15-859(D) Randomized Algorithms

Homework # 1 **due: September 29, 1998**

Groundrules:

- Homeworks will generally consist of *exercises*, easier problems designed to give you practice, and *problems*, that may be harder, trickier, and/or somewhat open-ended. You should do the exercises by yourself, but you may work with a friend on the harder problems if you want. One exception: no fair working with someone who has already figured out (or already knows) the answer. If you work with a friend, then write down who you are working with.
- If you've seen a problem before (sometimes I'll give problems that are "famous"), then say that in your solution (it won't affect your score, I just want to know). Also, if you use any sources other than the textbook, write that down too (it's fine to look up a complicated sum or inequality or whatever, but don't look up an entire solution).

Reading: Chapters 1 and 2 of M&R.

Exercises:

1. Prove that $\mathbf{E}[XY] = \mathbf{E}[X] \cdot \mathbf{E}[Y]$ for independent discrete random variables X and Y.
2. Prove that $\mathbf{BPP} \subseteq \mathbf{P/poly}$.
3. Imagine you are on a street corner waiting to flag a taxi. Suppose that each minute, there is a probability p (e.g., $p = 1/10$) that you will see a taxi, independent of how long you have been waiting. What's the expected waiting time until you see a taxi? What's the variance? (Prove your answers.)

Problems:

4. Quickselect is the following simple algorithm for finding the kth smallest element in an unsorted set S (in the book, this is called Find and is described on p.15).

   Quickselect(S, k):

   1. Pick a pivot element p at random from S.
   2. By comparing p to each element of S, split S into two pieces: $S_1 = \{x \in S : x < p\}$ and $S_2 = \{x \in S : x > p\}$.
   3. If $|S_1| = k - 1$ then output p.
      If $|S_1| > k - 1$ then output Quickselect(S_1, k).
      If $|S_1| < k - 1$ then output Quickselect$(S_2, k - |S_1| - 1)$.

Prove that the expected number of comparisons made by Quickselect on a set S of size n is at most $3.5n$. (Half credit for proving it's at most $4n$.) You may assume that initially $k = n/2$ (we are trying to find the median of S), which is the worst case.

5. (Due to T. Cover and M. Rabin). Consider the following game. A friend writes down two numbers on two slips of paper and then randomly puts one in one hand and the other in the other hand. You get to pick a hand and see the number in it. You then can either keep the number you saw or else return it and get the other number. Say you end up with the number x and the other number was y. Then, your gain is $x - y$.

   For a given (possibly randomized) strategy S, let $\mathbf{E}_{x,y}(S)$ denote its expected gain, given that the two numbers are x and y. For instance, if S is a deterministic strategy, then $\mathbf{E}_{x,y}(S) = \frac{1}{2}\cdot$(gain of S given that it is initially shown x and the other number is y) + $\frac{1}{2}\cdot$(gain of S given that it is initially shown y and the other number is x).

   1. Consider the strategy $S =$ "if the first number I see is ≥ -17, then I keep it, else I switch." What is $\mathbf{E}_{x,y}(S)$ in terms of x and y?
   2. Give a randomized strategy S such that $\mathbf{E}_{x,y}(S) > 0$ for all $x \neq y$.

6. (Due to E. Lauer). Suppose that in the above game, your friend picks his two numbers by choosing the pair $\{3^{i-1}, 3^i\}$ with probability $1/2^i$ (for $i \geq 1$). In class, we saw that it seems that no matter what number you see, you should switch, since for any x, given that you see x your expected gain of switching is greater than your gain of not switching. Of course, this is weird since the "always switch" strategy is equivalent to the "always keep" strategy.

   What is going on here? Try to give as satisfying an explanation as you can. Here are some interesting random variable to consider. Define X_i to be the probability that you are initially shown the number 3^i, times the expected gain of your strategy given that you are initially shown 3^i. Define Y_i to be the probability that the pair of numbers chosen is $\{3^i, 3^{i+1}\}$, times the expected gain of your strategy given that this is the pair chosen.