Odd goings on:
In a party with n people, it is known that for every nonempty subset S of people, there is at least one person, inside or outside S, such that this person has an odd number of friends in S. Prove that n is even.
Solution: The proof given here relies on linear algebra over the field $\mathbb{F}_2$. It would be nice to have an alternative proof that did not do so, implicitly or explicitly.
Let $G = (V, E)$ be the graph defined by the friendship relation i.e. V is the set of people and (u, v) is an edge iff u and v are friends. Let A be the adjacency matrix of G. This is then n by n, with 0's on the main diagonal and 1 in position (i, j) if the i-th and j-th vertices are adjacent. Note that A is symmetric. We will consider it to be a matrix over $\mathbb{F}_2$, the finite field of order 2.
The key observation is that the given condition is equivalent to A being nonsingular. Indeed, let S be any nonempty subset of vertices, and let v be the 0/1 indicator vector of S. That is, v is a column vector of n elements, and there is a 1 in the i-th position iff the i-th vertex is in S. Then, the condition implies that the product Av is not the all-zero vector. That's because there must be some vertex, say the j-th, which has an odd number of neighbors in S. Then the j-th position of the product Av will be 1. (Everything is mod 2.)
Therefore, we conclude that for every v except for the all-zero vector, the matrix product $Av \neq 0$. This means that A is a nonsingular, symmetric matrix over $\mathbb{F}_2$ with 0's on the main diagonal. We now claim that this requires the dimension n to be even.
Suppose for the sake of contradiction that n is odd. Let us now show that the determinant of A is 0. This is will contradict the non-singularity of A, and this works over any field (not just the Real/Complex). Over $\mathbb{F}_2$, the determinant of a 0/1 matrix has a particularly nice form, because signs are irrelevant: -1 = +1 mod 2. Thus, the determinant counts the number of ways there are to pick exactly n positions in the n by n matrix A, such that we pick exactly one position per row and one position per column, and all n entries in the picked positions are 1. Call such a selection a Special Selection. It is easy to see that this matches the formal ordinary definition of the determinant, which was

$$\sum_{\pi} sign(\pi) \prod_{i} A_{i,\pi(i)}$$

where the sum is over the permutations π of $\{1, 2, \ldots, n\}$.
We must show that assuming n is odd, the number of Special Selections is even. For this, we will show how to pair up the Special Selections among themselves, with none left over. The pairing is by reflection over the main diagonal. Since A is symmetric, it is clear that the reflection of a Special Selection is still Special. It remains to check that no Special Selection reflects to become itself again. Here we use the fact that A has only 0's on the main diagonal, so the only way a Special Selection could reflect to become itself would be if all of the n selected positions decomposed into pairs that were in positions that were mirror images across the main diagonal. But this is impossible because n is odd.

Hence no Special Selection reflects back to itself, and we conclude that in the determinant calculation, we have an even number of Special Selections, yielding a determinant of 0.

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