Celebrity Problem

Celebrity Invanessa Marriot is being chased by the notorious paparazzi Gracias Prego. She has shrunk herself to a point x in the unit square $[0,1]^2$. Not to be out-done, Gracias has shrunk himself to a point $y \neq x$ in the same square. He has the latest laser camera and would love to get a picture of Invanessa in her reduced state. He points his camera and it emits a zero thickness light ray. If it hits a wall it will be reflected. He wants to shoot the ray so that it reaches Invanessa. She however, can place guards that are points $Z_1, Z_2, \ldots,$ in the square and if the beam tries to go through one of the guards it is blocked. How many guards does Invanessa need so that no beam from Gracias can reach her? Will a finite number do?

Solution: The following solution was provide by Karthik Lakshmanan.

The first thing we do is remove the complexity due to reflections. We consider $[0,1]^2$ to be just one square in a partition of the plane into unit squares. If two squares share an edge then one is the reflection of the other in that edge (see diagram). A light ray from Gracias is now just a line. When it meets a wall, instead of reflecting, it moves into the reflected copy of the square that is adjacent to it. Thus instead of one Invanessa to aim at, Gracias has a countable number to aim at. On the other hand, each guard can protect a countable number of Invanessa's.

In the diagram, the green mark represent the actual location of Invanessa and the red marks are the "virtual" copies. If the co-ordinates of Invanessa are (a, b) then Gracias needs to choose one of the targets $(2m \pm a, 2n \pm b)$ for $m, n \in Z = \{0, \pm 1, \pm 2, \ldots, \}$.

If the co-ordinates of Gracias are (c, d) then this means that the laser beams must be one of the lines:

$$(y - d)(2m \pm a - c) = (x - c)(2n \pm b - d).$$

The points

$$(x, y) = (m \pm a/2 + c/2,\ n \pm b/2 + d/2) \tag{1}$$

lie on these lines.

It follows that placing guards at the 16 points

$$\begin{aligned}
&(a/2 \pm c/2, b/2 \pm d/2)\\
&(a/2 \pm c/2, 1 - (b/2 \pm d/2))\\
&(1 - (a/2 \pm c/2), b/2 \pm d/2)\\
&(1 - (a/2 \pm c/2), 1 - (b/2 \pm d/2))
\end{aligned}$$

will protect Invanessa from Gracias.

Here we have assumed that $a < c$, if not interchange a and c. Similarly for b, d.

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