Received from Noga Alon, Feb 19, 1997

Expansion and eigenvalue gap (remark)

In [1] it is shown that the second largest eigenvalue of any d regular expander is bounded away from d. Here is a short proof of the main point in the somewhat easier fact that edge expansion implies an eigenvalue gap. This clearly supplies a certain estimate for vertex expansion as well.

Let $G = (V, E)$ be a d regular graph on n vertices denoted $\{1, 2, \ldots, n\}$, let $x_1 \geq x_2 \geq \ldots \geq x_n$ be nonnegative reals, and assume at least half of them are 0. Assume also that for every set U of at most half the vertices of G there are at least $c|U|$ edges between U and its complement, where c is some positive constant. Since $x_j = 0$ for all $j \geq n/2$,

$$\sum_{ij\in E} |x_i^2 - x_j^2| = \sum_{ij\in E, i<j} (x_i^2 - x_j^2) \geq \sum_{i:i<n/2} (x_i^2 - x_{i+1}^2)ci = c\sum_{i=1}^{n} x_i^2.$$

Therefore, by Cauchy Schwartz (twice)

$$c^2(\sum_{i=1}^{n} x_i^2)^2 \leq (\sum_{ij\in E} |x_i^2 - x_j^2|)^2 = (\sum_{ij\in E} |x_i - x_j||x_i + x_j|)^2$$

$$\leq (\sum_{ij\in E} (x_i - x_j)^2) \cdot (\sum_{ij\in E} (x_i + x_j)^2) \leq (\sum_{ij\in E} (x_i - x_j)^2) \cdot 2d(\sum_{i=1}^{n} x_i^2).$$

Hence

$$\sum_{ij\in E} (x_i - x_j)^2 \geq \frac{c^2}{2d} \sum_{i=1}^{n} x_i^2,$$

implying (with a slight extra effort) that the second largest eigenvalue of G is at most $d - \frac{c^2}{2d}$.

References

[1] N. Alon, Eigenvalues and expanders, Combinatorica 6(1986), 83-96.