

**Let there be light:**

Darkness has descended on the plane of Mopok. All electricity has been cut-off by the Trydan Corporation. All they have to light the plane are  $2k$  lighthouses that run on cooking oil. Here  $k \geq 1$  is a positive integer. They can be kept going indefinitely, the Mopoks love their french fries, but each lighthouse can only illuminate a sector of  $\frac{360}{k}$  degrees. The lamp of each lighthouse can be rotated but it must be fixed before the light is turned on. Can the lights be rotated so that the whole plane is covered?

**Solution** We give a solution due to Galperin and Galperin [1] and independently to Günter Rote [2] that was brought to our attention by Günter Rote. It is one of three applications of graph matching presented in a short paper by Gerhard Woeginger [3]. Our presentation is taken from [3].

We can prove that the plane of Mopok can be illuminated under quite general circumstances: Suppose that there are  $n$  lighthouses i.e. no longer restrict their number to be even and that  $i$ th lighthouse illuminates a sector of angle  $\alpha_i$  where  $\sum_{i=1}^n \alpha_i = 2\pi$ . Fix any point as origin and arbitrarily partition the plane into  $n$  sectors  $W_1, W_2, \dots, W_n$  where  $W_i$  has angle  $\alpha_i$ . We show that the sectors can be translated to the lighthouses at  $P_1, P_2, \dots, P_n$  so that the plane is illuminated. Let  $\vec{v}_k$  denote the vector of length  $1/\cos(\alpha_k/2)$  in the direction of the angle-bisector of  $W_k$ . We write  $\langle \vec{a}, \vec{b} \rangle$  to denote the inner product of the two vectors  $\vec{a}, \vec{b}$  and we write  $\|\vec{a}\|$  to denote the length of vector  $\vec{a}$ . Translating sector  $W_k$  from the origin to some other anchor point  $P$  yields the region  $W_k[P]$ .

**Lemma 1.** *Consider two points  $P$  and  $Q$ , and two integers  $k, \ell$  with  $1 \leq k, \ell \leq n$ . If  $Q$  lies inside  $W_k[P]$  but outside  $W_\ell[P]$ , then  $\langle \vec{PQ}, \vec{v}_k \rangle > \langle \vec{PQ}, \vec{v}_\ell \rangle$ .*

**Proof** Let  $\beta$  denote the angle between  $\vec{PQ}$  and  $\vec{v}_k$ . Then  $Q \in W_k[P]$  yields  $|\beta| \leq \alpha k/2$ , which implies  $\cos(\beta) \geq \cos(\alpha k/2)$ . Whence

$$\langle \vec{PQ}, \vec{v}_k \rangle = \cos(\beta) \cdot \|\vec{PQ}\| \cdot \|\vec{v}_k\| = \cos(\beta) \cdot \|\vec{PQ}\| / \cos(\alpha k/2) \geq \|\vec{PQ}\|.$$

A symmetric argument centered around the angle between  $\vec{PQ}$  and vector  $\vec{v}_\ell$  yields that  $\langle \vec{PQ}, \vec{v}_\ell \rangle < \|\vec{PQ}\|$ .  $\square$

We now fix some arbitrary point  $Q$  in the plane, and we define the  $Q$ -cost of assigning sector  $W_k$  to point  $P_i$  as  $\langle P_i Q, \vec{v}_k \rangle$ . We compute a matching  $\mathcal{M}$  between sectors and points that maximizes the total  $Q$ -cost. By renumbering the points, we may assume that for  $k = 1, \dots, n$  matching  $\mathcal{M}$  assigns sector  $W_k$  to point  $P_k$ , and that hence the translated sectors are  $W_k[P_k]$ . Here is the first beautiful observation:

**Lemma 2.** *If matching  $\mathcal{M}$  maximizes the  $Q$ -cost, then point  $Q$  is covered by one of the translated sectors.*

**Proof** Suppose not. Observe that for every point  $P_i$ , there exists some sector  $W_k$  with  $Q \in W_k[P_i]$ ; we denote this situation by  $i \rightarrow k$ . Clearly, the relation  $\rightarrow$  contains some directed cycle  $c_1 \rightarrow c_2 \rightarrow \dots \rightarrow c_s \rightarrow c_{s+1} = c_1$  for some  $s \geq 2$ . But then the following cyclic switch would increase the  $Q$ -cost of

$\mathcal{M}$ : For  $k = 1, \dots, s$  re-assign sector  $W_{c_{k+1}}$  from point  $P_{c_{k+1}}$  to point  $P_{c_k}$ . By Lemma 1, this contradicts the maximality of  $\mathcal{M}$ .  $\square$

Here is the second beautiful observation: The  $Q$ -costs do depend on the choice of  $Q$ , but the optimal matching  $\mathcal{M}$  does not. Indeed, let  $\pi$  be some assignment of sectors to points and consider two points  $Q_1$  and  $Q_2$ . Then the difference between the two objective values of  $\pi$  under  $Q_1$ -costs and under  $Q_2$ -costs equals

$$\begin{aligned} \sum_{k=1}^n \langle P_k \vec{Q}_1, \vec{v}_{\pi(k)} \rangle - \sum_{k=1}^n \langle P_k \vec{Q}_2, \vec{v}_{\pi(k)} \rangle &= \\ \sum_{k=1}^n \langle P_k \vec{Q}_1, \vec{v}_{\pi(k)} - P_k \vec{Q}_2, \vec{v}_{\pi(k)} \rangle &= \sum_{k=1}^n \langle Q_2 \vec{Q}_1, \vec{v}_{\pi(k)} \rangle = \langle Q_2 \vec{Q}_1, \sum_{k=1}^n \vec{v}_k \rangle. \end{aligned}$$

Since this difference is independent of the assignment  $\pi$ , matching  $\mathcal{M}$  maximizes the  $Q$ -cost for every possible point  $Q$ . Hence, by Lemma 2 every possible point  $Q$  is covered by one of the translated sectors.

## References

- [1] V. Galperin and G. Galperin, Osveschenije ploskosti prozhektorami Illuminating a plane with spotlights. Kvant 11 (1981) 28-30. (In Russian).
- [2] G. Rote, Two applications of point matching, Abstracts of the 25th European Workshop on Computational Geometry (EuroCG'09), Brussels, March 2009, 187-189.
- [3] G.J. Woeginger, Match, match, match and match again, OPTIMA 73 (2007) 6-8.