# Lovasz Local Lemma

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## Motivation

Suppose an experiment can fail if any one of $n$ "bad" events $\mathcal{E}_1 \ldots \mathcal{E}_n$ occurs.

- Assume $\Pr(\mathcal{E}_i) \leq \frac{1}{2}$.

- We want to know if there is a non-zero probability that the experiment will succeed.

In general,

$$\Pr(\text{success}) = \Pr(\bigcap_{i=1}^{n} \bar{\mathcal{E}}_i)$$

- If the events $\mathcal{E}_1 \dots \mathcal{E}_n$ are independent :

$$\Pr(\text{success}) = \prod_{i=1}^{n} \Pr(\bar{\mathcal{E}}_i) \geq \frac{1}{2^n} > 0$$

- If the events are not independent, we can try using the union-bound :

$$\begin{aligned}
\Pr(\text{success}) &= 1 - \Pr(\overline{\bigcap_{i=1}^{n} \bar{\mathcal{E}}_i}) \\
&= 1 - \Pr(\bigcup_{i=1}^{n} \mathcal{E}_i) \\
&\geq 1 - \frac{n}{2}
\end{aligned}$$

which is too weak when $n \geq 2$.

- The Lovasz Local Lemma guarantees us that an experiment will succeed with non-zero probability when the events $\mathcal{E}_1 \dots \mathcal{E}_n$ are "almost independent" under certain conditions.

## Dependency Graph

Recall that an event $\mathcal{E}_i$ is mutually independent of a set $S$ of events if $\Pr[\mathcal{E}_i | \bigcap_{j \in T} \mathcal{E}_j] = \Pr[\mathcal{E}_i]$, where $T \subseteq S$.

$G$ is a dependency graph for the set of events $\mathcal{E}_1 \ldots \mathcal{E}_n$, if there is a vertex for each event, and an event $\mathcal{E}_i$ is mututally independent of all other events $\mathcal{E}_j$ such that $(\mathcal{E}_i, \mathcal{E}_j)$ is not an edge of $G$.

**Lemma 1 (Lovasz Local Lemma)** *Let* $G(V, E)$ *be a dependency graph for events* $\mathcal{E}_1 \dots \mathcal{E}_n$. *Suppose there exist* $x_i \in [0, 1)\ \forall i$ *such that*

$$Pr[\mathcal{E}_i] \leq x_i \prod_{(i,j)\in E} (1 - x_j)$$

*Then*

$$Pr[\bigcap_{i=1}^{n} \bar{\mathcal{E}}_i] \geq \prod_{i=1}^{n} (1 - x_i) > 0$$

**Corollary 1** *Let* $\mathcal{E}_1 \dots \mathcal{E}_n$ *be events in a probability space, with* $Pr[\mathcal{E}_i] \leq p\ \forall i$. *If each event is mutually independent of all other events except for at most* $d$ *events, and if* $ep(d+1) \leq 1$, *then* $Pr[\bigcap_{i=1}^{n} \bar{\mathcal{E}}_i] > 0$.

**Proof of the Corollary.**

Let

$$x_i = \frac{1}{d+1}$$

The conditions for the Local Lemma are satisfied if for each $i$

$$\begin{aligned} p &\leq x_i \prod_{(i,j)\in E} (1 - x_j) \\ &= \frac{1}{d+1}(1 - \frac{1}{d+1})^k \qquad (k \leq d) \end{aligned} \tag{1}$$

Now, $(1 - \frac{1}{d+1})^k > 1/e$ for $k \leq d$. Hence, rearranging terms, the conditions for the Local Lemma are satisfied if $ep(d+1) \leq 1$.

## Application

**Theorem 1** *There exists a satisfying truth assignment for any instance of $k$-SAT for $k \geq 10$ in which each variable is contained in at most $2^{k/2}$ clauses.*

*Proof.*

- Consider a random assignment of values to the variables, each variable assigned to 0 or 1 with probability 1/2.
- Let $\mathcal{E}_i$ denote the event that the $i$th clause is not satisfied by the random assignment. $\Pr[\mathcal{E}_i] = 2^{-k}$.
- $\mathcal{E}_i$ is mutually independent of all but atmost $k2^{k/2}$ other events.

- Apply the corollary with $d = k2^{k/2}$.

$$\begin{aligned} ep(d+1) &= e2^{-k}(k2^{k/2}+1) \\ &= e(k/2^{k/2}+1/2^{k}) \\ &> 1 \end{aligned}$$

  for $k \geq 10$.

Can construct an algorithm to find the satisfying assignment, but with weaker conditions. Similar to the method of conditional expectations.

## Some more applications

- Finding efficient, congestion free packet routing.

- Fault-tolerance in hypercubes

- Finding Disjoint paths in expanders.