Card Tricks

A group of N people is being tested for their ability to cooperate and plan. A game is set up as follows. A deck of N cards is secretly created. The numbers $1 \ldots N$ appear on the front of the cards, and the N peoples' names appear on the back of the cards.

A room is prepared with the cards laid out on a table, number side up. The people are brought into the room one at a time. A person's goal when visiting the room is to find her own card. She is allowed to turn over up to $N/2$ cards in this quest. After each visit to the room, the cards are returned to their original state.

The group wins if every single one of them finds her own card. The group loses if any one of them fails. The group can meet and plan their strategy before the game starts, but once the game starts, no further communication is allowed.

The problem is to find a strategy for the group which gives a significant probability of victory i.e. more than some positive constant.

Wait a minute! It's easy to see that (no matter what) each person has at most a 50% chance of finding her own name. This clearly implies that the group collectively, can win with probability at most $(1/2)^N$. So the puzzle is impossible, right?

Solution:

The specious analysis above, of course, assumes that the events are independent. The goal is to come up with a strategy that makes these events very dependent, so that although each one has at most a 50% chance of suceeding, the whole group is almost in "lockstep" so that they kind of all succeed together or all fail together.

The wording of the following solution was adapted from the one submitted by Chris Peikert, Abhi Shelat and Grant Wang of MIT.

If N is even we can be victorious with probability at least $1 - \ln(2) \cong 0.3068$. If N is odd the bound is $1 - \ln(2) - 1/(N-1)$. (Although this says nothing for $N = 3$, the algorithm below works with probability $1/6$ in that case.)

The strategy is simple: In the pre-game conference, the players randomly assign themselves unique "names" $1, 2, \ldots, N$. From now on, we dispense with the original names. So if a player sees one of the original names, she just replaces it mentally with the appropriate number.

When player i enters the room, she starts with the card labelled i, and just "follows the numbers" she gets. That is, if card i has the name of player j on the back, then she will next flip card j, which will have the name of some player k on the back, so the next card to flip will be k, etc.

A permutation π can be decomposed into cycles. Because player i just follows a cycle within π, she will succeed iff the cycle containing i has length $\leq N/2$. This is because the card sending her back to card i is the one with her name on it, and she finds that card as soon as she has traversed the entire cycle! Therefore, we want to know the probability that *all* the cycles of π have short length.

We can compute exactly the number of permutations containing a cycle of length $\ell > N/2$ as follows: first, there are $\binom{N}{N-\ell}$ ways to choose elements not in the cycle, and $(N-\ell)!$ ways to permute them amongst themselves. Then, there are $(\ell-1)!$ ways to get one cycle among the ℓ elements. Multiplying, we get $N!/\ell$, so the probability of a permutation having a cycle of length exactly ℓ is $1/\ell$.

For a moment assume N is even. We want to evaluate the probability that there's a cycle of length $> N/2$. This probability is just the sum of the probability that there is a cycle of each length $\ell = N, N-1, \ldots, (N/2)+1$, because these events are disjoint. This is just $H_N - H_{N/2}$, where H_i is the ith harmonic number. So to lower bound the probability that there is no long cycle, we need to upper bound $H_N - H_{N/2}$. By looking at the curve $1/x$ it's easy to see that:

$$H_N - H_{N/2} \leq \int_{N/2}^{N} \frac{1}{x} dx = \ln(N) - \ln(N/2) = \ln(2)$$

So the probability that there is no long cycle is at least $1 - \ln(2)$.

If N is odd, we need to get an upper bound on $H_N - H_{(N-1)/2}$. Integrating as above we get

$$H_N - H_{(N-1)/2} \leq \int_{(N-1)/2}^{N} \frac{1}{x} dx = \ln(N) - \ln(\frac{N-1}{2}) = \ln(2 + \frac{2}{N-1})$$

Because the slope of $\ln(x)$ is $1/2$ at $x = 2$, and ln is convex we can write:

$$\ln(2 + \frac{2}{N-1}) \leq \ln(2) + \frac{1}{N-1}$$

And the probability of success is at least 1 minus this.

A correct solution was also submitted by Andrey N. Chernikov of The College of William and Mary. This problem was adapted from one communicated to us by Michael Albert and Noga Alon.