish the proof, apply the above construction to $4 \cdot k \cdot x$ to obtain integers a and b such that $a < 4 \cdot k \cdot x < b$ and $b - a = 2$. Then $a/(4k) < x < b/(4k)$ and $a/(4k) - b/(4k) = 1/(2k) < 1/k$. ■

Corollary 5.5.13 *Let F be an Archimedean field. For every $x \in F$ there exists a rational sequence $a \colon \mathbb{N} \to \mathbb{Q}$ such that $-2^{-k} < x - a_k < 2^{-k}$ for all $k \in \mathbb{N}$.*

Proof. By Proposition 5.5.12, for every $k \in \mathbb{N}$ there exist $c, d \in \mathbb{Q}$ such that $c < x < d$ and $c - d < 2^{-k}$, and so $-2^{-k} < x - (c + d)/2 < 2^{-k}$. By Number Choice, there is a map $a \colon \mathbb{N} \to \mathbb{Q}$ such that $2^{-k} < x - a_k < 2^{-k}$ for all $k \in \mathbb{N}$. ■

Corollary 5.5.14 *Let F be an Archimedean field. For all $x, y \in F$, if $x < y$ then there exists $q \in \mathbb{Q}$ such that $x < q < y$.*

Proof. Because F is Archimedean, there exists $k \in \mathbb{N}$ such that $y - x > 1/k > 0$. By Proposition 5.5.12, there exists $q \in \mathbb{Q}$ such that $-1/(4k) < (x + y)/2 - q < 1/(4k)$, therefore $x < q < y$. ■

Definition 5.5.15 Let F be an Archimedean field. A sequence $a \colon \mathbb{N} \to F$ *converges to* $x \in F$, written $\langle a_n \rangle_{n \in \mathbb{N}} \to x$, when

$$\forall\, k \in \mathbb{N} \,.\, \exists\, n \in \mathbb{N} \,.\, \forall\, m \in \mathbb{N} \,.\, (-1/k < x - a_{n+m} < 1/k) \;.$$

We say that x is a *limit* of $\langle a_n \rangle_{n \in \mathbb{N}}$. A sequence is *converging* if there exists a limit of it. A *Cauchy sequence* $a \colon \mathbb{N} \to F$ is a sequence that satisfies

$$\forall\, k \in \mathbb{N} \,.\, \exists\, n \in \mathbb{N} \,.\, \forall\, m, p \in \mathbb{N} \,.\, (-1/k < a_{n+m} - a_{n+p} < 1/k) \;.$$

It is straightforward to verify that every convergent sequence is a Cauchy sequence. An Archimedean field is *Cauchy complete* when every Cauchy sequence converges.

Proposition 5.5.16 *In an Archimedean field, a converging sequence has exactly one limit.*

Proof. Suppose $\langle a_n \rangle_{n\in\mathbb{N}} \to x$ and $\langle a_n \rangle_{n\in\mathbb{N}} \to y$. We show that, for all $z \in F$, $z < x$ if, and only if, $z < y$. Suppose $z < x$. There exists $k \in \mathbb{N}$ such that $x - z > 1/k > 0$. There exists $n \in \mathbb{N}$ such that $-1/(4k) < a_n - x < 1/(4k)$ and $-1/(4k) < y - a_n < 1/(4k)$. Now we see that

$$-1/(2k) = -1/(4k) - 1/(4k) < (y - a_n) + (a_n - x) = y - x ,$$

therefore $z < z + 1/(2k) < x - 1/(2k) < y$. The converse is proved similarly. By Proposition 5.5.7, $x = y$. ■

For an Archimedean field F, let $\mathsf{Conv}(F)$ be the space of converging sequences in F,

$$\mathsf{Conv}(F) = \left\{a\colon \mathbb{N} \to F \;\middle|\; \forall\, k \in \mathbb{N}\,.\, \exists\, n \in \mathbb{N}\,.\, \forall\, m \in \mathbb{N}\,.\, (-1/k < x - a_{n+m} < 1/k)\right\} .$$

By Proposition 5.5.16, for every $a \in \mathsf{Conv}(F)$ there exists a unique $x \in F$ such that $\langle a_n \rangle_{n\in\mathbb{N}} \to x$. By Unique Choice, there exists a map $\lim_F \colon \mathsf{Conv}(F) \to F$, called the *limit operator*, such that, for all $a \in \mathsf{Conv}(F)$, $\langle a_n \rangle_{n\in\mathbb{N}} \to \lim_F a$.

Corollary 5.5.17 *If F is a computable field then* $\lim_F$ *is computable.*

Proof. The operator $\lim_F$ is defined by Unique Choice in terms of the injection $\mathbb{Q} \to F$ and the basic operations on F. Therefore, $\lim_F$ is computable provided that F is. ■

Theorem 5.5.18 *Every two (computable) Cauchy complete Archimedean fields are canonically (and computably) isomorphic.*

Proof. Let F and G be Cauchy complete Archimedean fields. Define a relation $\rho \subseteq F \times G$ by

$$\rho(x, y) \longleftrightarrow \exists\, a \in \mathbb{Q}^{\mathbb{N}}\,.\, (x = \lim_F a \wedge y = \lim_G a) .$$

It suffices to show that ρ is a bijective correspondence. By Unique Choice we then obtain the desired isomorphisms.

First we prove that ρ is total. Suppose $x \in F$. By Corollary 5.5.13, there exists a Cauchy sequence $a\colon \mathbb{N} \to \mathbb{Q}$ such that $x = \lim_F a$. Let $y = \lim_G a$. Then $\rho(x, y)$ holds. A symmetric argument shows that for every $y \in G$ there exists $x \in F$ such that $\rho(x, y)$.

To prove uniqueness, suppose $\rho(x, y)$ and $\rho(x, z)$ hold. Then there exist Cauchy sequences $a, b\colon \mathbb{N} \to \mathbb{Q}$ such that $x = \lim_F a = \lim_F b$, $y = \lim_G a$ and $z = \lim_G b$. Because $\lim_F a = \lim_F b$, it follows that a and b coincide, therefore $y = \lim_G a = \lim_G b = z$. A symmetric arguments shows that $\rho(x, z)$ and $\rho(y, z)$ implies $x = y$.

Once again, computability of the isomorphisms follows from computability of F and G because the isomorphisms are defined by Unique Choice from the basic operations on F and G. ■

Let us now show that $\mathbb{R}$ is a computable Cauchy complete Archimedean field. The real numbers inherit the algebraic structure of a ring from the space of Cauchy Sequences, as follows:

$$0 = [0]_{n \in \mathbb{N}} \;, \qquad 1 = [1]_{n \in \mathbb{N}} \;, \qquad -[a_n] = [-a_n] \;,$$
$$[a_n] + [b_n] = [a_n + b_n] \;,$$
$$[a_n] \cdot [b_n] = [a_n \cdot b_n] \;.$$

Of course, we would need to check that these operations are well defined, i.e., that pointwise addition and multiplication on $\mathcal{C}$ map Cauchy sequences to Cauchy sequences, and preserve the coincidence relation. The proofs are analogous to classical presentations of this subject and are omitted. Additionally, the real numbers inherit from the rationals the absolute value map, $|\Box| \colon \mathbb{R} \to \mathbb{R}$, which is defined by

$$|[a_n]| = [|a_n|] \;.$$

Define an order relation on $\mathbb{R}$ by

$$[a_n] < [b_n] \longleftrightarrow \exists\, j,k \in \mathbb{N} \,.\, \forall\, m,n \in \mathbb{N} \,.\, (b_{j+n} - a_{j+m} > 1/k) \;.$$

We need to show that $<$ is well defined. Suppose $a \approx a'$, $b \approx b'$, and there exist $j,k \in \mathbb{N}$ such that $b_{j+n} - a_{j+m} > 1/k$ for all $m,n \in \mathbb{N}$. There is $i \geq j$ such that $|{a'}_{i+n} - a_{i+n}| < 1/(3k)$ and $|b_{i+n} - {b'}_{i+n}| < 1/(3k)$, for all $n \in \mathbb{N}$. Therefore, for all $n,m \in \mathbb{N}$,

$$\begin{aligned} |{a'}_{i+m} - {b'}_{i+n}| \geq \\ & |a_{i+m} - b_{i+n}| - |{a'}_{i+m} - a_{i+n}| - |{b'}_{i+n} - b_{i+n}| \\ & \qquad > 1/k - 2/(3k) = 1/(3k) \;. \end{aligned}$$

This proves that $[a'] < [b']$, and so $<$ is well defined. We can also express $<$ in terms of rapidly converging sequences as

$$[a_n]_{\mathrm{r}} < [b_n]_{\mathrm{r}} \longleftrightarrow \exists\, k \in \mathbb{N} \,.\, b_k - a_k > 2^{-k+2} \;.$$

Recall that $x \leq y$ is defined as $\neg(y < x)$. By Markov's Principle, it can be expressed equivalently in terms of Cauchy sequences as

$$[a_n] \leq [b_n] \longleftrightarrow \forall\, j,k \in \mathbb{N} \,.\, \exists\, n,m \in \mathbb{N} \,.\, (a_{j+n} - b_{j+m} \leq 1/k) \;,$$

or in terms of rapidly converging sequences as

$$[a_n]_{\mathrm{r}} \leq [b_n]_{\mathrm{r}} \longleftrightarrow \forall\, k \in a_k - b_k \leq 2^{-k+2} . \;.$$

As we mentioned earlier, the apartness relation and inequality coincide on $\mathbb{R}$.

Proposition 5.5.19 *For all $x, y \in \mathbb{R}$, $x \neq y \longleftrightarrow (x < y \vee y < x)$.*

Proof. If $x < y$ then $x \neq y$, and if $y < x$ then $x \neq y$. Therefore $x < y \vee y < x$ implies $x \neq y$. Conversely, suppose $x \neq y$. There exist rapidly converging sequences a and b such that $x = [a_n]$, $y = [b_n]$, and $\neg(a \sim b)$. By Markov's Principle, there exists $k \in \mathbb{N}$ such that $|a_k - b_k| > 2^{-k+2}$. Because $<$ is decidable on the rational numbers, it follows that $a_k < b_k$ or $b_k < a_k$. We prove that $a_k < b_k$ implies $x < y$, and a symmetric argument shows that $b_k < a_k$ implies $y < x$. If $a_k < b_k$ then, for all $m, n \in \mathbb{N}$,

$$b_{k+n} - a_{k+m} \geq |a_k - b_k| - |b_{k+n} - b_k| - |a_{k+m} - a_k| > \\ |a_k - b_k| - 2^{-k+1} - 2^{-k+1} = |a_k - b_k| - 2^{-k+2} .$$

Therefore, $x = [a_n] < [b_n] = [y]$ because there is a positive integer whose reciprocal value is smaller than $|a_k - b_k| - 2^{-k+2}$. ■

Theorem 5.5.20 *The real numbers are a computable Cauchy complete Archimedean ordered field.*

Proof. The algebraic structure of a ring is inherited from the space of Cauchy sequences. Zero, one, addition, multiplication, and negation are computable because they are defined explicitly in terms of λ-abstraction from the corresponding operations on rationals. This leaves us with proving that inverse $^{-1}$ is well defined. Suppose $x \in \mathbb{R}$ and $x \neq 0$. There is a rapidly converging sequence $\langle a_n \rangle$ such that $x = [a_n]$. Because $x \neq 0$, by Markov's Principle, there exists $k \in \mathbb{N}$ such that $|a_k| > 2^{-k+2}$. For every $m \geq k$, $|a_m - a_k| < 2^{-k+1}$ and so $|a_m| > 2^{-k+1}$. Define the sequence $\langle b_n \rangle$ by $b_n = 1/a_{k+n}$, and $\langle c_n \rangle$ by $c_n = a_{k+n}$. Clearly, $[c_n] = [a_n] = x$, and $[b_n \cdot c_n] = 1$. It is not hard to verify that $\langle b_n \rangle$ is a Cauchy sequence, hence we can define $x^{-1} = [b_n]$. This proves that $\mathbb{R}$ is a computable field.

The order relation $<$ satisfies the axioms from Definition 5.5.6. We omit the proofs since they are not complicated and can be found in [TvD88a, Chapter 5, Section 2]. Instead, we show that $<$ is semidecidable. Define a map $l_0 \colon \mathcal{C}_\mathrm{r} \times \mathcal{C}_\mathrm{r} \to 2^{\mathbb{N}}$ by

$$(l_0\langle a, b\rangle)n = \begin{cases} 1 & \text{if } b_n - a_n > 2^{-n+2} , \\ 0 & \text{otherwise} . \end{cases}$$

The sequence $l_0\langle a, b\rangle$ contains a 1 if, and only if, $[a_n]_\mathrm{r} < [b_n]_\mathrm{r}$. Define a map $l \colon \mathbb{R}_\mathrm{r} \times \mathbb{R}_\mathrm{r} \to \Sigma$ by

$$l\,\langle [a_n]_\mathrm{r}, [b_n]_\mathrm{r}\rangle = [l_0\langle a, b\rangle]_\Sigma .$$

By using the characterization of $<$ in terms of rapidly converging sequences, we can verify that l is well defined and that, for all $x, y \in \mathbb{R}$,

$$l, \langle x, y\rangle = \top \longleftrightarrow x < y .$$

Clearly, l is computable.

The ordered field $\mathbb{R}$ is Archimedean. Indeed, if $x \in \mathbb{R}$, there exists a rapidly converging sequence $\langle a_n \rangle$ such that $x = [a_n]$. There exists an integer $n > a_0 + 3$, and so $x < n$. By Proposition 5.5.9, $\mathbb{R}$ is Archimedean.

Lastly, we prove that $\mathbb{R}$ is Cauchy complete. Suppose $a \colon \mathbb{N} \to \mathbb{R}$ is a Cauchy sequence. Because $\mathbb{N}$ is projective there exists a map $b \colon \mathbb{N} \to \mathcal{C}_\mathrm{r}$ such that $ak = [(bk)n]_{n \in \mathbb{N}}$ for all $k \in \mathbb{N}$. Consider the sequence $c \in \mathcal{C}$ defined by $c_n = b\,n\,n$. We claim that c is a Cauchy sequence and that $\langle a_n \rangle_{n \in \mathbb{N}} \to [c_n]$. For an arbitrary $k \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that $12k < 2^n$ and $|a_{n+m} - a_n| < 1/(3k)$ for all $m \in \mathbb{N}$, from which we get

$$\begin{aligned}|c_{n+m} - c_n| = |b(n+m)(n+m) - b\,n\,n| \le \\ |b(n+m)(n+m) - a_{n+m}| + |a_{n+m} - a_n| + |a_n - b\,n\,n| < \\ 2^{-n-m+2} + \frac{1}{3k} + 2^{-n+2} < \frac{1}{3k} + \frac{1}{3k} + \frac{1}{3k} = \frac{1}{k}\,.\end{aligned}$$

Therefore, c is a Cauchy sequence. Let $x = [c_n]$. For every $k \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that $8k < 2^n$ and $|b(n+m)(n+m) - x| < 1/(2k)$ for all $m \in \mathbb{N}$, hence

$$\begin{aligned}|a_{n+m} - x| \le |a_{n+m} - b(n+m)(n+m)| + |b(n+m)(n+m) - x| < \\ 2^{-n-m+2} + \frac{1}{2k} < \frac{1}{2k} + \frac{1}{2k} = \frac{1}{k}\,.\end{aligned}$$

Therefore, $\langle a_n \rangle_{n \in \mathbb{N}} \to x$. ■

Remark 5.5.21 If we put Theorems 5.5.18 and 5.5.20 together we obtain the result that every computable Cauchy complete Archimedean field is computably isomorphic to $\mathbb{R}$. When this statement is interpreted in $\mathsf{Mod}(\mathbb{B}, \mathbb{B}_\sharp)$ it is precisely a result by Hertling [Her99] on the effective categoricity of the structure of real numbers. We see again the benefit of developing analysis in the logic of modest sets. We have proved an intuitionistic version of a standard theorem about the real numbers that can be found in any textbook on algebra, and its interpretation in $\mathsf{Mod}(\mathbb{B}, \mathbb{B}_\sharp)$ is a recent result about computability on the real numbers. In addition, our theorem can be interpreted in *any* category of modest sets, not just $\mathsf{Mod}(\mathbb{B}, \mathbb{B}_\sharp)$, to give us analogous results for other models of computation.

Hertling [Her99] observes that there are several possibilities for defining the structure of real numbers. He proves [Her99, Proposition 3.4] that the structure $R_1 = (\mathbb{R}, 0, 1, +, -, \cdot, <, \mathsf{CauchyLim})$ is *not* effectively categorical, even though one might expect it to be. He concludes that the problems with the limit operator $\mathsf{CauchyLim}$ arise because Cauchy sequences do not have a known rate of convergence. He proves that the correct structure to take is $R_2 = (\mathbb{R}, 0, 1, +, -, \cdot, <, \mathsf{NormLim})$, where only the rapidly converging Cauchy sequences are taken. We can explain the difference between R_1 and R_2 in the logic of modest sets very easily: R_2 is the space of Cauchy reals $\mathbb{R}$, whereas R_1 is the space of "not-not-Cauchy reals", defined by

$$\mathcal{C}_{\neg\neg} = \left\{ a \in \mathbb{Q}^{\mathbb{N}} \mid \neg\neg\forall\, k \in \mathbb{N}\,.\,\exists\, n \in \mathbb{N}\,.\,\forall\, m \in \mathbb{N}\,.\,|a_{n+m} - a_n| < 1/k \right\}\,, \qquad R_1 = \mathcal{C}_{\neg\neg}/\!\approx\,.$$

It is only to be expected that inserting a gratuitous double negation in the definition will destroy the computational structure of the space of Cauchy sequences. The space R_1 is *not* isomorphic to $\mathbb{R}$, and this is why it is a bad representation of the reals numbers. The logic of modest sets is guiding us in choosing the correct computational structure–where there are choices, it is best to follow the one that is most *logical.*

Here are some further properties of the order relation on reals.

Proposition 5.5.22 *For all $x, y, z \in \mathbb{R}$:*

(1) $x < y \longrightarrow x \le y$

(2) $x \le y \wedge y < z \longrightarrow x < z$

(3) $x < y \wedge y \le z \longrightarrow x < z$

(4) $x < y \wedge y < z \longrightarrow x < z$

(5) $x \le y \wedge y \le z \longrightarrow x \le z$

(6) $x < y \longrightarrow (x < z \vee z < y)$

(7) $x \le y \longleftrightarrow \neg\neg(x < y \vee x = y)$

(8) $\neg\neg(x \le y \vee y \le x)$

(9) $\neg\neg(x < y \vee x = y \vee x > y)$.

Proof. We omit detailed proofs, as they are straightforward and can be found in [TvD88a, Chapter 5, Proposition 2.11]. ∎

Theorem 5.5.23 *Every Archimedean field F is a subfield of the real numbers $\mathbb{R}$. If F is a computable field then the injection $F \to \mathbb{R}$ is computable.*

Proof. Let F be an Archimedean field. We define a map $i \colon F \to \mathbb{R}$ as follows. By Corollary 5.5.13, for every $x \in F$, there exists a *rational* converging sequence $a \colon \mathbb{N} \to \mathbb{Q}$ such that $\lim_F a = x$. Define $ix = \lim_{\mathbb{R}} a$. This is well defined because $\lim_F a = x = \lim_F b$ implies that a and b coincide, hence they have the same limit in $\mathbb{R}$.

If F is computable then i is computable because it is defined by Unique Choice in terms of the basic operations on F and $\mathbb{R}$, which are computable. ∎

Corollary 5.5.24 *In an Archimedean field, apartness coincides with inequality.*

Proof. Let $i \colon F \to \mathbb{R}$ be the canonical inclusion. If $x \lessgtr y$ then $ix \lessgtr iy$. By Proposition 5.5.19, $ix \neq iy$, therefore $x \neq y$ since i is injective. ∎

Corollary 5.5.25 *An Archimedean field F has a semidecidable order relation. If F is a computable field, then it is a computable ordered field.*

Proof. Let $l \colon \mathbb{R} \times \mathbb{R} \to \Sigma$ be the characteristic map of $<$ on $\mathbb{R}$. Let $i \colon F \to \mathbb{R}$ be the injection from Theorem 5.5.23. For all $x, y \in F$,

$$x < y \longleftrightarrow ix < iy \longleftrightarrow l \langle ix, iy \rangle = \top ,$$

therefore $l \circ (i \times i) \colon F \times F \to \Sigma$ is the characteristic map of $<$ on F. If F is computable then i is computable, and so is $l \circ (i \times i)$. ∎

We end this section by defining some more maps on the real numbers. As mentioned earlier, we have the absolute value map $|\square|\colon \mathbb{R} \to \mathbb{R}$, which is characterized by the property that, for all $x \in \mathbb{R}$,

$$-|x| \le x \le |x| \,, \qquad \forall\, y \in \mathbb{R} \,.\, (-y \le x <\le y \longrightarrow |x| \le y) \,.$$

It follows that for all $x \in \mathbb{R}$, $|x| \geq 0$, $|x| = 0 \longleftrightarrow x = 0$, $x \geq 0 \longleftrightarrow x = |x|$, and $x \leq 0 \longleftrightarrow x = -|x|$. From absolute value, we can define the maximum $\max : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and the minimum $\min : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$\max(x, y) = (x + y + |x - y|)/2 \,, \qquad \min(x, y) = (x + y - |x - y|)/2 \,,$$

These two maps are characterized by the property that, for all $x, y, \in \mathbb{R}$,

$$\forall\, z \in \mathbb{R} \,.\, (\max(x, y) < z \longleftrightarrow x < z \wedge y < z) \,,$$
$$\forall\, z \in \mathbb{R} \,.\, (z < \min(x, y) \longleftrightarrow z < x \wedge z < y) \,.$$

Lastly, we define the square root function. In fact, what we define is the square root of an absolute value, in order to avoid problems with negative numbers. Let $\mathbb{Q}_+ = \{a \in \mathbb{Q} \mid a > 0\}$, and let $\mathbb{Q}_+^0 = \{a \in \mathbb{Q} \mid a \geq 0\}$.

Lemma 5.5.26 *For every $p, q \in \mathbb{Q}_+^0$ such that $p < q$ there exists $r \in \mathbb{Q}_+$ such that $p - q < r^2 < p + q$. Thus, by Number Choice there exists a function $r\colon \mathbb{Q}_+^0 \times \mathbb{Q}_+^0 \to \mathbb{Q}_+$ such that $p - q < (r\,\langle p, q\rangle)^2 < p + q$ for all $p, q \in \mathbb{Q}_+$.*

Proof. Left as an exercise in number theory. ∎

Define a map $f\colon \mathcal{C} \to \mathcal{C}$ by $(fa)n = r\,\langle |a_n|, 2^{-n}\rangle$. It is a simple matter of juggling inequalities to show that fa is a Cauchy sequence and that f respects coincidence of Cauchy sequences. Thus, f induces a map $\mathbb{R} \to \mathbb{R}$, which is the map we are looking for, i.e.,

$$x \mapsto \sqrt{|x|} \,.$$

It is characterized by the property that $\sqrt{x^2} = |x|$ for all $x \in \mathbb{R}$.

5.5.4 Discontinuity of Real Maps

We prove a theorem that relates WCP to non-existence of discontinuous real maps. In an intuitionistic setting the statement (a) "there are no discontinuous real maps" is weaker than the statement (b) "all real maps are continuous". The former only claims that we will never encounter an explicit discontinuity, whereas the latter gives us evidence that all maps are continuous. In the logic of modest sets neither statement holds in general, but under the very reasonable assumption that the underlying computational model does not decide all Π^1_1 statements, the weaker one holds. The stronger statement holds only in specific models, such as $\mathsf{Mod}(\mathbb{B}, \mathbb{B}_\sharp)$.

Which statement, (a) or (b), is the one that people usually have in mind when they say (c) "in our model of computation all real maps are continuous"? Note that (a) and (b) are stated in the internal logic, whereas (c) is expressed in classical set theory. The interpretation of (a) expressed in set theory is "the computational model $\mathsf{Mod}(\mathbb{A}, \mathbb{A}_\sharp)$ does not contain any discontinuous real maps". The interpretation of (b) expressed in set theory is "in the computational model $\mathsf{Mod}(\mathbb{A}, \mathbb{A}_\sharp)$, given a program that computes a real map f, we can effectively compute a modulus of continuity for f". So it is the statement (a) that is equivalent to (c), not (b).

Theorem 5.5.27 *The space $2^{\mathbb{N}}$ is decidable if, and only if, $\mathbb{R}$ is decidable.*

Proof. Let $o = \lambda n : \mathbb{N} . 0$. If $\mathbb{R}$ is decidable then the characteristic map of equality on $2^{\mathbb{N}}$ can be defined as

$$\mathsf{eq}_{2^{\mathbb{N}}} \langle f, g \rangle = \begin{cases} 1 & \text{if } \sum_{i=0}^{\infty} |f i - g i| \cdot 2^{-i} = 0 \;, \\ 0 & \text{otherwise} \;. \end{cases}$$

Conversely, let $x, y \in \mathbb{R}$. There exist rapidly converging sequences $a, b \in \mathcal{C}_r$ such that $x = [a]$ and $y = [b]$. Now $x = y$ holds if, and only if, $\forall\, n \in \mathbb{N} .\, |a_n - b_n| \leq 2^{-n+2}$. Define a map $f \colon \mathbb{N} \to 2$ by

$$f n = \begin{cases} 0 & \text{if } |a_n - b_n| \leq 2^{-n+2} \;, \\ 1 & \text{otherwise} \;. \end{cases}$$

By assumption $2^{\mathbb{N}}$ is decidable, hence $f = o$ or $f \neq o$. If $f = o$ then $x = y$, and if $f \neq o$, then $x \neq y$. ■

Definition 5.5.28 A map $f \colon \mathbb{R} \to \mathbb{R}$ is *discontinuous at* $x \in \mathbb{R}$ when

$$\exists\, \epsilon > 0 \,.\, \forall\, \delta > 0 \,.\, \exists\, y \in \mathbb{R} \,.\, (|x - y| < \delta \wedge |f x - f y| > \epsilon) \;.$$

A map $f \colon \mathbb{R} \to \mathbb{R}$ is *discontinuous* when there exists $x \in \mathbb{R}$ such that f is discontinuous at x.

Equivalently, we could state discontinuity at x as

$$\exists\, k \in \mathbb{N} .\, \forall\, m \in \mathbb{N} .\, \exists\, y \in \mathbb{R} .\, \Big(|x - y| < 2^{-m} \wedge |f x - f y| > 2^{-k} \Big) \;.$$

Lemma 5.5.29 *Suppose $\langle a_n \rangle_{n \in \mathbb{N}}$ is a convergent sequence in $\mathbb{R}$. Then there exists a map $f \colon \mathbb{N}^+ \to \mathbb{R}$ such that $f n = a_n$ for all $n \in \mathbb{N}$, and $f \infty = \lim_{n \to \infty} a_n$.*

Proof. Define a map $s \colon \mathbb{N}^+ \to \mathbb{N} \to \mathbb{R}$ by

$$s\, x\, m = \begin{cases} a_m & \text{if } m \leq x \;, \\ a_x & \text{otherwise} \;. \end{cases}$$

This is well defined because $\leq$ is a decidable relation by Proposition 5.3.26, and $\neg(m \leq x)$ implies $x \in \mathbb{N}$. For all $x \in \mathbb{N}^+$ and for all $n, m \in \mathbb{N}$, $|s\, x\, n - s\, x\, m| \leq |a_n - a_m|$, as is easily established by considering four cases: (a) $m \leq x \wedge n \leq x$, (b) $m \not\leq x \wedge n \leq x$, (c) $m \leq x \wedge n \not\leq x$, (d) $m \not\leq x \wedge \not\leq n \leq x$. Let us show that, for every $x \in \mathbb{N}^+$, $\langle s\, x\, m \rangle_{m \in \mathbb{N}}$ is a Cauchy sequence. Because $\langle a_n \rangle_{n \in \mathbb{N}}$ is a Cauchy sequence, for every $k \geq 1$ there exists $n \in \mathbb{N}$ such that, for all $m \in \mathbb{N}$, $|a_m - a_n| < 1/k$, therefore

$$|s\, x\, (m + n) - s\, x\, m| \leq |a_{m+n} - a_m| < \frac{1}{k} \;.$$

Now we can define a map $f \colon \mathbb{N}^+ \to \mathbb{R}$ by

$$f x = \lim_{n \to \infty} (s\, x\, n) \;.$$

It is straightforward to check that f has the desired properties. ■

Theorem 5.5.30 *The principle* WCP *holds if, and only if, there does* not *exist a discontinuous map* $\mathbb{R} \to \mathbb{R}$.

Proof. Suppose f is discontinuous. We may assume that f is discontinuous at 0 and that $f0 = 0$, i.e., there exists $k \in \mathbb{N}$ such that

$$\forall\, m \in \mathbb{N} \,.\, \exists\, y \in \mathbb{R} \,.\, \left(|y| < 2^{-m} \wedge |fy| > 2^{-k}\right) \ .$$

By Number Choice, there exists a map $y \colon \mathbb{N} \to \mathbb{R}$ such that, for all $m \in \mathbb{N}$,

$$|y_m| < 2^{-m} \text{ and } |fy_m| > 2^{-k} \ .$$

Hence $\lim_{m\to\infty} y_m = 0$, and by Lemma 5.5.29 there exists a map $g \colon \mathbb{N}^+ \to \mathbb{R}$ such that $g\,m = y_m$, for all $m \in \mathbb{N}$, and $g\,\infty = 0$. By Corollary 5.5.25, the predicate $x < 2^{-k-1}$, $x \in \mathbb{R}$, is semidecidable. Let $r \colon \mathbb{R} \to \Sigma$ be its characteristic map, i.e., $\forall\, x \in \mathbb{R} \,.\, \left(x < 2^{-k-1} \longleftrightarrow rx = \top\right)$. Consider the map $h = r \circ f \circ g \colon \mathbb{N}^+ \to \Sigma$. We have $h\,\infty = r(f0) = r0 = \top$ and, for all $m \in \mathbb{N}$, $hm = r(fy_m) = \bot$. The map h contradicts condition (5.6) from Theorem (5.3.28), hence WCP does not hold.

Conversely, suppose $f \colon \mathbb{N} \to 2$ and $fn = 1$ for all $n \in \mathbb{N}$. If it were the case that $f\infty = 0$, then WCP would fail, and by Theorem 5.5.27 the equality on reals would be decidable, so we could define a discontinuous map $g \colon \mathbb{R} \to \mathbb{R}$ by

$$gx = \begin{cases} 1 & \text{if } x = 0 \ , \\ 0 & \text{if } x \neq 0 \ . \end{cases}$$

By assumption, there is no discontinuous map $\mathbb{R} \to \mathbb{R}$, therefore $f\,\infty = 1$. ∎

5.6 Metric Spaces

In this section we review the basic theory of metric spaces and relate it to topological bases and continuity. The intuitionistic theory of metric spaces is well developed, so we can just follow a standard text on the subject [TvD88a, TvD88b, BB85]. The original part of this section is the relationship between metric and (intrinsic) topology, and the statements about computability.

Definition 5.6.1 A *metric* on a space A is a map $d \colon A \times A \to \mathbb{R}$ such that, for all $x, y, z \in A$:

(1) $d(x, y) \geq 0$,

(2) $d(x, y) = 0$ if, and only if, $x = y$,

(3) $d(x, y) = d(y, x)$,

(4) $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality).

When d is a metric on A, we say that (A, d) is a *metric space*. A *computable metric space* is one whose metric is computable.

Example 5.6.2 (Euclidean Metric) The real numbers $\mathbb{R}$ form a computable metric space for the *Euclidean metric*,

$$d(x,y) = |x-y| \ .$$

More generally, for every $n \in \mathbb{N}$, the Euclidean metric on $\mathbb{R}^n$ is defined by

$$d(x,y) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2} \ .$$

Clearly, it is computable. It is straightforward to check that d satisfies the first three conditions. The triangle inequality for d follows from Minkowski's inequality, proved in the following lemma:

$$d(x,z) = \sqrt{\sum_{i=1}^{n}((x_i - y_i) - (y_i - z_i))^2} \leq \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2} + \sqrt{\sum_{i=1}^{n}(y_i - z_i)^2} = d(x,y) + d(y,z) \ .$$

Lemma 5.6.3 (Minkowski's Inequality) *Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ be real numbers. Then*

$$\sqrt{\sum_{i=1}^{n}(x_i - y_i)^n} \leq \sqrt{\sum_{i=1}^{n} x_i^2} + \sqrt{\sum_{i=1}^{n} y_i^2} \ .$$

Proof. This follows from Cauchy-Schwarz inequality, which is proved in the next lemma:

$$\sum_{i=1}^{n}(x_i - y_i)^2 = \sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} y_i^2 - 2\sum_{i=1}^{n} x_i y_i \leq \sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} y_i^2 + 2\sqrt{\sum_{i=1}^{n} x_i^2} \cdot \sqrt{\sum_{i=1}^{n} y_i^2} = \left(\sqrt{\sum_{i=1}^{n} x_i^2} + \sqrt{\sum_{i=1}^{n} y_i^2}\right)^2 \ .$$

■

Lemma 5.6.4 (Cauchy-Schwarz Inequality) *Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ be real numbers. Then*

$$\left|\sum_{i=1}^{n} x_i y_i\right| \leq \sqrt{\sum_{i=1}^{n} x_i^2} \cdot \sqrt{\sum_{i=1}^{n} y_i^2} \ .$$

Proof. The inequality is equivalent to

$$\sum_{i=1}^{n} x_i^2 \cdot \sum_{i=1}^{n} y_i^2 - \left(\sum_{i=1}^{n} x_i y_i\right)^2 =$$

$$\frac{1}{2} \cdot \sum_{i=1}^{n} x_i^2 \cdot \sum_{i=1}^{n} y_i^2 + \frac{1}{2} \cdot \sum_{i=1}^{n} x_i^2 \cdot \sum_{i=1}^{n} y_i^2 - \sum_{i=1}^{n} x_i y_i \cdot \sum_{i=1}^{n} x_i y_i =$$

$$\sum_{i,j=1}^{n} \frac{1}{2}\left(x_i^2 y_j^2 + x_j^2 x_i^2 - 2 x_i y_j x_j y_i\right) = \sum_{i,j=1}^{n} \frac{1}{2}\left(x_i y_j - x_j y_i\right)^2 \geq 0 \ .$$

■

The point of showing these manipulations is, apart from beautiful LaTeX 2_ε typesetting, that they are identical to the standard ones, found in any textbook on analysis. The reason behind this is that Cauchy-Schwarz inequality, Minkowski's Inequality, and the triangle inequality are just universally quantified inequalities, and so they are negative formulas, whose set-theoretic interpretation is the same as the interpretation in modest sets.

Example 5.6.5 (Discrete Metric) If A is a decidable space, then the transposition of the characteristic map of equality is a metric, i.e.,

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \ , \\ 1 & \text{if } x \neq y \ . \end{cases}$$

This is the *discrete metric.* Note that the underlying space must be decidable, otherwise the definition of discrete metric is invalid, since we cannot define it by cases. The discrete metric is computable if, and only if, the characteristic map for equality is computable.

Definition 5.6.6 Let (A, d_A) and (B, d_B) be metric spaces. The *product metric* $d_{A\times B} \colon (A \times B) \times (A \times B) \to \mathbb{R}$ is defined by

$$d_{A\times B}(\langle u, x\rangle, \langle v, y\rangle) = \sqrt{d_A(u, v)^2 + d_B(x, y)^2} \ .$$

The proof that the product metric really is a metric is analogous to the proof that the Euclidean metric on $\mathbb{R}^n$ is a metric. It is straightforward to extend the definition of product metric to a product of finitely many spaces. The Euclidean metric on $\mathbb{R}^n$ is just the product metric formed from n copies of the Euclidean metric on $\mathbb{R}$. The product metric of computable metrics is computable.

Definition 5.6.7 Let (A, d_A) be a metric space and $B \subseteq A$ a subspace. The *subspace metric* $d_B \colon B \times B \to \mathbb{R}$ is defined, for all $x, y \in B$, by

$$d_B(x, y) = d_A(x, y) \ .$$

Definition 5.6.8 In a metric space (A, d), the *open ball* with radius $r \in \mathbb{R}$, $r > 0$, and centered at $x \in A$ is the subspace

$$\mathrm{B}(x, r) = \left\{y \in A \mid d(x, y) < r\right\} \ .$$

The *closed ball* with radius r, centered at x, is the subspace

$$\overline{\mathrm{B}}(x, r) = \left\{y \in A \mid d(x, y) \leq r\right\} \ .$$

Note that open and closed balls are regular subspaces because inequality on reals is stable. Furthermore, an open ball $\mathrm{B}(x,r)$ is an open subspace of A in the intrinsic topology of A, because inequality on reals is not only stable but also semidecidable. Similarly, a closed ball is a closed subspace in the intrinsic topology.

Definition 5.6.9 A subspace $B \subseteq A$ of a metric space (A,d) is *dense* when for all $x \in A$ and $k \in \mathbb{N}$ there exists $y \in B$ such that $d(x,y) < 1/k$. A metric space (A,d) is *separable* if it contains a dense countable subspace.

Example 5.6.10 The reals with the Euclidean metric are a separable metric space because $\mathbb{Q} \subseteq \mathbb{R}$ is a dense countable subset. A product of separable metric spaces is again a separable metric space.

Caution, a subspace of a separable metric space need not be separable. Classically, this is the case, but the proof requires the use of the axiom of choice.

Proposition 5.6.11 *A separable metric space is countably based. More precisely, let (A,d) be a separable metric space with a dense countable subspace $S \subseteq A$. The family of open balls $\{\mathrm{B}(a,1/k) \mid a \in S \wedge k \in \mathbb{N} \setminus \{0\}\}$ forms a countable basis on A, called* the metric basis generated *by S.*

Proof. Let $B = S \times (\mathbb{N} \setminus \{0\})$ and define $\in_B$ to be the relation

$$x \in_B \langle a,k\rangle \longleftrightarrow x \in \mathrm{B}(a,1/k)\ .$$

Since $x \in_B \langle a,k\rangle$ is equivalent to $d(a,x) < 1/k$, $\in_B$ is semidecidable. Because S and $\mathbb{N} \setminus \{0\}$ are countable, so is B. Therefore, B is a prebasis on A. Because S is a dense countable subspace of A, there exists for every $x \in A$ some $a \in A$ such that $d(x,s) < 1$, hence $x \in \mathrm{B}(a,1)$. Hence, A is open with respect to B. By Corollary 5.4.6 it suffices to show that there is a strong inclusion for B. Define $\prec$ on B by

$$\langle a,k\rangle \prec \langle b,m\rangle \longleftrightarrow d(a,b) + \frac{1}{k} < \frac{1}{m}\ .$$

Clearly, this is a semidecidable relation. It is easy to check that $\langle a,k\rangle \prec \langle b,m\rangle$ implies $\mathrm{B}(a,1/k) \subseteq \mathrm{B}(b,1/m)$. Suppose $x \in \mathrm{B}(a,1/k)$ and $x \in \mathrm{B}(b,1/m)$. There exists a positive integer n such that $d(x,a) + 1/n < 1/m$ and $d(x,b) + 1/n < 1/k$. Because S is dense, there exists $c \in S$ such that $d(x,c) < 1/(3n)$. Then $x \in \mathrm{B}(c,1/(3n))$, $\langle c,3n\rangle \prec \langle a,k\rangle$ and $\langle c,3n\rangle \prec \langle b,k\rangle$. ■

Note that the metric topology depends on the choice of the dense countable subspace S. However, this dependency is inessential, because different choices of dense countable subspaces result in canonically homeomorphic countably based spaces, which is an easy consequence of the following proposition.

Proposition 5.6.12 *Suppose $S \subseteq A$ is a dense countable subspace of a metric space (A,d). For all $x \in A$, $r \in \mathbb{R}$, $r > 0$, the open ball $\mathrm{B}(x,r)$ is open with respect to the metric basis generated by S.*

Proof. Let $\langle a_\square, k_\square\rangle \colon \mathbb{N} \to A \times (\mathbb{N} \setminus \{0\})$ be an enumeration of the metric basis generated by S. Let $x \in A$, $r \in \mathbb{R}$, and $r > 0$. Define $C \in \mathcal{O}(\mathbb{N})$ by

$$n \in C \longleftrightarrow d(x, a_n) + 1/k_n < r \ .$$

Let $T = \{y \in A \mid \exists\, n \in C \,.\, y \in \mathrm{B}(a_n, 1/k_n)\}$, which is open with respect to the basis by Proposition 5.4.4. The open ball $\mathrm{B}(x,r)$ is open with respect to the basis because, for all $y \in A$, $y \in T$ if, and only if, $y \in \mathrm{B}(x,r)$. ■

Proposition 5.6.13 *The metric topology is Hausdorff. Therefore, the intrinsic topology of a metric space is Hausdorff.*

Proof. Let (A,d) be a metric space. Suppose $x \neq y$. Then $d(x,y) > 0$. There exists $k \in \mathbb{N}$ such that $1/k < d(x,y)$. Now $x \in \mathrm{B}(x, 1/(3k))$, $y \in \mathrm{B}(y, 1/(3k))$, $\mathrm{B}(x,1/(3k)) \cap \mathrm{B}(y,1/(3k)) = \emptyset$, and open balls are always open in the intrinsic topology. ■

Continuous and Uniformly Continuous Maps

In a metric space, pointwise continuity of maps is equivalent to the usual ϵ-δ continuity.

Proposition 5.6.14 *Let (A, d_A) and (B, d_B) be metric spaces. A map $f \colon A \to B$ is pointwise continuous in the metric topology if, and only if, it is ϵ-δ continuous, which means that for all $x \in A$,*

$$\forall\, \epsilon > 0 \,.\, \exists\, \delta > 0 \,.\, \forall\, y \in A \,.\, (d_A(x,y) < \delta \longrightarrow d_B(fx, fy) < \epsilon) \ .$$

Proof. The ϵ-δ continuity states that for every $\epsilon > 0$ there exists $\delta > 0$ such that $f_*\mathrm{B}(x,\delta) \subseteq \mathrm{B}(fx,\epsilon)$. Hence, pointwise continuity in the metric topology implies ϵ-δ continuity.

Conversely, suppose f is ϵ-δ continuous, and $fx \in \mathrm{B}(y,\eta)$. There exists $\epsilon > 0$ such that $d(y,x) + \epsilon < \eta$. Then $x \in \mathrm{B}(fx,\epsilon) \subseteq \mathrm{B}(y,\eta)$. There exists $\delta > 0$ such that $x \in \mathrm{B}(x,\delta)$ and $f_*\mathrm{B}(x,\delta) \subseteq \mathrm{B}(fx,\epsilon) \subseteq \mathrm{B}(y,\eta)$. ■

Definition 5.6.15 Let (A, d_A) and (B, d_B) be metric spaces. A map $f \colon A \to B$ is *uniformly continuous* when

$$\forall\, \epsilon > 0 \,.\, \exists\, \delta > 0 \,.\, \forall\, x, y \in A \,.\, (d_A(x,y) < \delta \longrightarrow d_B(fx, fy) < \epsilon) \ .$$

A map $f \colon A \to B$ is *locally uniformly continuous* when for every $x \in A$ there is an open ball centered at x in which f is uniformly continuous:

$$\forall\, x \in A \,.\, \exists\, \eta > 0 \,.\, \forall\, \epsilon > 0 \,.\, \exists\, \delta > 0 \,.\, \forall\, y, z \in \mathrm{B}(x,\eta) \,.\, (d_A(y,z) < \delta \longrightarrow d_B(fy, fz) < \epsilon) \ .$$

Proposition 5.6.16 *A uniformly continuous map is locally uniformly continuous. A locally uniformly continuous map is continuous.*

Proof. Obvious. ■

We denote the space of uniformly continuous maps $f\colon A \to B$ by $\mathcal{C}_{\mathrm{u}}(X,Y)$. It is defined by

$$\mathcal{C}_{\mathrm{u}}(A,B) = \{f\colon A \to B \mid \forall\,\epsilon > 0\,.\,\exists\,\delta > 0\,.\,\forall\,x,y \in A\,.\,(d_A(x,y) < \delta \longrightarrow d_B(fx,fy) < \epsilon)\}\ .$$

We also write $\mathcal{C}_{\mathrm{u}}(A) = \mathcal{C}_{\mathrm{u}}(A,\mathbb{R})$.

5.6.1 Complete Metric Spaces

The definitions of convergent and Cauchy sequences generalize from the reals to arbitrary metric spaces in a straightforward fashion.

Definition 5.6.17 Let (A,d) be a metric space. A sequence $a\colon \mathbb{N} \to A$ *converges to* $x \in A$ when

$$\forall\,k \in \mathbb{N}\,.\,\exists\,n \in \mathbb{N}\,.\,\forall\,m \in \mathbb{N}\,.\,d(x, a_{n+m}) < 1/k\ .$$

We say that x is *the limit* of the sequence a, and that a is a *convergent sequence*. A *Cauchy sequence* in a metric space (A,d) is a sequence $a\colon \mathbb{N} \to A$ such that

$$\forall\,k \in \mathbb{N}\,.\,\exists\,n \in \mathbb{N}\,.\,\forall\,m,p \in \mathbb{N}\,.\,(d(a_{n+m}, a_{n+p}) < 1/k)\ .$$

It is straightforward to check that every convergent sequence is a Cauchy sequence. A metric space is *complete* when every Cauchy sequence is convergent. A subspace $S \subseteq A$ of a metric space is *complete* when every convergent sequence in S converges to a limit in S.

Let $\mathsf{Cauchy}(A)$ be the space of convergent sequences in a complete metric space A,

$$\mathsf{Cauchy}(A) = \left\{a \in A^{\mathbb{N}} \mid \forall\,k \in \mathbb{N}\,.\,\exists\,n \in \mathbb{N}\,.\,\forall\,m,p \in \mathbb{N}\,.\,(d(a_{n+m}, a_{n+p}) < 1/k)\right\}\ .$$

Just like in the case of Cauchy complete ordered fields, we can check that a Cauchy sequence has exactly one limit. By Unique Choice we obtain the *limit operator* $\lim\colon \mathsf{Cauchy}(A) \to A$, characterized by $\langle a_n\rangle_{n\in\mathbb{N}} \to \lim a$, for all $a \in \mathsf{Cauchy}(a)$. When A is a computable metric space, the limit operator is computable as well.

5.6.2 Totally Bounded Metric Spaces

Complete totally bounded metric spaces play the role of compact spaces in constructive mathematics.

Definition 5.6.18 Let (A,d) be a metric space. An *ϵ-net* for A is a finite sequence $a_0,\dots,a_{n-1} \in A$ such that for every $x \in A$ there exists $i \in \{0,\dots,n-1\}$ such that $d(x,a_i) < \epsilon$. A metric space (A,d) is *totally bounded* when for every $k \in \mathbb{N}$ there exists a $1/k$-net for A.

Note that every totally bounded metric space is separable.

Proposition 5.6.19

(1) $\mathbb{R}$ is not totally bounded.

(2) A closed bounded interval is complete and totally bounded.

(3) If (A,d_A) and (B,d_B) are totally bounded then so is their metric product $(A \times B, d_{A\times B})$.

Proof. (1) and (2) are easy. (3) Let $\epsilon > 0$. Let $a_0, \dots, a_{m-1} \in A$ be a $\sqrt{\epsilon}$-net for A, and let $b_0, \dots, b_{n-1} \in B$ be a $\sqrt{\epsilon}$-net for B. Then $\{\langle a_i, b_j \rangle \mid i = 0, \dots, m-1; j = 0, \dots, n-1\}$ is an ϵ-net for $A \times B$. ■

An *upper bound* for $f \colon A \to \mathbb{R}$ is $M \in \mathbb{R}$ such that $fx \leq M$ for all $x \in A$. A *least upper bound*, also called the *supremum*, is an upper bound M_0 for f such that for all $k \in \mathbb{N}$ there exists $x \in A$ such that $M_0 \leq f(x) + 2^{-k}$. It is obvious that a map has at most one least upper bound. The notions of a *lower bound* and the *greatest lower bound*, also called the *infimum*, are defined analogously.

Proposition 5.6.20 *Let (A, d) be an inhabited totally bounded metric space. Every uniformly continuous map $f \colon A \to \mathbb{R}$ has a supremum M and an infimum m. In addition, for every $\epsilon > 0$ there exists $x \in A$ such that $|fx - M| < \epsilon$, and there exists $y \in A$ such that $|fy - m| < \epsilon$.*

Proof. Let $k \in \mathbb{N}$. Because f is uniformly continuous there exists $\delta > 0$ such that $d_A(x, y) < \delta$ implies $|fx - fy| < 2^{-k}$. Let $a_0, \dots, a_n \in A$ be a δ-net, and let

$$N = \max(fa_0, \dots, fa_n) .$$

The number N is well defined because A is inhabited and so $n > 0$. For every $x \in A$ there exists i such that $d_A(x, a_i) < \delta$, hence $|fx - fa_i| < 2^{-k}$ and

$$fx < fa_i + 2^{-k} \leq N + 2^{-k} .$$

Thus, $N + 2^{-k}$ is an upper bound for f, and there exists i such that $fa_i = N$. By Number Choice, there is a sequence $M \colon \mathbb{N} \to \mathbb{R}$ such that, for every $n \in \mathbb{N}$, $M_n + 2^{-n}$ is an upper bound for f and there exists $x \in A$ such that $fx = M_n$. For all $k, j \in \mathbb{N}$, $M_k \leq M_j + 2^{-j}$ and $M_j \leq M_k + 2^{-k}$, hence $|M_j - M_k| < 2^{-\min(j,k)}$. Therefore, $\langle M_n \rangle_{n \in \mathbb{N}}$ is a Cauchy sequence. Let $M = \lim_{n \to \infty} M_n$. For every $k \in \mathbb{N}$, $|M - M_k| \leq 2^{-k}$ and $M_k \leq M$. Thus, by construction of M_k, for every $k \in \mathbb{N}$ there exists $x \in A$ such that $|M - fx| < 2^{-k}$. I remains to show that M is an upper bound for f. Suppose that for $x \in A$ it were the case that $M < fx$. Then there would exist $k \in \mathbb{N}$ such that $2^{-k} < ((fx) - M)/2$, from which it would follow

$$(fx) - M = (fx) - M_k + M_k - M < 2^{-k} + 2^{-k} < (fx) - M .$$

This is a contradiction, therefore $fx \leq M$. We have proved that M is the supremum of f, and it is clear from the construction that for every $k \in \mathbb{N}$ there exists $x \in A$ such that $|fx - M| \leq 2^{-k}$. The infimum m of f is equal to $-K$ where K is the supremum of $-f$. ■

Let (A, d) be a totally bounded metric space. The *supremum operator* $\sup_A \colon \mathcal{C}_\mathrm{u}(A) \to \mathbb{R}$ is defined by

$$\sup_A f = \mathsf{the}\, M \in \mathbb{R} \,.\, (M \text{ is the supremum of } f) .$$

The infimum operator inf is defined analogously.

By Theorem 2.3.1 $\sup_A$ is a computable map. The *uniform metric* d_u on $\mathcal{C}_\mathrm{u}(A)$ is defined by

$$d_\mathrm{u}(f, g) = \sup_A (\lambda x \colon A \,.\, |fx - gx|) .$$

Proposition 5.6.21 *Suppose (A, d) is an inhabited totally bounded metric space. There exist $a \colon \mathbb{N} \to A$ and $\beta \colon \mathbb{N} \to \mathbb{N}$ such that, for every $k \in \mathbb{N}$, $a0, \dots, a(\beta k)$ is a 2^{-k}-net.*

Proof. For every $k \in \mathbb{N}$ there exists a 2^{-k}-net $a_{k,0}, \dots, a_{k,n_k} \in A$. By Number Choice we can concatenate all these finite sequences together into one infinite sequence $a\colon \mathbb{N} \to A$ and define β by $\beta 0 = n_0$, $\beta(k+1) = (\beta k) + n_{k+1}$. ∎

Definition 5.6.22 A metric space (M, d) is connected when, for every pair of inhabited metrically open subsets $A, B \subseteq M$, $A \cup B = M$ implies that $A \cap B$ is inhabited.

Proposition 5.6.23 *(a) A closed interval* $[a, b]$ *is connected. (b) An open interval* (a, b) *is connected. (c)* $\mathbb{R}$ *is connected.*

Proof. We only prove (c), since the proofs of (a) and (b) are very similar. Suppose $A, B \subseteq \mathbb{R}$ are metrically open, $a_0 \in A$, $b_0 \in B$, and $A \cup B = \mathbb{R}$. Let ρ be the binary relation on $\mathbb{R}^2$ defined by

$$\begin{aligned}\rho(\langle x, y\rangle, \langle u, v\rangle) \longleftrightarrow {} & ((x+y)/2 \in A \longrightarrow u = (x+y)/2 \wedge v = y) \wedge \\ & ((x+y)/2 \in B \longrightarrow u = x \wedge v = (x+y)/2)\ .\end{aligned}$$

By assumption $A \cup B = \mathbb{R}$, hence for every $\langle x, y\rangle \in \mathbb{R}^2$ there exists $\langle u, v\rangle \in \mathbb{R}^2$ such that $\rho(\langle x, y\rangle, \langle u, v\rangle$. By Dependent Choice 3.6.3 there exists a sequence $\langle x_n, y_n\rangle_{n\in\mathbb{N}}$ such that $x_0 = a_0$, $y_0 = b_0$, and $\rho(\langle x_n, y_n\rangle, \langle x_{n+1}, y_{n+1}\rangle)$ for all $n \in \mathbb{N}$. It follows that $\langle x_n\rangle_{n\in\mathbb{N}}$ and $\langle y_n\rangle_{n\in\mathbb{N}}$ are Cauchy sequences with the same limit because $|x_{n+1} - y_{n+1}| \le |x_n - y_n|/2$. Let $c = \lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$. Because A is metrically open and by construction $x_n \in A$ for all $n \in \mathbb{N}$, we see that $c \in A$. For the same reason $c \in B$, therefore $c \in A \cap B$. ∎

Theorem 5.6.24 (Intermediate Value Theorem) *Let* (A, d) *be a connected metric space,* $a, b \in A$*, and* $f\colon A \to \mathbb{R}$ *a uniformly continuous map. For every* $\xi \in [fa, fb]$ *and every* $\epsilon > 0$ *there exist* $c \in A$ *such that* $|fc - \xi| < \epsilon$.

Proof. Let $U = \{x \in A \mid fx > \xi - \epsilon\}$ and $V = \{x \in A \mid fx < \xi + \epsilon\}$. The subspaces U and V are metrically open because they are inverse images of open intervals $(\xi - \epsilon, \infty)$ and $(-\infty, \xi + \epsilon)$ under the uniformly continuous map f. The subspaces U and V are inhabited by b and a, respectively. Because $\xi - \epsilon < \xi + \epsilon$, $\xi - \epsilon < fx$ or $fx < \xi + \epsilon$ for all $x \in A$, therefore $A = U \cup V$. Because A is connected there exists $c \in U \cap V$. This the c we are looking for. ∎

Example 5.6.25 In Example 2.2.9 we defined the space of paths $\mathsf{Path}(F)$ for a fan $F \in \mathsf{Fan}$. Let $\delta\colon \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{+}$ be defined coinductively by

$$\delta\,\langle p, q\rangle = \mathtt{if}\ p0 \neq q0\ \mathtt{then}\ 0\ \mathtt{else}\ \mathsf{s}(\delta\,\langle \lambda n.\, p(n+1), \lambda n.\, q(n+1)\rangle)\ .$$

Thus, $\delta\langle p, q\rangle = n$ if $pn \neq qn$ and $pi = qi$ for all $i < n$, and $\delta\langle p, q\rangle = \infty$ if $p = q$. Define a map $d_F\colon \mathsf{Path}(F) \times \mathsf{Path}(F) \to \mathbb{R}$ by

$$d_F(p, q) = 2^{-\delta(p,q)}\ ,$$

where $2^{-\infty} = 0$. The map d_F is well defined by Lemma 5.5.29. We claim that d_F is a metric on $\mathsf{Path}(F)$. The only non-obvious part is the triangle inequality. Suppose it were the case that $d_F(p, q) > d_F(p, r) + d_F(r, q)$ for $p, q, r \in \mathsf{Path}(F)$. Then $d_F(p, q) > 0$, hence $\delta(p, q) < \infty$. It also follows that $\delta(p, q) < \delta(p, r)$ and $\delta(p, q) < \delta(q, r)$. But this implies $p(\delta(p, q)) \neq q(\delta(p, q)) =$

$r(\delta(p,q)) = p(\delta(p,q))$, which is impossible. Therefore $d_F(p,q) \leq d_F(p,q) + d_F(r,q)$. Moreover, $(\mathsf{Path}(F), d_F)$ is a complete totally bounded metric space.

First we show that it is complete. Suppose $\langle p_n \rangle_{n \in \mathbb{N}}$ is a Cauchy sequence. For every $k \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $d_F(p_{m+i}, p_m) < 2^{-k}$ for all $i \in \mathbb{N}$. By Number choice there exists a map $c \colon \mathbb{N} \to \mathbb{N}$ such that $d_F(p_{(ck)+i}, p_{ck}) < 2^{-k}$ for all $i, k \in \mathbb{N}$. Define $q \colon \mathbb{N} \to \mathbb{N}$ by $qk = p_{ck}k$. It follows that $m \geq k$ implies $qk = p_m k$. Hence $q \in \mathsf{Path}(F)$ because, for every $i \leq k \in \mathbb{N}$, $qi = p_{ck}i$ for all $i \leq k$. This also shows that $d_F(q, p_{ck}) < 2^{-k}$ for all $k \in \mathbb{N}$, therefore $\lim_{n\to\infty} p_n = q$.

It remains to show that $\mathsf{Path}(F)$ is totally bounded, but this is straightforward because all paths in F of the form

$$n_0, n_1, \ldots, n_k, 0, 0, \ldots$$

can be listed and form a 2^{-k}-net. There are only finitely many of them because F is finitely branching.

Example 5.6.26 Continuing the previous example, let $C \in \mathsf{Fan}$ be the unique fan that satisfies the corecursive equation $C = [C, C]$. The fan C is the full infinite binary tree, and the space $\mathsf{Path}(C)$ is isomorphic to the Cantor space $2^{\mathbb{N}}$. This gives us a metric d on $2^{\mathbb{N}}$,

$$d(f,g) = 2^{-\delta(f,g)} \ ,$$

where δ is as in the previous example. Therefore, the Cantor space is a complete totally bounded metric space. We call this metric the *standard metric* on the Cantor space. Unless otherwise stated, whenever $2^{\mathbb{N}}$ is considered as a metric space we have the standard metric in mind.

Example 5.6.27 (Hilbert Space ℓ^2) We conclude this section with an example of a famous space, the Hilbert space ℓ^2. It is worth noting that the construction of ℓ^2 matches the classical one. We omit most proofs of the basic properties of ℓ^2, since they closely follow the usual ones.

The field of complex numbers $\mathbb{C}$ is the space $\mathbb{R} \times \mathbb{R}$ with the usual basic operations on complex numbers. The standard metric on $\mathbb{C}$ is defined as the absolute value of the difference, $\lambda\langle z, w\rangle : \mathbb{C}^2 \,.\, |z - w|$. The complex numbers form a complete metric space.

Let $S \colon \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ be the *partial sum operator*,

$$(Sa)n = \sum_{i=0}^{n} a_i \ .$$

The space ℓ^2 is defined to be

$$\ell^2 = \left\{ a \in \mathbb{C}^{\mathbb{N}} \;\middle|\; S(\lambda n : \mathbb{N} \,.\, |a_n|^2) \in \mathsf{Cauchy}(\mathbb{R}) \right\} \ .$$

In words, a complex sequence $a \colon \mathbb{N} \to \mathbb{C}$ is a point of ℓ^2 if, and only if, the infinite sum $\sum_{n=0}^{\infty} |a_n|^2$ converges. The space ℓ^2 is a complex vector space, where the addition and multiplication by a scalar are defined coordinate-wise. The *scalar product* $\langle \Box, \Box \rangle_2 \colon \ell^2 \times \ell^2 \to \mathbb{C}$ is defined by

$$\langle a, b \rangle_2 = \sum_{n=0}^{\infty} a_n \overline{b_n} = \lim\nolimits_{\mathbb{C}} \left(S(\lambda n : \mathbb{N} \,.\, a_n \overline{b_n}) \right) \ .$$

The scalar product is computable, as it is a composition of computable maps. The ℓ^2-norm $\|\square\|_2$ and the ℓ^2-metric d_2 are defined by

$$\|a\|_2 = \sqrt{\langle a, a\rangle_2}\ ,$$
$$d_2(a, b) = \|a - b\|_2\ .$$

These operations are computable because the scalar product is computable. The space ℓ^2 is a complete separable metric space. It is separable because it contains a countable dense subspace $Q \subseteq \ell^2$, where $Q = \mathsf{List}_{\mathbb{Q}}$ is the space of finite sequence of rational numbers, and the inclusion $\mathsf{i}_Q \colon Q \to \ell^2$ is defined by

$$\mathsf{i}_Q[a_0, \ldots, a_{k-1}] = \lambda n : \mathbb{N}.\ (\texttt{if}\ k < n\ \texttt{then}\ a_k\ \texttt{else}\ 0)\ .$$

Let us compute a representation of ℓ^2. For this purpose, we first unravel the defining predicate $S(\lambda n : \mathbb{N}.\, |a_n|^2) \in \mathsf{Cauchy}(\mathbb{R})$. It is equivalent to

$$\forall\, k \in \mathbb{N}.\, \exists\, n \in \mathbb{N}.\, \forall\, m \in \mathbb{N}.\, \left(\textstyle\sum_{i=n}^{n+m} |a_i|^2 < 2^{-k}\right)\ .$$

By Number Choice, this is equivalent to

$$\exists\, t \in \mathbb{N}^{\mathbb{N}}.\, \forall\, k, m \in \mathbb{N}.\, \left(\textstyle\sum_{i=tk}^{(tk)+m} |a_i|^2 < 2^{-k}\right)\ .$$

Thus, ℓ^2 is isomorphic to the quotient

$$\left\{\langle a, t\rangle \in \mathbb{C}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \;\middle|\; \forall\, k, m \in \mathbb{N}.\, \left(\textstyle\sum_{i=tk}^{(tk)+m} |a_i|^2 < 2^{-k}\right)\right\} /\!\sim\ ,$$

where $\langle a, t\rangle \sim \langle b, u\rangle$ if, and only if, $a = b$. The map t gives us the rate of convergence of $S(\lambda n : \mathbb{N}.\, |a_n|^2)$—if we want to compute $\sum_{n=0}^{\infty} |a_n|^2$ with accuracy 2^{-k}, it suffices to sum the first $t\,k$ terms.

5.7 Hyperspaces

Computability of points in a space is not the only kind of computability we are interested in. Often we are concerned with computability of *subspaces* of a space. An example of this is the space $\mathcal{O}(A) = \mathbf{\Sigma}^A$ of open subspaces of A. The logic of modest sets cannot handle *general* subspaces of a space. In order to study completely arbitrary subspaces, we would have to add a *powerset* operator to the logic, which would turn it into a logic of *realizability toposes*. However, we avoid doing that, and investigate instead how much can be done with modest sets alone. Of course, this means that we must restrict attention to special kinds of subspaces.

A *hyperspace* H over a space A is a space H together with a *membership* relation $\in_H \colon A \times H$. We think of the points of H as representing subspaces of A. If for $x \in A$ and $h \in H$ we have $x \in_H h$ then we say that x belongs to h. Every $h \in H$ determines a subspace $|h| = \{x \in A \mid x \in_H h\}$. Thus, a hyperspace can be viewed as a dependent type $\{|h| \mid h \in H\}$, where each space in the family is a subspace of A.

5.7.1 The Hyperspace of Open Subspaces

For any space A the space $\mathcal{O}(A) = \Sigma^A$ is the *hyperspace of intrinsically open subspaces of* A. The membership relation is simply the evaluation map, i.e., for $x \in A$, $U \in \mathcal{O}(A)$, $x \in_{\mathcal{O}(A)} U$ if, and only if, $Ux = \top$.

Suppose (X, B) is a countably based space. By Proposition 5.4.4 a subspace $S \subseteq X$ is open with respect to B if, and only if, there exists $C \in \mathcal{O}(\mathbb{N})$, called a countable union predicate, such that

$$\forall x \in X . (x \in S \longleftrightarrow \exists n \in C . (x \in B_n)) \ .$$

There is a preorder $\sqsubseteq$ on $\mathcal{O}(\mathbb{N})$ defined by

$$C \sqsubseteq D \longleftrightarrow \bigcup_{n \in C} B_n \subseteq \bigcup_{n \in D} B_n \ .$$

The relation $\sim$ defined by $C \sim D \longleftrightarrow C \sqsubseteq D \land D \sqsubseteq C$ is an equivalence relation on $\mathcal{O}(\mathbb{N})$. The *hyperspace of open subspaces* $\mathcal{O}(X, B)$ is the space

$$\mathcal{O}(X, B) = (\Sigma^{\mathbb{N}})/{\sim} \ ,$$

with the membership relation

$$x \in_{\mathcal{O}(X,B)} [C]_{\sim} \longleftrightarrow x \in \bigcup_{n \in C} B_n \ .$$

The countable union operator $\bigcup \colon \mathcal{O}(X, B)^{\mathbb{N}} \to \mathcal{O}(X, B)$ is defined as follows. Suppose $U \colon \mathbb{N} \to \mathcal{O}(X, B)$. Because $\mathbb{N}$ is projective there exists $C \colon \mathbb{N} \to \Sigma^{\mathbb{N}}$ such that $Un = [Cn]$ for all $n \in \mathbb{N}$. Let

$$\textstyle\bigcup_{n \in \mathbb{N}} U_n = \left[\bigvee_{n \in \mathbb{N}} C_n\right] \ .$$

Because $\bigvee$ is computable, it follows that $\bigcup$ is computable as well.

By Definition 5.4.2 the intersection of two basic open subspaces $B_m \cap B_n$ is open. More precisely,

$$\forall m, n \in \mathbb{N} . \exists C \in \Sigma^{\mathbb{N}} . \forall x \in X . (x \in B_m \land x \in B_n \longleftrightarrow \exists k \in C . x \in B_k) \ .$$

By Number Choice there exists a map $C \colon \mathbb{N} \times \mathbb{N} \to \Sigma^{\mathbb{N}}$ such that, for all $m, n \in \mathbb{N}$, $x \in X$, $x \in B_m \land x \in B_n$ if, and only if, $\exists k \in C(m, n) . x \in B_k$. Now we can define binary intersection $\square \cap \square \colon \mathcal{O}(X, B) \times \mathcal{O}(X, B) \to \mathcal{O}(X, B)$ by

$$[C] \cap [D] = \left[\textstyle\bigvee_{m \in C} \bigvee_{n \in D} C(m, n)\right] \ .$$

5.7.2 The Hyperspace of Formal Balls

Edalat and Heckmann [EH98] introduced the domain of formal balls to study computability on metric spaces, and Edalat and Sünderhauf [ES99a] applied to formal-ball model to computability in Banach spaces. The space of formal-balls is easily defined in the logic of modest sets.

Let M be a metric space and let $\mathbb{R}_+ = \left\{x \in \mathbb{R} \mid x \geq 0\right\}$ be the space of non-negative reals. The hyperspace of *formal balls* over M is the space $\mathsf{FB}(M) = M \times \mathbb{R}_+$ with the membership relation

$$x \in_{\mathsf{FB}(M)} \langle y, r \rangle \longleftrightarrow d_M(x, y) \leq r \ .$$

We see that the point $\langle y, r\rangle \in \mathsf{FB}(M)$ represents the closed ball $\overline{\mathrm{B}}(y, r)$. The *formal order* on $\mathsf{FB}(M)$ is the relation $\sqsubseteq$ defined by

$$\langle x, r\rangle \sqsubseteq \langle y, s\rangle \longleftrightarrow d_M(x, y) \leq r - s \;.$$

The order is "formal" because in general $\langle x, r\rangle \sqsubseteq \langle y, s\rangle$ implies $\overline{\mathrm{B}}(x, r) \subseteq \overline{\mathrm{B}}(y, s)$, but not vice versa. The metric space M can be embedded in $\mathsf{FB}(M)$ via the embedding $e\colon M \to \mathsf{FB}(M)$ defined by $ex = \langle x, 0\rangle$. The image of e is exactly the subspace of maximal elements of $\mathsf{FB}(M)$.

5.7.3 Complete Located Subspaces

Let (M, d) be a metric space and $S \subseteq M$ a subspace. Classically, we define the *distance map* $d(\Box, S)\colon M \to \mathbb{R}$ for S by

$$d(x, S) = \inf \left\{ d(x, y) \;\middle|\; y \in S \right\} \;. \tag{5.12}$$

Classically the distance map for a set is well defined as long as the set is non-empty. In a constructive setting we need to be careful about the interpretation of the infimum in (5.12) because not every inhabited subspace of reals that is bounded below has an infimum. In addition, we want to avoid talking about powersets since the logic of modest sets does not have a powerset operator.

We say that $a \in \mathbb{R}$ is the *infimum* of $A \subseteq \mathbb{R}$, and write $a = \inf A$, when $a \leq x$ for every $x \in A$, and for every $\epsilon \in \mathbb{R}$ there exists $x \in A$ such that $x < a + \epsilon$. Note that we did *not* define an infimum operator that maps subspaces to their infima, even though the notation $a = \inf A$ suggests so.

Let (M, d) be a metric space and $S \subseteq M$ a subspace. We say that S is *located* when it has a *distance map*, which is a map $d(\Box, S)\colon M \to \mathbb{R}$ such that, for all $x \in M$,

$$d(x, S) = \inf \left\{ t \in \mathbb{R} \;\middle|\; \exists\, y \in S \,.\, t = d(x, y) \right\} \;.$$

Suppose $S \subseteq M$ is located. Then its metric closure

$$\overline{S} = \left\{ x \in M \;\middle|\; \forall\, k \in \mathbb{N} \,.\, \exists\, y \in S \,.\, d(x, y) < 1/k \right\}$$

is located as well and it has the same distance map as S. The metric closure of a located subspace can be recovered from its distance map because

$$\overline{S} = \left\{ x \in M \;\middle|\; d(x, S) = 0 \right\} \;.$$

This suggests that we can define the space of metrically closed located spaces to be the space of all distance maps.

Definition 5.7.1 Let (M, d) be a metric space. We say that $f\colon M \to \mathbb{R}$ is a *distance map* when its zero-space $\mathsf{Z}(f) = \left\{ x \in M \;\middle|\; fx = 0 \right\}$ is located and f is the distance map for $\mathsf{Z}(f)$.

Formally, $f\colon M \to \mathbb{R}$ is a distance map when

$$\forall\, x \in M \,.\, \forall\, y \in M \,.\, (fy = 0 \Longrightarrow fx \leq d(x, y)) \tag{5.13}$$

and

$$\forall\, x \in M \,.\, \forall\, k \in \mathbb{N} \,.\, \exists\, y \in M \,.\, \left(fy = 0 \land d(x, y) \leq (fx) + 2^{-k} \right) \;. \tag{5.14}$$

If $f, g\colon M \to \mathbb{R}$ are distance maps, it is obvious that $f = g$ if, and only if, $\mathsf{Z}(f) = \mathsf{Z}(g)$, because a distance map and its zero-space uniquely determine each other.

Proposition 5.7.2 *(a) A distance map is uniformly continuous. (b) The zero-space of a distance map is complete.*

Proof. (a) Suppose $S \subseteq M$ is a located space. For all $x, y \in M$, $d(x, S) \leq d(x, y) + d(y, S)$, therefore $d(\Box, S)$ is uniformly continuous. (b) If S is located and $\langle a_n \rangle_{n \in \mathbb{N}}$ is a convergent sequence in S then $d(\lim_{n\to\infty} a_n, S) = \lim_{n\to\infty} d(a_n, S) = 0$, because $d(\Box, S)$ is continuous. ∎

Definition 5.7.3 The hyperspace $\mathsf{Loc}(M)$ of *complete located subspaces* over an inhabited metric space M is the space

$$\mathsf{Loc}(M) = \left\{ f \in \mathbb{R}^M \;\middle|\; f \text{ is a distance map} \right\}$$

with the membership relation defined by $x \in_{\mathsf{Loc}(M)} f \longleftrightarrow fx = 0$.

There is an embedding $e \colon M \to \mathsf{Loc}(M)$, defined by $ex = \lambda y : M \,.\, d(x, y)$. To see that e is an embedding, suppose $f \in \mathsf{Loc}(M)$ and there $\neg\neg$-exists $x \in M$ such that $f = \lambda y : M \,.\, d(x, y)$. Because M is inhabited, by (5.14) there exists $z \in M$ such that $fz = 0$. But then $0 = fz = d(x, z)$, from which we conclude $\neg\neg(x = z)$, therefore $x = z$.

Let us compute a representation for $\mathsf{Loc}(M)$. For this purpose we need to know what the realizers for (5.14) are. We need not worry about (5.13) because it is a stable statement. By Intensional Choice, (5.14) is equivalent to

$$\exists\, t \in M^{\mathsf{r}M \times \mathbb{N}} \,.\, \forall\, a \in \mathsf{r}M \,.\, \forall\, k \in \mathbb{N} \,.\, d([a], ta) \leq (f[a]) + 2^{-k} \,.$$

Therefore, a representation for $\mathsf{Loc}(M)$ is the space

$$\begin{aligned}\big\{ \langle f, t \rangle \in \mathbb{R}^M \times M^{\mathsf{r}M \times \mathbb{N}} \;\big|\; & (\forall\, x \in M \,.\, \forall\, y \in M \,.\, (fy = 0 \Longrightarrow fx \leq d_M(x, y))) \;\land \\ & \forall\, x \in M \,.\, \forall\, k \in \mathbb{N} \,.\, d(x, tx) \leq (fx) + 2^{-k} \big\} / {\sim} \,,\end{aligned}$$

where $\langle f, t \rangle \sim \langle g, u \rangle$ if, and only if, $f = g$.

5.7.4 The Upper Space

Next we consider the hyperspace of complete totally bounded subspaces of a metric space, known as the *upper space*.

Proposition 5.7.4 *An inhabited complete totally bounded subspace of a metric space is located.*

Proof. Suppose $S \subseteq M$ is an inhabited totally bounded subspace of a metric space M. Let $a \colon \mathbb{N} \to M$ and $\beta \colon \mathbb{N} \to \mathbb{N}$ be as in Proposition 5.6.21. It is not hard to see that the distance map $d(\Box, S)$ can be defined by

$$d(x, S) = \lim_{k \to \infty} \left(\min \left(d(x, a_i) \;\middle|\; 0 \leq i \leq \beta k \right) \right) \,.$$

∎

The *upper space* $\mathsf{Upper}(M)$ of an inhabited metric space M is the hyperspace that corresponds to the inhabited complete totally bounded subspaces of M. By Proposition 5.7.4, we can define it as a subspace of $\mathsf{Loc}(M)$,

$$\mathsf{Upper}(M) = \{f \in \mathsf{Loc}(M) \mid \mathsf{Z}(f) \text{ is totally bounded}\} .$$

Recall that $\mathsf{Z}(f)$ is the zero-space of f, $\mathsf{Z}(f) = \{x \in M \mid fx = 0\}$.

The embedding $e\colon M \to \mathsf{Loc}(M)$ restricts to an embedding $e\colon M \to \mathsf{Upper}(M)$ because a singleton is a complete totally bounded subspace.

It is known that the complete totally bounded metric subspaces are exactly the uniformly continuous quotients of Cantor space $2^{\mathbb{N}}$, see [TvD88b, VII.4.4]. We can use this fact to prove that the inhabited, complete, totally bounded subspaces of a *complete* metric space M are precisely the uniformly continuous subquotients of M.

Theorem 5.7.5 *A subspace $S \subseteq M$ of a complete metric space M is inhabited, complete, and totally bounded if, and only if, there exists a uniformly continuous map $f \in \mathcal{C}_{\mathrm{u}}(2^{\mathbb{N}}, M)$ such that*

$$S = \left\{x \in M \mid \exists\, \alpha \in 2^{\mathbb{N}} \,.\, f\alpha = x\right\} .$$

Proof. For a proof see [TvD88b, VII.4]. ∎

It follows that the upper space of an inhabited complete metric space M is a quotient

$$\mathsf{Upper}(M) = \mathcal{C}_{\mathrm{u}}(2^{\mathbb{N}}, M)/\!\sim , \tag{5.15}$$

where $\sim$ is defined as follows. Define a relation $\sqsubseteq$ on $\mathcal{C}_{\mathrm{u}}(2^{\mathbb{N}}, M)$ by

$$f \sqsubseteq g \longleftrightarrow \forall\, \alpha \in 2^{\mathbb{N}} \,.\, \exists\, \beta \in 2^{\mathbb{N}} \,.\, f\alpha = g\beta ,$$

and let $\sim$ be the equivalence relation

$$f \sim g \longleftrightarrow f \sqsubseteq g \wedge g \sqsubseteq f .$$

Suppose $f\colon 2^{\mathbb{N}} \to M$ is a uniformly continuous map. For every $k \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that, for all $\alpha, \beta \in 2^{\mathbb{N}}$, $d(\alpha, \beta) \leq 2^{-m}$ implies $d(f\alpha, f\beta) \leq 2^{-k}$. The finite collection of all binary sequences $\alpha \in 2^{\mathbb{N}}$ such that $\alpha j = 0$ for all $j > m$ forms a 2^{-m} net in $2^{\mathbb{N}}$. Therefore, when this collection is mapped to M by f, it forms a 2^{-k}-net for $[f]_{\sim}$. We denote this 2^{-k}-net by $\mathsf{net}(f, k)$.

Suppose M is a complete separable metric space and $B_n = B(a_n, r_n)$, $n \in \mathbb{N}$ is the standard countable basis for M. There is a semidecidable predicate $\subseteq\colon \mathsf{Upper}(M) \times \mathcal{O}(M, B) \to \Sigma$ such that $(S \subseteq U) = \top$ if, and only if, $|S| \subseteq |U$. Here $|S| \subseteq |U|$ means that for all $x \in X$, $x \in S$ implies $x \in U$. For suppose $S = [f] \in \mathsf{Upper}(M)$ and $U = [C] \in \mathcal{O}(M, B)$. Then $|S| \subseteq |U|$ if, and only if, there exists $k \in \mathbb{N}$ such that, for all $a \in \mathsf{net}(f, k)$, there exists $j \in \mathbb{N}$ such that $d(b_i, a_j) < r_j$. This is obviously a semidecidable predicate.

Remark 5.7.6 Suppose we wanted to implement a data structure for the upper space of a complete, inhabited metric space M. What does representation (5.15) suggest? We would be mistaken to think that an appropriate representation of a complete totally bounded subspace of M is a uniformly continuous map $f\colon 2^{\mathbb{N}} \to M$. Such a mistake happens when we interpret the meaning of $\mathcal{C}_{\mathrm{u}}(2^{\mathbb{N}}, M) \subseteq M^{2^{\mathbb{N}}}$ classically rather than intuitionistically. We must also include a witness for the uniform continuity of f. Therefore, a representation of a complete totally bounded subspace of M is a *pair* $\langle f \colon 2^{\mathbb{N}} \to M, m \colon \mathbb{N} \to \mathbb{N}\rangle$, where m is the modulus of uniform continuity for f. With the modulus of continuity we can compute an ϵ-net for $\mathsf{im}(f)$, in other words, we can approximate the space represented by f arbitrarily well in the Hausdorff metric on $\mathsf{Upper}(M)$.

5.7.5 The Hyperspace of Solids

Edalat and Lieutier [EL99] considered a space of *solids* in domain-theoretic setting. In this section we explore the hyperspace of solids in the logic of modest sets.

For an arbitrary space X we can develop a theory of *partial solids* by taking the hyperspace ${2_\bot}^X$ of partial Boolean predicates with the membership predicate defined by $x \in_{{2_\bot}^X} S \longleftrightarrow Sx = 1$. Then the basic operations of complement, union, and intersection easily turn out to be computable because the correspond to Boolean operations on $2_\bot$. However, that is about all we can do in such a general setting. Instead, let us consider the hyperspace of solids on a separable metric space.

Let M be a separable metric space and let $B_n = B(a_n, r_n)$, $n \in \mathbb{N}$, be an enumeration of open balls that form a countable basis for M. A *partial solid* in M is a pair $\langle U, V\rangle \in \mathcal{O}(M, B)$ of disjoint metrically open subspaces of M. The domain of partial solids in M is the space

$$\mathsf{Solid}(M) = \left\{\langle U, V\rangle \in \mathcal{O}(M,B) \times \mathcal{O}(M,B) \;\middle|\; \forall\, x \in X \,.\, \neg(x \in U \wedge x \in V)\right\} .$$

The complement, union, and intersection of solids can be easily defined in terms of union and intersection on $\mathcal{O}(X, B)$ as

$$\begin{aligned}
\langle U, V\rangle^{\mathrm{c}} &= \langle V, U\rangle , \\
\langle U, V\rangle \cup \langle U', V'\rangle &= \langle U \cup U', V \cap V'\rangle , \\
\langle U, V\rangle \cap \langle U', V'\rangle &= \langle U \cap U', V \cup V'\rangle .
\end{aligned}$$

More interestingly, there is an inclusion predicate $\subseteq \colon \mathsf{Upper}(M) \times \mathsf{Solid}(M) \to 2_\bot$ such that, for all $S \in \mathsf{Upper}(M)$ and $\langle U, V\rangle \in \mathsf{Solid}(M)$, $(S \subseteq \langle U, V\rangle) = 1$ if, and only if, $|S| \subseteq |U|$, and $(S \subseteq \langle U, V\rangle) = 0$ if, and only if, $|S| \subseteq |V|$. The inclusion predicate can be defined by

$$(S \subseteq \langle U, V\rangle) = h\langle (S \subseteq U), (S \subseteq V)\rangle ,$$

where $h \colon \left\{\langle x, y\rangle \in \Sigma \times \Sigma \;\middle|\; x \wedge y = \bot\right\} \to 2_\bot$ is the isomorphism from Proposition 5.3.35.

We compute a representation for $\mathsf{Solid}(\mathbb{R}^n)$. Let $B = \left\{\mathrm{B}(a_n, r_n) \;\middle|\; n \in \mathbb{N}\right\}$ be a countable basis for $\mathbb{R}^n$ consisting of all open balls with rational radii centered at rational points. Suppose $U, V \subseteq \mathbb{R}^n$ are metrically open. Then $U = [C]_{\mathcal{O}(\mathbb{R}^n, B)}$ and $V = [D]_{\mathcal{O}(\mathbb{R}^n, B)}$ for some $C, D \in \mathcal{O}(\mathbb{N})$. The subspaces U and V are disjoint if, and only if, whenever $n \in C$ and $m \in D$ then the open balls $B(a_n, r_n)$ and $B(a_m, r_m)$ are disjoint, which is equivalent to $d(a_n, a_m) \geq r_n + r_m$. The pair $\langle C, D\rangle \in \Sigma^{\mathbb{N}} \times \Sigma^{\mathbb{N}}$ determines a map $f_{\langle C, D\rangle} \colon \mathbb{N} \to 2_\bot$, defined by

$$f_{C,D}n = h\langle Cn, Dn\rangle .$$

Conversely, every map $g \colon \mathbb{N} \to 2_\bot$ determines a pair $\langle C_g, D_g\rangle \in \mathcal{O}(\mathbb{N}) \times \mathcal{O}(\mathbb{N})$, defined by

$$C_g n = r(gn) , \qquad D_g n = s(gn) ,$$

where $r, s \colon 2_\perp \to \Sigma$ are the maps determined by $r1 = \top$, $r0 = r\perp = \perp$, and $s0 = \top$, $s1 = s\perp = \perp$. More precisely, r is the the total map that corresponds to the partial map $r \colon 2 \rightharpoonup 1$ defined on $\{1\} \subseteq 2$ by $r1 = \star$, and a similar construction works for s. We see now that every $S \in {2_\perp}^{\mathbb{N}}$ such that

$$\forall\, n, m \in \mathbb{N}\,.\,(Sn = 0 \wedge Sm = 1 \longrightarrow d(a_n, a_m) \geq r_n + r_m)$$

determines a solid $\langle S_0, S_1 \rangle \in \mathsf{Solid}(\mathbb{R}^n)$, where

$$S_0 = \bigcup \left\{ \mathrm{B}(()a_n, r_n) \bigm| n \in \mathbb{N} \wedge Sn = 0 \right\} \ , \qquad S_1 = \bigcup \left\{ \mathrm{B}(()a_n, r_n) \bigm| n \in \mathbb{N} \wedge Sn = 1 \right\} \ .$$

Conversely, every solid is determined by some such S. Putting all this together, we obtain the following representation for $\mathsf{Solid}(\mathbb{R}^n)$:

$$\mathsf{Solid}(\mathbb{R}^n) = \left\{ S \in {2_\perp}^{\mathbb{N}} \bigm| \forall\, n, m \in \mathbb{N}\,.\,(Sn = 0 \wedge Sm = 1 \longrightarrow d(a_n, a_m) \geq r_n + r_m) \right\} \Big/ {\approx} \ ,$$

where $\approx$ is defined by

$$S \approx T \longleftrightarrow S_0 = T_0 \wedge S_1 = T_1 \ .$$

5.8 Two Applications of Banach's Fixed Point Theorem

In this section we prove a computable version of Banach's Fixed Point Theorem. Then we look at two applications: the Newton-Raphson method for zero-finding, and Picard's Theorem about unique existence of local solutions of ordinary differential equations. We also recall briefly the basic theory of integration and differentiation in a constructive setting from [BB85] and [TvD88a, VI.2]. The main difference between the classical and constructive theory of differential calculus is that *uniformly* differentiable maps are used in the constructive version.

The proofs of the theorems are essentially the same as in the classical setting. The proofs of Banach's Fixed Point Theorem and Picard's Theorem are the ones I learned in my undergraduate course on ordinary differential equations.

5.8.1 Banach's Fixed Point Theorem

A *contraction* between metric spaces (A, d_A) and (B, d_B) is a map $f \colon A \to B$ for which there exists $\alpha \in \mathbb{R}$, $0 < \alpha < 1$, called a *contraction factor* for f, such that

$$d_B(fx, fy) \leq \alpha \cdot d_A(x, y) \ . \qquad (x, y \in A)$$

Proposition 5.8.1 *A contraction is uniformly continuous.*

Proof. Suppose $f \colon A \to B$ is a contraction with contraction factor α. For every $\epsilon > 0$, $d_A(x, y) < \epsilon/\alpha$ implies $d_B(x, y) \leq \alpha \cdot d_A(x, y) < \epsilon$. ∎

Let $\mathsf{Contr}(A)$ be the space of all contractions on (A, d):

$$\mathsf{Contr}(A) = \left\{ f \in A^A \;\middle|\; \exists\, \alpha \in (0,1) \,.\, \forall\, x, y \in A \,.\, d(fx, fy) \leq \alpha \cdot d(x,y) \right\} \;.$$

Theorem 5.8.2 (Banach's Fixed Point Theorem) *Let (A, d) be an inhabited complete metric space. Every contraction on A has a unique fixed point. There exists a* fixed-point operator $\mathsf{fix}\colon \mathsf{Contr}(A) \to A$ *such that, for all $f \in \mathsf{Contr}(A)$,*

$$f(\mathsf{fix}\, f) = \mathsf{fix}\, f \;.$$

If $\#A$ is inhabited then fix *is computable.*

Proof. Because A is non-empty there exists a point $a \in A$. In case $\#A$ is non-empty, we know that there is $a \in \#A$. Let $f \in \mathsf{Contr}(A)$ and let α be a contraction factor for f. Define a sequence

$$x_0 = a \;, \qquad x_{n+1} = f x_n \;.$$

From the inequality

$$d(x_{n+2}, x_{n+1}) = d(f x_{n+1}, f x_n) \leq \alpha \cdot d(x_{n+1}, x_n)$$

we obtain the estimate $d(x_{n+2}, x_{n+1}) \leq \alpha^n d(fa, a)$, hence $\langle x_n \rangle$ is a Cauchy sequence. Let $x = \lim_{n\to\infty} x_n$. Then x is a fixed point of f:

$$fx = f(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} f x_n = \lim_{n\to\infty} x_{n+1} = x \;.$$

We can exchange the limit and f because f is uniformly continuous. To see that x is the unique fixed point of f, suppose $fy = y$ for some $y \in A$. Then

$$d(x, y) = d(fx, fy) \leq \alpha \cdot d(x, y) \;,$$

hence $d(x, y) \leq 0$. Since also $d(x, y) \geq 0$, $d(x, y) = 0$, therefore $x = y$. By Unique Choice, we obtain the fixed-point operator $\mathsf{fix}\colon \mathsf{Contr}(A) \to A$. It can be explicitly defined as

$$\mathsf{fix}\, f = \lim(\lambda n \,{:}\, \mathbb{N} \,.\, f^n a) \;.$$

where f^n is the n-fold composition of f, defined by $f^0 = 1_A$, $f^{n+1} = f \circ f^n$. Since we proved that every contraction has a unique fixed point, fix does not depend on the choice of a. If there is $a \in \#A$, then fix is computable. ∎

5.8.2 Differentiation and Integration

We briefly recall constructive theory of differentiation and integration. In a constructive setting the *uniformly differentiable* maps are better behaved than the usual point-wise differentiable maps. We encountered a similar situation in the theory of continuous maps where uniform continuity led to a more satisfactory theory than point-wise continuity.

Definition 5.8.3 Let $a < b$. A map $f\colon [a, b] \to \mathbb{R}$ is *uniformly differentiable* with derivative $f'\colon [a, b] \to \mathbb{R}$ when

$$\forall\, \epsilon > 0 \,.\, \exists\, \delta > 0 \,.\, \forall\, x, y \in [a, b] \,.\, \big(|x - y| < \delta \longrightarrow |(f'x)\cdot(x - y) - (fy - fx)| < \epsilon \cdot |x - y|\big) \;. \qquad (5.16)$$

Relation (5.16) is abbreviated as $\mathsf{der}(f, f')$.

We define the space of *differentiable maps* $\mathcal{C}^{(1)}[a,b]$ to be

$$\mathcal{C}^{(1)}[a,b] = \left\{ f \in \mathbb{R}^{[a,b]} \;\middle|\; \exists\, f' \in \mathbb{R}^{[a,b]} \,.\, \mathsf{der}(f, f') \right\} .$$

It can be proved that whenever $f \in \mathcal{C}^{(1)}[a,b]$ then both f and its derivative are uniformly continuous [TvD88a, VI.2.2]. Thus, $\mathcal{C}^{(1)}[a,b]$ is a subspace of $\mathcal{C}_{\mathrm{u}}[a,b]$. It is easy to show that the derivative f' is unique whenever it exists. Therefore by Unique Choice, we obtain the *derivative operator*

$$D \colon \#\Big(\mathcal{C}^{(1)}[a,b] \longrightarrow \mathcal{C}_{\mathrm{u}}[a,b]\Big) .$$

We also write f' for Df, f'' for $D^2 f$, and $f^{(n)}$ for $D^n f$. We define the spaces of *n-times uniformly differentiable maps* $\mathcal{C}^{(n)}[a,b]$ inductively by

$$\mathcal{C}^{(0)}[a,b] = \mathcal{C}_{\mathrm{u}}[a,b] , \qquad \mathcal{C}^{(n+1)}[a,b] = \left\{ f \in \mathcal{C}^{(n)}[a,b] \;\middle|\; \exists\, g \in \mathbb{R}^{[a,b]} \,.\, \mathsf{der}[D^n f, g] \right\} .$$

The space of uniformly smooth maps is defined by

$$\mathcal{C}^{(\infty)}[a,b] = \left\{ f \in \mathbb{R}^{[0,1]} \;\middle|\; \forall\, n \in \mathbb{N} .\, f \in \mathcal{C}^{(n)}[a,b] \right\} .$$

The derivative operator D can be restricted to a map

$$D \colon \mathcal{C}^{(n+1)}[a,b] \longrightarrow \mathcal{C}^{(n)}[a,b] , \qquad D \colon \mathcal{C}^{(\infty)}[a,b] \longrightarrow \mathcal{C}^{(\infty)}[a,b] .$$

The derivative operator satisfies the usual properties, such as

$$\begin{aligned}
D(\alpha \cdot f + \beta \cdot g) &= \alpha \cdot Df + \beta \cdot Dg , && (\alpha, \beta \in \mathbb{R}) \\
D(f \cdot g) &= Df \cdot g + f \cdot Dg , \\
D(f \circ g) &= (Df \circ g) \cdot Dg , \\
D(\lambda x .\, x^n) &= \lambda x .\, \left(n x^{n-1}\right) . && (n \in \mathbb{N})
\end{aligned}$$

A uniformly differentiable map can be expanded into a Taylor's series.

Theorem 5.8.4 (Taylor's Series) *Let $f \in \mathcal{C}^{(n+1)}[a,b]$ and let the reminder R be defined by*

$$R = f(b) - \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} \cdot (b-a)^k .$$

There exists $\xi \in [a,b]$ such that

$$\left| R - \frac{f^{(n+1)}(\xi)}{n!} \cdot (b-\xi)^n \cdot (b-a) \right| \leq \epsilon .$$

Proof. See [TvD88a, Theorem 6.2.5] or [BB85]. The difference between this theorem and the classical one is that in the classical setting the remainder R is *equal* to $f^{(n+1)}(\xi) \cdot (b-\xi)^{n+1}/(n+1)!$ for some $\xi \in [a,b]$. Also note that in the constructive version we have $n!$ instead of $(n+1)!$, $(\xi - a)^n$ instead of $(\xi - a)^{n+1}$, and there is an extra factor $(b-a)$. ∎

The Riemann integral of a uniformly continuous map f is defined by the usual Riemann sums,

$$\int_a^b f = \lim_{N\to\infty} \sum_{i=0}^{N} f(x_k) \cdot (x_{k+1} - x_k) ,$$

where $x_0 \leq \cdots \leq x_n$ are suitable partitions of $[a, b]$. The value of the limit does not depend on the partitions, see [TvD88a, Chapter 6, Section 2]. We obtain a computable uniformly continuous operator

$$\int_\square^\square \square \colon \#\left(\mathcal{C}_{\mathrm{u}}[a,b] \longrightarrow \mathbb{R}\right)$$

that satisfies the usual identities, such as

$$\int_a^b (\alpha \cdot f + \beta \cdot g) = \alpha \int_a^b f + \beta \int_a^b g , \qquad (\alpha, \beta \in \mathbb{R})$$

$$\int_a^b f + \int_b^c f = \int_a^c f , \qquad (a \leq b \leq c)$$

$$\int_a^b (f \cdot Dg) = (fb)(gb) - (fa)(ga) - \int_a^b (Df \cdot g) , \qquad (f, g \in \mathcal{C}^{(1)}[a,b])$$

$$\int_a^b \lambda x \,.\, x^n = \frac{b^{n+1} - a^{n+1}}{n+1} . \qquad (n \in \mathbb{N})$$

Theorem 5.8.5 (Fundamental Theorem of Calculus) *Let $f \in \mathcal{C}_{\mathrm{u}}[a,b]$. Then the map*

$$Fx = \int_a^x f \qquad (x \in [a,b])$$

is uniformly differentiable on $[a, b]$ and $F' = f$. If G is any uniformly differentiable map such that $G' = f$ then $F - G$ is a constant map.

Proof. See [BB85, Theorem 2.6.10]. ■

5.8.3 The Newton-Raphson Method

Before proceeding with the derivation of the Newton-Raphson method we prove that a stricly increasing continuous map that attains a negative and a positive value attains zero exactly once. We find the unique zero with the *trisection* method. We cannot use the classical *bisection* method, because it is not the case that every real number is either non-negative or non-positive.

Proposition 5.8.6 (Trisection Method) *Let $f \colon [a,b] \to \mathbb{R}$ be a strictly increasing continuous map such that $fa < 0$ and $fb > 0$. Then there exists exactly one $c \in [a,b]$ such that $fc = 0$.*

Proof. There exists at most one zero of f because f is injective: if $x \neq y$ then $x < y$ or $y < x$, therefore $fx < fy$ or $fy < fx$, hence $fx \neq fy$.

We find c with the trisection method. By Dependent Choice we can define sequences $\langle x_n \rangle_{n \in \mathbb{N}}$ and $\langle y_n \rangle_{n \in \mathbb{N}}$ as follows. Let $x_0 = a$ and $y_0 = b$. To define x_{n+1} and y_{n+1} from x_n and y_n, consider $t = (2x_n + y_n)/3$ and $s = (x_n + 2y_n)/3$. Since $t < s$, $ft < fs$, therefore $ft < 0$ or $0 < fs$. If $ft < 0$ let $x_{n+1} = t$ and $y_{n+1} = y_n$, and if $0 < fs$ let $x_{n+1} = x_n$ and $y_{n+1} = s$.

Since $|x_n - y_n| \leq (2/3)^n \cdot (b - a)$, it follows that $\langle x_n \rangle_{n \in \mathbb{N}}$ and $\langle y_n \rangle_{n \in \mathbb{N}}$ are Cauchy sequences with the same limit $c = \lim_{n \to \infty} x_n = \lim_{n \to \infty} y_n$. By construction fx_n is an increasing sequence of negative numbers, and fy_n is a decreasing sequences of positive numbers. Because f is continuous we get $fc = \lim_{n \to \infty} fx_n \leq 0 \leq \lim_{n \to \infty} fy_n = fc$, therefore $fc = 0$. ∎

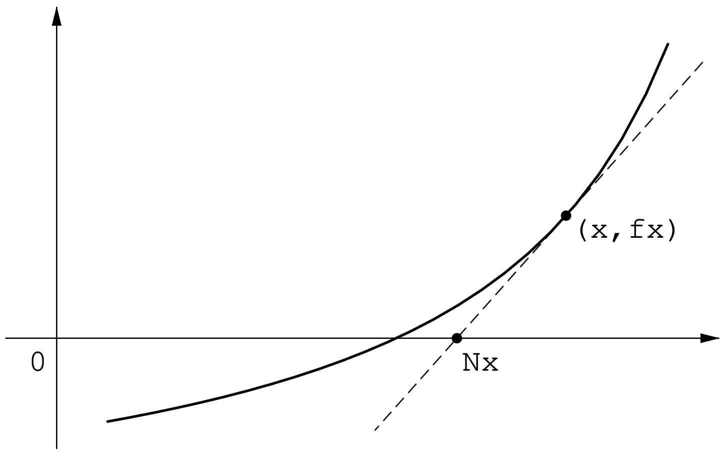

Figure 5.2: Newton-Raphson method

Let $a < b$, $m > 0$, $M > 0$ and $f \in \mathcal{C}^{(3)}[a, b]$ such that $fa < 0$, $fb > 0$, $f'x \geq m > 0$, and $f''x \geq 0$ for all $x \in [a, b]$. Since f'' is uniformly continuous on $[a, b]$ it has a supremum M on $[a, b]$ by Proposition 5.6.20. A picture of a typical such map is shown in Figure 5.2. The map f is strictly increasing: if $x < y$ then, by the Fundamental Theorem of Calculus,

$$fy - fx = \int_x^y f' > \int_x^y m = m \cdot (y - x) > 0 \,.$$

By Proposition 5.8.6 there exists exactly one $c \in [a, b]$ such that $fc = 0$. Define the *iterator* map $N \colon [c, b] \to \mathbb{R}$ by

$$Nx = x - \frac{fx}{f'x} \,.$$

Let us prove that there exists an interval $[c, b_1] \subseteq [c, b]$ on which N is a contraction. Since f'' is uniformly continuous it has a supremum M by Proposition 5.6.20. For every $x \in [c, b]$ we have

$$|N'x| = \left| \frac{(fx)(f''x)}{(f'x)^2} \right| \leq \frac{M}{m^2} \cdot |fx| \,.$$

Because $fc = 0$ and f is continuous, there exists $b_1 \in (c, b]$ such that, for every $x \in [c, b_1]$, $|fx| < m^2/(2M)$. Then for all $x \in [c, b_1]$,

$$|N'x| \leq \frac{M}{m^2} \cdot \frac{m^2}{2M} = \frac{1}{2} \; .$$

By the Fundamental Theorem of Calculus, for all $x, y \in [c, b_1]$,

$$\begin{aligned} |Ny - Nx| = |N(\max(x,y)) - N(\min(x,y))| &= \left| \int_{\min(x,y)}^{\max(x,y)} N' \right| \\ &\leq \int_{\min(x,y)}^{\max(x,y)} |N'| \leq \int_{\min(x,y)}^{\max(x,y)} \frac{1}{2} = (\max(x,y) - \min(x,y))/2 = |y - x|/2 \; . \end{aligned}$$

Therefore, N is a contraction on $[c, b_1]$. Let us show that if $c \leq x \leq b_1$ then $c \leq Nx \leq b_1$. Obviously, $Nx = x - (fx)/(f'x) \leq x \leq b_1$. The inequality $c \leq Nx$ is equivalent to $fx \leq (x - c) \cdot f'x$, and since $fc = 0$, this is the same as

$$fx \leq fc + (x - c) \cdot f'x \; .$$

Because f'' is non-negative, f' is increasing, hence:

$$fx = fc + \int_c^x f' \leq fc + \int_c^x f'x = fc + (x - c) \cdot f'x \; ,$$

as required. Define the sequence $\langle x_n \rangle_{n \in \mathbb{N}}$ by $x_0 = b_1$ and $x_{n+1} = Nx_n$. This is well defined because $N \colon [c, b_1] \to [c, b_1]$. By Banach's Fixed Point Theorem 5.8.2, N has exactly one fixed point and $\langle x_n \rangle_{n \in \mathbb{N}}$ converges to it with a geometric rate. The unique fixed point of N is c because $Nc = c - (fc)/(f'c) = c$.

5.8.4 Picard's Theorem for Ordinary Differential Equations

Let $H \colon [a, b] \times \mathbb{R} \to \mathbb{R}$ be a uniformly continuous map that is Lipshitz in the second argument, which means that there exists $\lambda \in \mathbb{R}$ such that, for all $x \in [a, b]$, $y, z \in \mathbb{R}$, $|H(x, y) - H(x, z)| \leq \lambda \cdot |z - y|$. Consider the ordinary differential equation

$$f'x = H(x, fx) \; , \qquad fx_0 = y_0 \; . \qquad (f \in \mathcal{C}^{(1)})$$

Here $x_0 \in (a, b)$ and $y_0 \in \mathbb{R}$. By the Fundamental Theorem of Calculus, the differential equation is equivalent to the integral equation

$$fx = y_0 + \int_{x_0}^x H(t, ft) \, dt \; . \qquad (f \in \mathcal{C}^{(1)})$$

Define an operator $\Phi \colon \mathcal{C}^{(1)}[a, b] \to \mathcal{C}^{(1)}[a, b]$ by

$$\Phi g = y_0 + \int_{x_0}^x H(t, gt) \, dt \; .$$

The integral equation can be written as a fixed point equation

$$f = \Phi f \; .$$

Let us show that on a sufficiently small interval (a_1, b_1) around x_0, Φ is a contraction:

$$|\Phi g - \Phi h| = \left| \int_{x_0}^{x} (H(t, gt) - H(t, ht))\, dt \right| \leq \left| \int_{x_0}^{x} \lambda \cdot |ht - gt|\, dt \right| \leq |x - x_0| \cdot \lambda \cdot \|h - g\|_\infty \,.$$

For $\epsilon < \min(1/\lambda, b - x_0, x_0 - a)$ the operator Φ is a contraction on the metric space $\mathcal{C}^{(1)}[x_0 - \epsilon, x_0 + \epsilon]$. By Banach's Fixed Point Theorem there exists a unique fixed point $f = \mathsf{fix}\,\Phi \in \mathcal{C}^{(1)}[x_0 - \epsilon, x_0 + \epsilon]$. Let us summarize what we have proved.

Theorem 5.8.7 (Picard's Theorem) *Let $H \colon [a, b] \to \mathbb{R}$ be a uniformly continuous map that is Lipschitz in the second argument, with a Lipschitz constant λ. Let $x \in (a, b)$ and $y_0 \in \mathbb{R}$. The ordinary differential equation*

$$f'x = H(x, fx)\,, \qquad\qquad f x_0 = y_0$$

has a unique solution in the interval $[x_0 - \epsilon, x_0 + \epsilon]$ for every $\epsilon < \min(1/\lambda, b - x_0, x_0 - a)$.

Bibliography

[ABS99] S. Awodey, L. Birkedal, and D.S. Scott. Local realizability toposes and a modal logic for computability. In L. Birkedal, J. van Oosten, G. Rosolini, and D.S. Scott, editors, *Tutorial Workshop on Realizability Semantics, FLoC'99, Trento, Italy, 1999*, volume 23 of *Electronic Notes in Theoretical Computer Science*. Elsevier, 1999.

[AC98] R.M. Amadio and P.-L. Curien. *Domains and Lambda-Calculi*, volume 46 of *Cambridge Tracts in Theoretical Computer Science*. Cambridge University Press, 1998.

[AL00] S. Abramsky and M. Lenisa. A fully complete PER model for ML polymorphic types. In P.G. Clote and H. Schwichtenberg, editors, *Computer Science Logic, 14th International Workshop, CSL 2000 Annual Conference of the EACSL Fischbachau, Germany, August 21-26, 2000 Proceedings*, volume 1862 of *Lecture Notes in Computer Science*. Springer, 2000.

[Bar85] H.P. Barendregt. *The Lambda Calculus. Its Syntax and Semantics*, volume 103 of *Studies in Logic and the Foundations of Mathematics*. North-Holland, revised edition, second printing edition, 1985.

[BB85] E. Bishop and D. Bridges. *Constructive Analysis*, volume 279 of *Grundlehren der math. Wissenschaften*. Springer-Verlag, 1985.

[BBS98] A. Bauer, L. Birkedal, and D.S. Scott. Equilogical spaces. *Preprint submitted to Elsevier*, 1998.

[BCSS97] L. Blum, F. Cucker, M. Shub, and S. Smale. *Complexity and Real Computation*. Springer Verlag, 1997.

[Bee85] M. Beeson. *Foundations of Constructive Mathematics*. Springer Verlag, 1985.

[Ber93] U. Berger. Total sets and objects in domain theory. *Annals of Pure and Applied Logic*, 60:91–117, 1993. Available at `http://www.mathematik.uni-muenchen.de/~berger/articles/apal/diss.dvi.Z`.

[Ber97a] U. Berger. *Continuous Functionals of Dependent and Transitive Types*. Habilitationsschrift, Ludwig-Maximilians-Universität München, 1997.

[Ber97b] U. Berger. Continuous functionals of dependent and transitive types. Short version. Available at `http://www.mathematik.uni-muenchen.de/~berger/habil/lc.dvi.Z.`, 1997.

[Ber00] Ulrich Berger. Effectivity and density in domains: A survey. In Giuseppe Rosolini Lars Birkedal, Jaap van Oosten and Dana S. Scott, editors, *Electronic Notes in Theoretical Computer Science*, volume 23. Elsevier Science Publishers, 2000.

[Bir99] L. Birkedal. *Developing Theories of Types and Computability.* PhD thesis, School of Computer Science, Carnegie Mellon University, December 1999. Available as CMU Technical Report: CMU-CS-99-173.

[Bla97a] J. Blanck. *Computability on Topological Spaces by Effective Domain Representations.* PhD thesis, Department of Mathematics, Uppsala University, 1997.

[Bla97b] J. Blanck. Domain representability of metric spaces. *Annals of Pure and Applied Logic*, 83(3):225–247, February 1997.

[Bla99] J. Blanck. Effective domain representations of $H(X)$, the space of compact subsets. *Theoretical Computer Science*, 219(1 & 2):19–48, May 1999.

[Bor94] F. Borceux. *Handbook of Categorical Algebra I. Basic Category Theory*, volume 51 of *Encyclopedia of Mathematics and Its Applications.* Cambridge University Press, 1994.

[BvORS99] L. Birkedal, J. van Oosten, G. Rosolini, and D.S. Scott. *Tutorial Workshop on Realizability Semantics, FLoC'99, Trento, Italy, 1999*, volume 23 of *Electronic Notes in Theoretical Computer Science.* Elsevier, 1999.

[BW99] Vasco Brattka and Klaus Weihrauch. Computability on subsets of Euclidean space I: Closed and compact subsets. *Theoretical Computer Science*, 219:65–93, 1999.

[Eda95] A. Edalat. Dynamical systems, measures and fractals via domain theory. *Information and Computation*, 120(1):32–48, 1995.

[Eda96] A. Edalat. Power domains and iterated function systems. *Information and Computation*, 124(2):182–197, 1996.

[Eda97] A. Edalat. Domains for computation in mathematics, physics and exact real arithmetic. *Bulletin of Symbolic Logic*, 3(4):401–452, December 1997.

[EGNR99a] Yu. L. Ershov, S.S. Goncharov, A. Nerode, and J.B. Remmel, editors. *Handbook of Recursive Mathematics, Volume 1.* Number 138 in Studies in Logic and the Foundations of Mathematics. Elsevier, 1999.

[EGNR99b] Yu. L. Ershov, S.S. Goncharov, A. Nerode, and J.B. Remmel, editors. *Handbook of Recursive Mathematics, Volume 2.* Number 138 in Studies in Logic and the Foundations of Mathematics. Elsevier, 1999.

[EH98] A. Edalat and R. Heckmann. A computational model for metric spaces. *Theoretical Computer Science*, 193(1 & 2):53–73, February 1998.

[EK99] A. Edalat and M. Krznarić. Numerical integration with exact real arithmetic. In Jirí Wiedermann, Peter van Emde Boas, and Mogens Nielsen, editors, *Automata, Languages and Programming, 26th International Colloquium, ICALP'99, Prague, Czech*

Republic, July 11-15, 1999, Proceedings, volume 1644 of *Lecture Notes in Computer Science*, pages 90–104. Springer, 1999.

[EL99] A. Edalat and A. Lieutier. Foundation of a computable solid modelling. In *Proceedings of ACM's Symposium on Solid Modeling'99*. ACM, 1999.

[EP97] A. Edalat and P.J. Potts. A new representation of for exact real numbers. *Electronic Notes in Theoretical Computer Science*, 6, 1997.

[Ers77] Y.L. Ershov. Model $\mathbb{C}$ for partial continuous functionals. In *Logic Colloquium 1976*, pages 455–467. North-Holland, 1977.

[ES99a] A. Edalat and P. Sünderhauf. Computable banach spaces via domain theory. *Theoretical Computer Science*, 219(1 & 2):169–184, May 1999.

[ES99b] A. Edalat and P. Sünderhauf. A domain-theoretic approach to real number computation. *Theoretical Computer Science*, 210(1):73–98, January 1999.

[Esc97] M.H. Escardó. PCF extended with real numbers: A domain-theoretic approach to higher-order exact real number computation. Technical Report ECS-LFCS-97-374, Department of Computer Science, University of Edinburgh, December 1997. PhD thesis at Imperial College of the University of London, 1996. `http://www.dcs.ed.ac.uk/lfcsreps/EXPORT/97/ECS-LFCS-97-374/`.

[Fra65] S.P. Franklin. Spaces in which sequences suffice. *Fund. Math.*, 57:107–115, 1965.

[GL00] Paul Gowland and David Lester. A survey of exact computer arithmetic. In Jens Blanck, Vasco Brattka, Peter Hertling, and Klaus Weihrauch, editors, *Computability and Complexity in Analysis*, volume 272 of *Informatik Berichte*, pages 99–115. FernUniversität Hagen, September 2000. CCA2000 Workshop, Swansea, Wales, September 17–19, 2000.

[GS90] C.A. Gunter and D.S. Scott. Semantic domains. In J. van Leeuwen, editor, *Handbook of Theoretical Computer Science*. Elsevier Science Publisher, 1990.

[Her99] P. Hertling. A real numbers structure that is effectively categorical. *Mathematical Logic Quarterly*, 45(2):147–182, 1999.

[HL00] J.D. Hamkins and A. Lewis. Infinite time turing machines. *Journal of Symbolic Logic*, 65(2):567–604, June 2000.

[Hyl82] J.M.E. Hyland. The effective topos. In A.S. Troelstra and D. Van Dalen, editors, *The L.E.J. Brouwer Centenary Symposium*, pages 165–216. North Holland Publishing Company, 1982.

[Hyl92] J.M.E. Hyland. First steps in synthetic domain theory. In A. Carboni, M.C. Pedicchio, and G. Rosolini, editors, *Category Theory '90*, volume 1144 of *Lectures Notes in Mathematics*, pages 131–156, Como, 1992. Springer-Verlag.

[JM89] P.T. Johnstone and I. Moerdijk. Local maps of toposes. *Proc. London Math. Soc.*, 3(58):281–305, 1989.

[Kle45] S.C. Kleene. On the interpretation of intuitionistic number theory. *Journal of Symbolic Logic*, 10:109–124, 1945.

[Kle59] S.C. Kleene. Countable functionals. In *Constructivity in Mathematics*, pages 81–100, 1959.

[Kre59] G. Kreisel. Interpretation of analysis by means of functionals of finite type. In A. Heyting, editor, *Constructivity in Mathematics*, pages 101–128. North-Holland, 1959.

[Kur52] C. Kuratowski. *Topologie.* Warszawa, 1952.

[KV65] S.C. Kleene and R.E. Vesley. *The Foundations of Intuitionistic Mathematics, especially in relation to recursive functions.* North-Holland Publishing Company, 1965.

[KW85] Christoph Kreitz and Klaus Weihrauch. Theory of representations. *Theoretical Computer Science*, 38:35–53, 1985.

[Lie99] P. Lietz. Comparing realizability over $\mathcal{P}\omega$ and K_2. Available at `http://www.mathematik.tu-darmstadt.de/~lietz/comp.ps.gz`, October 1999.

[Lon94] J.R. Longley. *Realizability Toposes and Language Semantics.* PhD thesis, University of Edinburgh, 1994.

[Lon99] J. Longley. Unifying typed and untyped realizability. Available at `http://www.dcs.ed.ac.uk/home/jrl/unifying.txt`, April 1999.

[McC84] D.C. McCarty. *Realizability and Recursive Mathematics.* D.Phil. Thesis, University of Oxford, 1984.

[MM92] S. Mac Lane and I. Moerdijk. *Sheaves in Geometry and Logic. A First Introduction to Topos Theory.* Springer-Verlag, New York, 1992.

[MS00] M. Menni and A. Simpson. Topological and limit-space subcategories of countably-based equilogical spaces. Submitted to Math. Struct. in Comp. Science, 2000.

[Nor98a] D. Normann. Categories of domains with totality. Available at `http://www.math.uio.no/~dnormann/`, June 1998.

[Nor98b] D. Normann. The continuous functionals of finite types over the reals. Preprint Series 19, University of Oslo, December 1998.

[PEE97] P.J. Potts, A. Edalat, and M.H. Escardó. Semantics of exact real arithmetic. In *Proceedings, 12th Annual IEEE Symposium on Logic in Computer Science, Warsaw, Poland, June 29–July 2, 1997*, pages 248–257. IEEE Computer Society Press, 1997.

[PER89] Marian B. Pour-El and J. Ian Richards. *Computability in Analysis and Physics.* Perspectives in Mathematical Logic. Springer, Berlin, 1989.

[Pho92] W.K.-S. Phoa. An introduction to fibrations, topos theory, the effective topos and modest sets. Technical Report ECS-LFCS-92-208, Department of Computer Science, University of Edinburgh, 1992.

[Plo98] G. Plotkin. Full abstraction, totality, and PCF. *Mathematical Structures in Computer Science*, 1998.

[Rey83] J.C. Reynolds. Types, abstraction and parametric polymorphism. In R.E.A. Mason, editor, *Information Processing 83, Proceedings of the IFIP 9th World Computer Congress, Paris, France, September 19–23, 1983*, pages 513–523. North-Holland/IFIP, 1983.

[Rey98] J.C. Reynolds. *Theories of Programming Languages.* Cambridge University Press, 1998.

[Ros86] G. Rosolini. *Continuity and Effectiveness in Topoi.* PhD thesis, University of Oxford, 1986.

[RS99] P. Rosolini and Th. Streicher. Comparing models of higher type computation. In L. Birkedal, J. van Oosten, G. Rosolini, and D.S. Scott, editors, *Tutorial Workshop on Realizability Semantics, FLoC'99, Trento, Italy, 1999*, volume 23 of *Electronic Notes in Theoretical Computer Science.* Elsevier, 1999.

[Sch96] H. Schwichtenberg. Density and choice for total continuous functionals. In P. Odifreddi, editor, *Kreiseliana. About and Around George Kreisel*, pages 335–365. A K Peters, 1996.

[Sch00] M. Schröder. Extended admissibility. Submitted for publication, May 2000.

[Sco72] D.S. Scott. Continuous lattices. In F.W. Lawvere, editor, *Toposes, Algebraic Geometry and Logic*, volume 274 of *Lecture Notes in Mathematics*, pages 97–136. Springer Verlag, 1972.

[Sco76] D.S. Scott. Data types as lattices. *SIAM Journal of Computing*, 5(3):522–587, 1976.

[Sco96] D.S. Scott. A new category? Unpublished Manuscript. Available at `http://www.cs.cmu.edu/Groups/LTC/`, December 1996.

[SHLG94] V. Stoltenberg-Hansen, I. Lindström, and E.R. Griffor. *Mathematical Theory of Domains.* Number 22 in Cambridge Tracts in Computer Science. Cambridge University Press, 1994.

[Soa87] R.I. Soare. *Recursively Enumerable Sets and Degrees.* Perspectives in Mathematical Logic. Springer-Verlag, 1987.

[Spr98] D. Spreen. On effective topological spaces. *The Journal of Symbolic Logic*, 63(1):185–221, March 1998.

[TvD88a] A.S. Troelstra and D. van Dalen. *Constructivism in Mathematics, An Introduction, Vol. 1.* Number 121 in Studies in Logic and the Foundations of Mathematics. North-Holland, 1988.

[TvD88b] A.S. Troelstra and D. van Dalen. *Constructivism in Mathematics, An Introduction, Vol. 2.* Number 123 in Studies in Logic and the Foundations of Mathematics. North-Holland, 1988.

[TZ99] J.V. Tucker and J.I. Zucker. Computation by 'while' programs on topological partial algebras. *Theoretical Computer Science*, 219:379–420, 1999.

[vOS98] J. van Oosten and A.K. Simpson. Axioms and (counter)examples in synthetic domain theory. Technical Report 1080, Department of Mathematics, Utrecht University, 1998.

[Wei85] Klaus Weihrauch. Type 2 recursion theory. *Theoretical Computer Science*, 38:17–33, 1985.

[Wei87] Klaus Weihrauch. *Computability*, volume 9 of *EATCS Monographs on Theoretical Computer Science*. Springer, Berlin, 1987.

[Wei95] Klaus Weihrauch. A simple introduction to computable analysis. Informatik Berichte 171, FernUniversität Hagen, Hagen, July 1995. 2nd edition.

[Wei00] Klaus Weihrauch. *Computable Analysis*. Springer-Verlag, 2000.

Index of Symbols

Categories

Combinatory Algebras

Logic

Spaces

Maps

Miscellaneous Symbols

$\square$	placeholder, e.g., $\langle\square,\square\rangle$ indicates the syntax of pairing
$x : A$	variable or point x has type A, 62
$x \in A$	variable or point x belongs to A, 62
$A \cong B$	A is isomorphic to B, 65
$A \subseteq B$	A is a subspace of B, 69
$A \subseteq_e B$	A is a subspace of B via map $e\colon A \to B$, 69
$A \cup B$	union of A and B, 164
$A \cap B$	intersection of A and B, 163
$\bigcup_{i\in I} A_i$	union of the family $(A_i)_{i\in I}$.
$\bigcap_{i\in I} A_i$	intersection of the family $(A_i)_{i\in I}$.
$x \vee y$	least upper bound of x and y
$x \wedge y$	greatest lower bound of x and y
$\bigvee_{x\in A} t(x)$	least upper bound, 164
$\bigwedge_{x\in A} t(x)$	greatest lower bound
$\mathcal{O}_X(a)$	open neighborhoods of $a \in X$
$\mathcal{O}(a)$	open enighborhoods of a
$\overline{U}$	topological closure of U
$\mathrm{B}(x, r)$	open ball with radius r and centered at x, 205
$\overline{\mathrm{B}}(x, r)$	closed ball with radius r and centered at x, 205
$\uparrow x$	upper set of x (the set of elements above x), 27, 24
$\downarrow x$	lower set of x (the set of elements below x)
$\bot$	bottom, the smallest element, 172
$\top$	top, the largest element
$x \ll y$	x is way below y
$x \approx_A y$	x is equivalent to y, w.r.t. PER $\approx_A$
$x \equiv_A y$	x is equivalent to y, w.r.t. equivalence relation $\equiv_A$
$\mathsf{E}_A x$	realizers of $x \in A$, existence predicate on A, 35
$x \lessgtr y$	apartness relation, x and y are apart, 192
$F \Longrightarrow G$	natural transformation from F to G
$\mathcal{K}(P)$	set of compact elements of a poset P, 27
$[\![\phi]\!]$	realizability interpretation of formula ϕ, 89
$[\![A]\!]$	realizability interpretation of space A, 89
$[\![t]\!]$	realizability interpretation of term t, 89
$\mathsf{Mod}(\mathbb{A}, \mathbb{A}_\sharp) \models \phi$	semantic validity, ϕ is valid in $\mathsf{Mod}(\mathbb{A}, \mathbb{A}_\sharp)$, 90
$\#_A(x)$	x is a computable ponit of A, 80
φ_k	k-th partial recursive function, 23
W_k	k-th r.e. set, 180
$a \Vdash_S x$	realizability relation on S, a realizes x, 35
$[a]^A_{\mathsf{r}}$	point of A that is realized by $a \in \mathsf{r}A$, 99
$\langle m, n\rangle$	encoding of pairs of natural numbers m and n, 24

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