Hat Problems

Here are some problems with hats. The scenarios are all very similar. At the start there are n people wearing hats.

Problem 1 Each hat is black or white. The people are standing in line. Person i can only see which types of hat persons $1, 2, \ldots, i-1$ are wearing. The inquisitor starts with person n and goes down the line, $n, n-1, \ldots, 1$ asking each person in turn what sort of hat they are wearing. Each person hears all the previous answers to the question but nothing more i.e. he/she does not hear directly whether a previous answer was correct. A wrong answer leads to elimination, which is painful and is to be avoided at all costs. The n people can decide on a strategy before the inquisition begins. Devise a strategy that will lead to as few eliminations as possible.

Solution: The following strategy results in at most one person being eliminated. When person n is interrogated he says his hat is black if the number of black hats he can see is odd and white otherwise. The answer might be wrong, but everybody has now been informed of the parity p_n of the number of black hats in front of n. Since person $n-1$ now knows p_n as well as p_{n-1}, he can derive the color of his own hat. Similarly person $n-2$ knows p_{n-1} and p_{n-2} and can derive the color of his own hat. All players after n can correctly state the color of their own hat.

Problem 2 Now our n hat wearing friends are standing in a circle and so everyone can see everybody else's hat. The hats have been assigned randomly and each allocation of hat colors is equally likely. At a certain moment in time each person must simultaneously shout "my hat is black" or "my hat is white" or "I haven't a clue". The team wins a big prize if at least one person gets the color of his hat right and no one gets it wrong (saying "I haven't a clue" is not getting it wrong). Of course, if anyone gets it wrong, the whole team is eliminated and this is painful. The prize is big enough to risk the pain and so devise a strategy which gives a good chance of success.

Try the same problem assuming there a q colors for the hats.

Solution Our solution uses the *probabilistic method* to prove the existence of a solution for which the our friends win with probability $1 - O(\frac{\ln n}{n})$. (The logarithm is not essential, see [2]).

Suppose that we partition $Q_n = \{0,1\}^n$ into 2 sets W, L which have the property that L is a *cover* i.e. if $x = x_1 x_2 \cdots x_n \in W = Q_n \backslash L$ then there is $y_1 y_2 \cdots y_n \in L$ such that $h(x, y) = 1$ where

$$h(x, y) = |\{j : x_j \neq y_j\}|$$

is the Hamming distance between x and y.

Assume that $0 \equiv White$ and $1 \equiv Black$. Person i knows x_j for $j \neq i$ (color of hat j) and if there is a unique value of x_i which places x in W then person i will declare that their hat has color i.

If indeed $x \in W$ then there is at least one person who will be in this situation and any such person will guess correctly.

Let $p = \frac{\ln n}{n}$. Choose L_1 randomly by placing $y \in Q_n$ into L_1 with probability p. Then let L_2 be those $z \in Q_n$ which are not at Hamming distance ≤ 1 from some member of L_1. Clearly $L = L_1 \cup L_2$ is a cover and

$$\mathbf{E}(|L|) = 2^n p + 2^n (1-p)^{n+1} \leq 2^n (p + e^{-np}) \leq 2^n \frac{2 \ln n}{n}.$$

So there must exist a cover of size at most $2^n \frac{2 \ln n}{n}$ and the players can win with probability at least $1 - \frac{2 \ln n}{n}$.
This argument is taken from [1] where using a similar argument, it is shown how to deal with q colors.
Problem 3 Our n hat wearing friends are again in a circle but this time the hats have been placed on their heads by Agar the Adversary who would like nothing better than to eliminate the lot of them. The hat wearers are allowed to think and there is a clock and after each minute passes, anybody is allowed to shout out what sort of hat they are wearing. Time is up after n minutes and anyone who hasn't declared will be eliminated. Also, if anyone declares wrongly, the whole group will be eliminated. Can they survive with any certainty?
Now consider the same situation where someone rushes into the room and truthfully shouts "there is someone wearing a black hat" or "you are all wearing white hats" before Agar can silence him. How does this help?
Solution Without the intruder it would be impossible to tell the difference say between one person wearing a Black hat and nobody wearing a Black hat.
Suppose then that there is an intruder. Assume that the intruder shouts there is someone wearing a black hat. Each person counts the number of Black hats that he sees. The strategy is: After k minutes have elapsed, if no-one has shouted then everyone who sees $k-1$ Black hats announces that he/she is wearing a Black hat.
The paradox to contemplate here is this: Suppose there are two or more black hats. Then the intruder's statement "there is someone wearing a black hat" is a fact that is known by everybody there! So of what value is it? How can making a statement that everybody knows already cause this chain of events to unfold? The answer is that the fact that there is at least one hat is known by everybody. But the announcement by the intruder becomes what is called *common knowledge*. That is, everybody knows it, and everybody knows that everybody knows it and everybody knows that everybody knows that everybody knows it, etc. This meta-information is used by the participants to derive the color of their own hat.

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This is the last puzzle by Alan and Danny. The puzzle page is being taken over by the **PUZZLE TOAD**.

References

[1] N. Alon, http://www.math.tau.ac.il/ nogaa/PDFS/extremalII.pdf.

[2] J. Aspnes, R. Beigel, M. Furst and S. Rudich, The expressive power of voting polynomials, *Combinatorica* 14 (1994)1-14.